# Derivations of Applied Mathematics

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# Preface

[You are reading a prepublished draft of the book, dated on the title page.]

I never meant to write this book. It emerged unheralded, unexpectedly.

The book began in 1983 when a high-school classmate challenged me to prove the Pythagorean theorem on the spot. I lost the dare, but looking the proof up later I recorded it on loose leaves, adding to it the derivations of a few other theorems of interest to me. From such a kernel the notes grew over time, until family and friends suggested that the notes might make the material for the book you hold.

The book is neither a tutorial on the one hand nor a bald reference on the other. It is a study reference, in the tradition of, for instance, Kernighan's and Ritchie's *The C Programming Language* [37]. In this book, you can look up some particular result directly, or you can begin on page one and read—with toil and commensurate profit—straight through to the end of the last chapter.

The reader who has come to look up a particular result will, I trust, already have turned ahead to it, so let me here address the other reader, who means to begin on page one. The book as a whole surveys the general mathematical methods common to engineering, architecture, chemistry and physics. The book thus is a marshal or guide. It concisely arrays and ambitiously reprises the mathematics the practicing scientist or engineer will likely once have met but may imperfectly recall, deriving the mathematics it reprises, filling gaps in one's knowledge while extending one's mathematical reach. For the prospective, college-bound scientist or engineer, to the extent to which study of the book proper is augmented by reflection, review, exercise and application, the book offers a concrete means to earn an enduring academic advantage.

Its focus on derivations is what principally distinguishes this book from the few others[1] of its class. No result is presented here but that it is

[1] Other books of the class include [14][34][3].

justified in a style engineers, scientists and other applied mathematicians will recognize—not indeed the high style of the professional mathematician, which serves other needs; but the long-established style of applications.

### Plan

Following its introduction in Ch. 1 the book comes in three parts. The first part begins with a brief review of classical algebra and geometry and develops thence the *calculus* of a single complex variable, this calculus being the axle as it were about which higher mathematics turns. The second part constructs the initially oppressive but broadly useful mathematics of *matrices* and *vectors,* without which so many modern applications (to the fresh incredulity of each generation of college students) remain analytically intractable—the jewel of this second part being the *eigenvalue* of Ch. 14. The third and final part, the most interesting but also the most advanced, introduces the the mathematics of the *Fourier transform, probability* and the *wave equation*—each of which is enhanced by the use of *special functions,* the third part's unifying theme.

Thus, the book's overall plan, though extensive enough to take several hundred pages to execute, is straightforward enough to describe in a single sentence. The plan is to derive as many mathematical results, useful to engineers and their brethren, as possible in a coherent train, recording and presenting the derivations together in an orderly manner in a single volume. What constitutes "useful" or "orderly" is a matter of perspective and judgment, of course. My own peculiar heterogeneous background in military service, building construction, electrical engineering, electromagnetic analysis and Debian development, my nativity, residence and citizenship in the United States, undoubtedly bias the selection and presentation to some degree. How other authors go about writing their books, I do not know, but I suppose that what is true for me is true for many of them also: we begin by organizing notes for our own use, then observe that the same notes might prove useful to others, and then undertake to revise the notes and to bring them into a form which actually is useful to others. Whether this book succeeds in the last point is for the reader to judge.

### Notation

The book deviates from—or cautiously improves, if you will endorse the characterization—the conventional notation of applied mathematics in one conspicuous respect which, I think, requires some defense here. The book

employs hexadecimal numerals.

Why not decimal only? There is nothing wrong with decimal numerals as such. I am for them, retaining especially a partial regard for the stately grandeur of the decimal numerals MDCLXVI of the famous Roman style. Decimal numerals are well in history and anthropology (man has ten fingers), finance and accounting (dollars, cents, pounds, shillings, pence: the base hardly matters), law and engineering (the physical units are arbitrary anyway); but they are merely serviceable in mathematical theory, never aesthetic. Custom is not always defaced, but sometimes adorned, by the respectful attendance of a prudent discrimination. It is in this spirit alone that hexadecimal numerals are given place here.

Admittedly, one might judge the last to be more excuse than cause. Yet, though a dreary train of sophists down the years, impatient of experience, eager to innovate, has indisputably abused such causes—in ways which the mature reader of a certain cast of mind will find all too familiar—such causes would hardly merit abuse did they not sometimes hide a latent measure of justice. It is to the justice, or at least to the aesthetic, rather than to the sophistry that I affect to appeal here.

There unfortunately really is no gradual way to bridge the gap to hexadecimal (shifting to base eleven, thence to twelve, etc., is no use). If one wishes to reach hexadecimal ground then one must leap. Twenty years of keeping my own private notes in hex have persuaded me that the leap justifies the risk. In other matters, by contrast, the book leaps seldom. The book in general walks a tolerably conventional applied mathematical line.

### Publication

The book belongs to the emerging tradition of open-source software where at the time of this writing it fills a void. Nevertheless it is a *book,* not a program. Lore among open-source developers holds that open development inherently leads to superior work. Well, maybe. Often it does in fact. Personally with regard to my own work, I should rather not make too many claims. It would be vain to deny that professional editing and formal peer review, neither of which the book enjoys, had substantial value. On the other hand, it does not do to despise the *amateur* (literally, one who does for the love of it: not such a bad motive, after all[2]) on principle, either—unless one would on the same principle despise a Socrates, a Washington, an Einstein or a Debian Developer [15]. Open source has a spirit to it which

[2]The expression is derived from an observation I seem to recall George F. Will making.

leads readers to be far more generous with their feedback than ever could be the case with a traditional, proprietary book. Such readers, among whom a surprising concentration of talent and expertise are found, enrich the work freely. This has value, too.

The book's open-source publication implies that it can neither go out of print nor grow hard to find. You can indeed copy and distribute the book yourself if you wish. Most readers naturally will be satisfied merely to read the book, but the point is that other writers can refer *their* readers hither without undue fear of imposition, thus saving scarce pages in the other writers' own books and articles, pages otherwise devoted to mathematical appendices rather omitted. After all, a reference one's reader can conveniently follow differs practically from a reference one's reader can follow with difficulty if at all.

### Edition

The book is extensively footnoted. Some of the footnotes unremarkably cite sources but many are discursive in nature, offering nonessential material which, though edifying, coheres insufficiently well to join the book's main narrative. The footnote is an imperfect messenger, of course. Catching the reader's eye, it can break the flow of otherwise good prose. Modern publishing promotes various alternatives to the footnote—numbered examples, sidebars, special fonts, colored inks, etc. Some of these are merely trendy. Others, like numbered examples, really do help the right kind of book; but for this book the humble footnote, long sanctioned by an earlier era of publishing, extensively employed by such sages as Gibbon [24] and Shirer [56], seems the most able messenger. In this book it shall have many messages to bear.

In typical science/engineering style, the book numbers its sections, tables, figures and formulas, but not its theorems, the last of which it generally sets in italic type. Within the book, a theorem is referenced by the number of the section that states it.

The book subjoins an alphabetical index as a standard convenience. Even so, the canny reader will avoid using the index (of this and most other books), which alone of the book's pages is not to be regarded as a proper part of the book. Such a reader will tend rather to consult the book's table of contents which is a proper part.

The book includes a bibliography listing works I have referred to while writing. This is as it should be. Mathematics however by nature promotes queer bibliographies, for its methods and truths are established by derivation

rather than authority. Much of the book consists of common mathematical knowledge or of proofs I have worked out with my own pencil from various ideas gleaned—who knows from where?—over the years. The latter proofs are perhaps original or semi-original from my personal point of view but it is unlikely that many if any of them are truly new. To the initiated, the mathematics itself often tends to suggest the form of the proof: if to me, then surely also to others who came before; and even where a proof is new the idea proven probably is not.

**Acknowledgements**

Among the bibliography's entries stands a reference [10] to my doctoral adviser G.S. Brown, though the book's narrative seldom explicitly invokes the reference. Prof. Brown had nothing directly to do with the book's development, for a substantial bulk of the manuscript, or of the notes underlying it, had been drafted years before I had ever heard of Prof. Brown or he of me, and my work under him did not regard the book in any case. However, the ways in which a good doctoral adviser influences his student are both complex and deep. Prof. Brown's style and insight touch the book in many places and in many ways, usually too implicitly coherently to cite.

Steady encouragement from my wife and children contribute to the book in ways only an author can properly appreciate.

More and much earlier than to Prof. Brown or to my wife and children, the book owes a debt to my mother and, separately, to my father, without either of whom the book would never have come to be. Admittedly, any author of any book might say as much in a certain respect, but it is no office of a mathematical book's preface to burden readers with an author's expressions of filial piety. No, it is in entirely another respect that I lay the matter here. My mother taught me at her own hand most of the mathematics I ever learned as a child, patiently building a foundation that cannot but be said to undergird the whole book today. More recently, my mother has edited tracts of the book's manuscript. My father generously financed much of my formal education but—more than this—one would have had to grow up with my brother and me in my father's home to appreciate the grand sweep of the man's curiosity, the general depth of his knowledge, the profound wisdom of his practicality and the enduring example of his love of excellence.

May the book deserve such a heritage.

THB

# Chapter 1

# Introduction

This is a book of applied mathematical proofs. If you have seen a mathematical result, if you want to know why the result is so, you can look for the proof here.

The book's purpose is to convey the essential ideas underlying the derivations of a large number of mathematical results useful in the modeling of physical systems. To this end, the book emphasizes main threads of mathematical argument and the motivation underlying the main threads, deëmphasizing formal mathematical rigor. It derives mathematical results from the purely applied perspective of the scientist and the engineer.

The book's chapters are topical. This first chapter treats a few introductory matters of general interest.

## 1.1 Applied mathematics

What is applied mathematics?

> Applied mathematics is a branch of mathematics that concerns itself with the application of mathematical knowledge to other domains. . . . The question of what is applied mathematics does not answer to logical classification so much as to the sociology of professionals who use mathematics. [41]

That is about right, on both counts. In this book we shall define *applied mathematics* to be correct mathematics useful to scientists, engineers and the like; proceeding not from reduced, well-defined sets of axioms but rather directly from a nebulous mass of natural arithmetical, geometrical and classical-algebraic idealizations of physical systems; demonstrable but generally lacking the detailed rigor of the professional mathematician.

## 1.2 Rigor

It is impossible to write such a book as this without some discussion of mathematical rigor. Applied and pure mathematics differ principally and essentially in the layer of abstract definitions the latter subimposes beneath the physical ideas the former seeks to model. Notions of mathematical rigor fit far more comfortably in the abstract realm of the professional mathematician; they do not always translate so gracefully to the applied realm. The applied mathematical reader should be aware of this difference.

### 1.2.1 Axiom and definition

Ideally, a professional mathematician knows or precisely specifies in advance the set of fundamental axioms he means to use to derive a result. A prime aesthetic here is irreducibility: no axiom in the set should overlap the others or be specifiable in terms of the others. Geometrical argument—proof by sketch—is distrusted. The professional mathematical literature discourages undue pedantry indeed, but its readers do implicitly demand a convincing assurance that its writers *could* derive results in pedantic detail if called upon to do so. Precise definition here is critically important, which is why the professional mathematician tends not to accept blithe statements such as that

$$\frac{1}{0} = \infty,$$

without first inquiring as to exactly what is meant by symbols like 0 and $\infty$.

The applied mathematician begins from a different base. His ideal lies not in precise definition or irreducible axiom, but rather in the elegant modeling of the essential features of some physical system. Here, mathematical definitions tend to be made up *ad hoc* along the way, based on previous experience solving similar problems, adapted implicitly to suit the model at hand. If you ask the applied mathematician exactly what his axioms are, which symbolic algebra he is using, he usually doesn't know; what he knows is that the bridge is founded in certain soils with specified tolerances, suffers such-and-such a wind load, etc. To avoid error, the applied mathematician relies not on abstract formalism but rather on a thorough mental grasp of the essential physical features of the phenomenon he is trying to model. An equation like

$$\frac{1}{0} = \infty$$

may make perfect sense without further explanation to an applied mathematical readership, depending on the physical context in which the equation

is introduced. Geometrical argument—proof by sketch—is not only trusted but treasured. Abstract definitions are wanted only insofar as they smooth the analysis of the particular physical problem at hand; such definitions are seldom promoted for their own sakes.

The irascible Oliver Heaviside, responsible for the important applied mathematical technique of phasor analysis, once said,

> It is shocking that young people should be addling their brains over mere logical subtleties, trying to understand the proof of one obvious fact in terms of something equally ... obvious. [46]

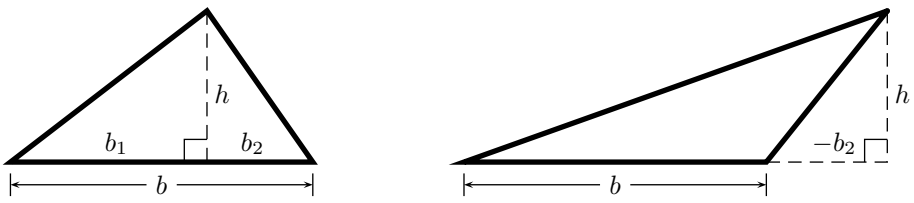
Exaggeration, perhaps, but from the applied mathematical perspective Heaviside nevertheless had a point. The professional mathematicians Richard Courant and David Hilbert put it more soberly in 1924 when they wrote,

> Since the seventeenth century, physical intuition has served as a vital source for mathematical problems and methods. Recent trends and fashions have, however, weakened the connection between mathematics and physics; mathematicians, turning away from the roots of mathematics in intuition, have concentrated on refinement and emphasized the postulational side of mathematics, and at times have overlooked the unity of their science with physics and other fields. In many cases, physicists have ceased to appreciate the attitudes of mathematicians. [14, Preface]

Although the present book treats "the attitudes of mathematicians" with greater deference than some of the unnamed 1924 physicists might have done, still, Courant and Hilbert could have been speaking for the engineers and other applied mathematicians of our own day as well as for the physicists of theirs. To the applied mathematician, the mathematics is not principally meant to be developed and appreciated for its own sake; it is meant to be *used.* This book adopts the Courant-Hilbert perspective.

The introduction you are now reading is not the right venue for an essay on why both kinds of mathematics—applied and professional (or pure)—are needed. Each kind has its place; and although it is a stylistic error to mix the two indiscriminately, clearly the two have much to do with one another. However this may be, this book is a book of derivations of applied mathematics. The derivations here proceed by a purely applied approach.

Figure 1.1: Two triangles.




### 1.2.2 Mathematical extension

Profound results in mathematics are occasionally achieved simply by extending results already known. For example, negative integers and their properties can be discovered by counting backward—3, 2, 1, 0—then asking what follows (precedes?) 0 in the countdown and what properties this new, negative integer must have to interact smoothly with the already known positives. The astonishing Euler's formula (§ 5.4) is discovered by a similar but more sophisticated mathematical extension.

More often, however, the results achieved by extension are unsurprising and not very interesting in themselves. Such extended results are the faithful servants of mathematical rigor. Consider for example the triangle on the left of Fig. 1.1. This triangle is evidently composed of two right triangles of areas

$$\begin{aligned} A_1 &= \frac{b_1 h}{2}, \\ A_2 &= \frac{b_2 h}{2} \end{aligned}$$

(each right triangle is exactly half a rectangle). Hence the main triangle's area is

$$A = A_1 + A_2 = \frac{(b_1 + b_2)h}{2} = \frac{bh}{2}.$$

Very well. What about the triangle on the right? Its $b_1$ is not shown on the figure, and what is that $-b_2$, anyway? Answer: the triangle is composed of the *difference* of two right triangles, with $b_1$ the base of the larger, overall one: $b_1 = b + (-b_2)$. The $b_2$ is negative because the sense of the small right triangle's area in the proof is negative: the small area is subtracted from

the large rather than added. By extension on this basis, the main triangle's area is again seen to be $A = bh/2$. The proof is exactly the same. In fact, once the central idea of adding two right triangles is grasped, the extension is really rather obvious—too obvious to be allowed to burden such a book as this.

Excepting the uncommon cases where extension reveals something interesting or new, this book generally leaves the mere extension of proofs—including the validation of edge cases and over-the-edge cases—as an exercise to the interested reader.

## 1.3 Complex numbers and complex variables

More than a mastery of mere logical details, it is an holistic view of the mathematics and of its use in the modeling of physical systems which is the mark of the applied mathematician. A *feel* for the math is the great thing. Formal definitions, axioms, symbolic algebras and the like, though often useful, are felt to be secondary. The book's rapidly staged development of complex numbers and complex variables is planned on this sensibility.

Sections 2.12, 3.10, 3.11, 4.3.3, 4.4, 6.2, 9.5 and 9.6.5, plus all of Chs. 5 and 8, constitute the book's principal stages of complex development. In these sections and throughout the book, the reader comes to appreciate that most mathematical properties which apply for real numbers apply equally for complex, that few properties concern real numbers alone.

Pure mathematics develops an abstract theory of the complex variable.[1] The abstract theory is quite beautiful. However, its arc takes off too late and flies too far from applications for such a book as this. Less beautiful but more practical paths to the topic exist;[2] this book leads the reader along one of these.

For supplemental reference, a bare sketch of the abstract theory of the complex variable is found in Appendix C.

## 1.4 On the text

The book gives numerals in hexadecimal. It denotes variables in Greek letters as well as Roman. Readers unfamiliar with the hexadecimal notation will find a brief orientation thereto in Appendix A. Readers unfamiliar with the Greek alphabet will find it in Appendix B.

[1] [4][20][57][31]

[2] See Ch. 8's footnote 8.

Licensed to the public under the GNU General Public Licence [22], version 2, this book meets the Debian Free Software Guidelines [16].

A book of mathematical derivations by its nature can tend to make dry, even gray reading. Delineations of black and white become the book's duty. Mathematics however should serve the demands not only of deduction but equally of insight, by the latter of which alone mathematics derives either feeling or use. Yet, though *this* book does try—at some risk to strict consistency of tone—to add color in suitable shades, to strike an appropriately lively balance between the opposite demands of logical progress and literary relief; nonetheless, neither every sequence of equations nor every conjunction of figures is susceptible to an apparent hue the writer can openly paint upon it, but only to that abeyant hue, that luster which reveals or reflects the fire of the reader's own mathematical imagination, which color otherwise remains unobserved. The book's subject and purpose thus restrict its overt style.

The book begins by developing the calculus of a single variable.

# Part I

# The calculus of a single variable

# Chapter 2

# Classical algebra and geometry

One learns arithmetic and the simplest elements of classical algebra and geometry as a child. Few readers presumably, on the present book's tier, would wish the book to begin with a treatment of $1 + 1 = 2$, or of how to solve $3x - 2 = 7$, or of the formal consequences of the congruence of the several angles produced when a line intersects some parallels. However, there are some basic points which do seem worth touching. The book starts with these.

## 2.1 Basic arithmetic relationships

This section states some arithmetical rules.

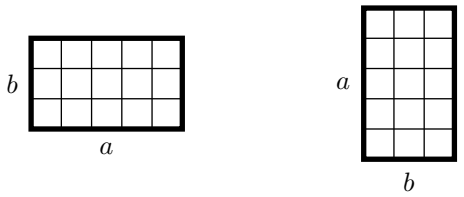
### 2.1.1 Commutivity, associativity, distributivity, identity and inversion

Table 2.1 lists several arithmetical rules, each of which applies not only to real numbers but equally to the complex numbers of § 2.12. Most of the rules are appreciated at once if the meaning of the symbols is understood. In the case of multiplicative commutivity, one imagines a rectangle with sides of lengths $a$ and $b$, then the same rectangle turned on its side, as in Fig. 2.1: since the area of the rectangle is the same in either case, and since the area is the length times the width in either case (the area is more or less a matter of counting the little squares), evidently multiplicative commutivity holds. A similar argument validates multiplicative associativity, except that here

Table 2.1: Basic properties of arithmetic.

| | | | |
|---|---|---|---|
| $a + b$ | $=$ | $b + a$ | Additive commutivity |
| $a + (b + c)$ | $=$ | $(a + b) + c$ | Additive associativity |
| $a + 0 = 0 + a$ | $=$ | $a$ | Additive identity |
| $a + (-a)$ | $=$ | $0$ | Additive inversion |
| $ab$ | $=$ | $ba$ | Multiplicative commutivity |
| $(a)(bc)$ | $=$ | $(ab)(c)$ | Multiplicative associativity |
| $(a)(1) = (1)(a)$ | $=$ | $a$ | Multiplicative identity |
| $(a)(1/a)$ | $=$ | $1$ | Multiplicative inversion |
| $(a)(b + c)$ | $=$ | $ab + ac$ | Distributivity |

Figure 2.1: Multiplicative commutivity.



we compute the *volume* of a three-dimensional rectangular box, which box we turn various ways.[1]

Multiplicative inversion lacks an obvious interpretation when $a = 0$. Loosely,

$$\frac{1}{0} = \infty.$$

But since $3/0 = \infty$ also, surely either the zero or the infinity, or both, somehow differ in the latter case.

Looking ahead in the book, we note that the multiplicative properties do not always hold for more general linear transformations. For example, matrix multiplication is not commutative and vector cross-multiplication is not associative. Where associativity does not hold and parentheses do not otherwise group, right-to-left association is notationally implicit:[2,3]

$$\mathbf{A} \times \mathbf{B} \times \mathbf{C} = \mathbf{A} \times (\mathbf{B} \times \mathbf{C}).$$

The sense of it is that the thing on the left ($\mathbf{A} \times$ ) *operates* on the thing on the right ($\mathbf{B} \times \mathbf{C}$). (In the rare case in which the question arises, you may want to use parentheses anyway.)

### 2.1.2 Negative numbers

Consider that

$$\begin{aligned}
(+a)(+b) &= +ab, \\
(+a)(-b) &= -ab, \\
(-a)(+b) &= -ab, \\
(-a)(-b) &= +ab.
\end{aligned}$$

The first three of the four equations are unsurprising, but the last is interesting. Why would a negative count $-a$ of a negative quantity $-b$ come to

[1] [57, Ch. 1]

[2] The fine C and C++ programming languages are unfortunately stuck with the reverse order of association, along with division inharmoniously on the same level of syntactic precedence as multiplication. Standard mathematical notation is more elegant:

$$abc/uvw = \frac{(a)(bc)}{(u)(vw)}.$$

[3] The nonassociative *cross product* $\mathbf{B} \times \mathbf{C}$ is introduced in § 15.2.2.

a positive product $+ab$? To see why, consider the progression

$$\begin{aligned}
&\vdots \\
(+3)(-b) &= -3b, \\
(+2)(-b) &= -2b, \\
(+1)(-b) &= -1b, \\
(0)(-b) &= 0b, \\
(-1)(-b) &= +1b, \\
(-2)(-b) &= +2b, \\
(-3)(-b) &= +3b, \\
&\vdots
\end{aligned}$$

The logic of arithmetic demands that the product of two negative numbers be positive for this reason.

### 2.1.3 Inequality

If[4]

$$a < b,$$

then necessarily

$$a + x < b + x.$$

However, the relationship between $ua$ and $ub$ depends on the sign of $u$:

$$\begin{aligned}
ua &< ub \qquad \text{if } u > 0; \\
ua &> ub \qquad \text{if } u < 0.
\end{aligned}$$

Also,

$$\frac{1}{a} > \frac{1}{b}.$$

### 2.1.4 The change of variable

The applied mathematician very often finds it convenient *to change variables,* introducing new symbols to stand in place of old. For this we have

[4]Few readers attempting this book will need to be reminded that $<$ means "is less than," that $>$ means "is greater than," or that $\leq$ and $\geq$ respectively mean "is less than or equal to" and "is greater than or equal to."

the *change of variable* or *assignment* notation[5]

$$Q \leftarrow P.$$

This means, "in place of $P$, put $Q$"; or, "let $Q$ now equal $P$." For example, if $a^2 + b^2 = c^2$, then the change of variable $2\mu \leftarrow a$ yields the new form $(2\mu)^2 + b^2 = c^2$.

Similar to the change of variable notation is the *definition* notation

$$Q \equiv P.$$

This means, "let the new symbol $Q$ represent $P$."[6]

The two notations logically mean about the same thing. Subjectively, $Q \equiv P$ identifies a quantity $P$ sufficiently interesting to be given a permanent name $Q$, whereas $Q \leftarrow P$ implies nothing especially interesting about $P$ or $Q$; it just introduces a (perhaps temporary) new symbol $Q$ to ease the algebra. The concepts grow clearer as examples of the usage arise in the book.

## 2.2 Quadratics

Differences and sums of squares are conveniently factored as

$$\begin{aligned} a^2 - b^2 &= (a + b)(a - b), \\ a^2 + b^2 &= (a + ib)(a - ib), \\ a^2 - 2ab + b^2 &= (a - b)^2, \\ a^2 + 2ab + b^2 &= (a + b)^2 \end{aligned} \tag{2.1}$$

(where $i$ is the *imaginary unit,* a number defined such that $i^2 = -1$, introduced in more detail in § 2.12 below). Useful as these four forms are,

[5]There appears to exist no broadly established standard mathematical notation for the change of variable, other than the = equal sign, which regrettably does not fill the role well. One can indeed use the equal sign, but then what does the change of variable $k = k + 1$ mean? It looks like a claim that $k$ and $k + 1$ are the same, which is impossible. The notation $k \leftarrow k + 1$ by contrast is unambiguous; it means to increment $k$ by one. However, the latter notation admittedly has seen only scattered use in the literature.

The C and C++ programming languages use `==` for equality and `=` for assignment (change of variable), as the reader may be aware.

[6]One would never write $k \equiv k + 1$. Even $k \leftarrow k + 1$ can confuse readers inasmuch as it appears to imply two different values for the same symbol $k$, but the latter notation is sometimes used anyway when new symbols are unwanted or because more precise alternatives (like $k_n = k_{n-1} + 1$) seem overwrought. Still, usually it is better to introduce a new symbol, as in $j \leftarrow k + 1$.

In some books, $\equiv$ is printed as $\triangleq$.

however, none of them can directly factor the more general quadratic[7] expression

$$z^2 - 2\beta z + \gamma^2.$$

To factor this, we *complete the square,* writing

$$\begin{aligned} z^2 - 2\beta z + \gamma^2 &= z^2 - 2\beta z + \gamma^2 + (\beta^2 - \gamma^2) - (\beta^2 - \gamma^2) \\ &= z^2 - 2\beta z + \beta^2 - (\beta^2 - \gamma^2) \\ &= (z - \beta)^2 - (\beta^2 - \gamma^2). \end{aligned}$$

The expression evidently has roots[8] where

$$(z - \beta)^2 = (\beta^2 - \gamma^2),$$

or in other words where[9]

$$z = \beta \pm \sqrt{\beta^2 - \gamma^2}. \tag{2.2}$$

This suggests the factoring[10]

$$z^2 - 2\beta z + \gamma^2 = (z - z_1)(z - z_2), \tag{2.3}$$

where $z_1$ and $z_2$ are the two values of $z$ given by (2.2).

It follows that the two solutions of the quadratic equation

$$z^2 = 2\beta z - \gamma^2 \tag{2.4}$$

are those given by (2.2), which is called *the quadratic formula.*[11] (*Cubic* and *quartic formulas* also exist respectively to extract the roots of polynomials of third and fourth order, but they are much harder. See Ch. 10 and its Tables 10.1 and 10.2.)

[7]The adjective *quadratic* refers to the algebra of expressions in which no term has greater than second order. Examples of quadratic expressions include $x^2$, $2x^2 - 7x + 3$ and $x^2 + 2xy + y^2$. By contrast, the expressions $x^3 - 1$ and $5x^2y$ are *cubic* not quadratic because they contain third-order terms. First-order expressions like $x + 1$ are *linear;* zeroth-order expressions like 3 are *constant.* Expressions of fourth and fifth order are *quartic* and *quintic,* respectively. (If not already clear from the context, *order* basically refers to the number of variables multiplied together in a term. The term $5x^2y = 5[x][x][y]$ is of third order, for instance.)

[8]A *root* of $f(z)$ is a value of $z$ for which $f(z) = 0$. See § 2.11.

[9]The symbol $\pm$ means "+ or −." In conjunction with this symbol, the alternate symbol $\mp$ occasionally also appears, meaning "− or +"—which is the same thing except that, where the two symbols appear together, $(\pm z) + (\mp z) = 0$.

[10]It suggests it because the expressions on the left and right sides of (2.3) are both quadratic (the highest power is $z^2$) and have the same roots. Substituting into the equation the values of $z_1$ and $z_2$ and simplifying proves the suggestion correct.

[11]The form of the quadratic formula which usually appears in print is

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a},$$

## 2.3 Integer and series notation

Sums and products of series arise so frequently in mathematical work that one finds it convenient to define terse notations to express them. The summation notation

$$\sum_{k=a}^{b} f(k)$$

means to let $k$ equal each of the integers $a, a+1, a+2, \ldots, b$ in turn, evaluating the function $f(k)$ at each $k$, then adding the several $f(k)$. For example,[12]

$$\sum_{k=3}^{6} k^2 = 3^2 + 4^2 + 5^2 + 6^2 = 0\text{x}56.$$

The similar multiplication notation

$$\prod_{j=a}^{b} f(j)$$

means *to multiply* the several $f(j)$ rather than to add them. The symbols $\sum$ and $\prod$ come respectively from the Greek letters for S and P, and may be regarded as standing for "Sum" and "Product." The $j$ or $k$ is a *dummy variable, index of summation* or *loop counter*—a variable with no independent existence, used only to facilitate the addition or multiplication of the series.[13] (Nothing prevents one from writing $\prod_k$ rather than $\prod_j$, incidentally. For a dummy variable, one can use any letter one likes. However, the general habit of writing $\sum_k$ and $\prod_j$ proves convenient at least in § 4.5.2 and Ch. 8, so we start now.)

---

which solves the quadratic $ax^2 + bx + c = 0$. However, this writer finds the form (2.2) easier to remember. For example, by (2.2) in light of (2.4), the quadratic

$$z^2 = 3z - 2$$

has the solutions

$$z = \frac{3}{2} \pm \sqrt{\left(\frac{3}{2}\right)^2 - 2} = 1 \text{ or } 2.$$

[12]The hexadecimal numeral 0x56 represents the same number the decimal numeral 86 represents. The book's preface explains why the book represents such numbers in hexadecimal. Appendix A tells how to read the numerals.

[13]Section 7.3 speaks further of the dummy variable.

The product shorthand

$$\begin{aligned} n! &\equiv \prod_{j=1}^{n} j, \\ n!/m! &\equiv \prod_{j=m+1}^{n} j, \end{aligned}$$

is very frequently used. The notation $n!$ is pronounced "$n$ factorial." Regarding the notation $n!/m!$, this can of course be regarded correctly as $n!$ divided by $m!$ , but it usually proves more amenable to regard the notation as a single unit.[14]

Because multiplication in its more general sense as linear transformation (§ 11.1.1) is not always commutative, we specify that

$$\prod_{j=a}^{b} f(j) = [f(b)][f(b-1)][f(b-2)]\cdots[f(a+2)][f(a+1)][f(a)]$$

rather than the reverse order of multiplication.[15] Multiplication proceeds from right to left. In the event that the reverse order of multiplication is needed, we will use the notation

$$\coprod_{j=a}^{b} f(j) = [f(a)][f(a+1)][f(a+2)]\cdots[f(b-2)][f(b-1)][f(b)].$$

Note that for the sake of definitional consistency,

$$\begin{aligned} \sum_{k=N+1}^{N} f(k) &= 0 + \sum_{k=N+1}^{N} f(k) = 0, \\ \prod_{j=N+1}^{N} f(j) &= (1) \prod_{j=N+1}^{N} f(j) = 1. \end{aligned}$$

This means among other things that

$$0! = 1. \tag{2.5}$$

[14] One reason among others for this is that factorials rapidly multiply to extremely large sizes, overflowing computer registers during numerical computation. If you can avoid unnecessary multiplication by regarding $n!/m!$ as a single unit, this is a win.

[15] The extant mathematical literature lacks an established standard on the order of multiplication implied by the "$\coprod$" symbol, but this is the order we will use in this book.

Context tends to make the notation

$$N, j, k \in \mathbb{Z}$$

unnecessary, but if used (as here and in § 2.5) it states explicitly that $N$, $j$ and $k$ are integers. (The symbol $\mathbb{Z}$ represents[16] the set of all integers: $\mathbb{Z} \equiv \{\ldots, -5, -4, -3, -2, -1, 0, 1, 2, 3, 4, 5, \ldots\}$. The symbol $\in$ means "belongs to" or "is a member of." Integers conventionally get the letters[17] $i$, $j$, $k$, $m$, $n$, $M$ and $N$ when available—though $i$ is sometimes avoided because the same letter represents the imaginary unit of § 2.12. Where additional letters are needed $\ell$, $p$ and $q$, plus the capitals of these and the earlier listed letters, can be pressed into service, occasionally joined even by $r$ and $s$. Greek letters are avoided, as—ironically in light of the symbol $\mathbb{Z}$—are the Roman letters $x$, $y$ and $z$. Refer to Appendix B.)

On first encounter, the $\sum$ and $\prod$ notation seems a bit overwrought, whether or not the $\in \mathbb{Z}$ notation also is used. Admittedly it is easier for the beginner to read "$f(1)+f(2)+\cdots+f(N)$" than "$\sum_{k=1}^{N} f(k)$." However, experience shows the latter notation to be extremely useful in expressing more sophisticated mathematical ideas. We will use such notation extensively in this book.

## 2.4 The arithmetic series

A simple yet useful application of the series sum of § 2.3 is the *arithmetic series*

$$\sum_{k=a}^{b} k = a + (a+1) + (a+2) + \cdots + b.$$

Pairing $a$ with $b$, then $a+1$ with $b-1$, then $a+2$ with $b-2$, etc., the average of each pair is $[a+b]/2$; thus the average of the entire series is $[a+b]/2$. (The pairing may or may not leave an unpaired element at the series midpoint $k = [a+b]/2$, but this changes nothing.) The series has $b-a+1$ terms. Hence,

$$\sum_{k=a}^{b} k = (b-a+1)\frac{a+b}{2}. \tag{2.6}$$

[16] The letter $\mathbb{Z}$ recalls the transitive and intransitive German verb *zählen,* "to count."

[17] Though Fortran is perhaps less widely used a computer programming language than it once was, it dominated applied-mathematical computer programming for decades, during which the standard way to declare an integer variable to the Fortran compiler was simply to let its name begin with `I`, `J`, `K`, `L`, `M` or `N`; so, this alphabetical convention is fairly well cemented in practice.

Table 2.2: Power properties and definitions.

$$
\begin{aligned}
z^n &\equiv \prod_{j=1}^{n} z, \quad n \geq 0 \\
z &= (z^{1/n})^n = (z^n)^{1/n} \\
\sqrt{z} &\equiv z^{1/2} \\
(uv)^a &= u^a v^a \\
z^{p/q} &= (z^{1/q})^p = (z^p)^{1/q} \\
z^{ab} &= (z^a)^b = (z^b)^a \\
z^{a+b} &= z^a z^b \\
z^{a-b} &= \frac{z^a}{z^b} \\
z^{-b} &= \frac{1}{z^b} \\
j, n, p, q &\in \mathbb{Z}
\end{aligned}
$$

Success with this arithmetic series leads one to wonder about the *geometric series* $\sum_{k=0}^{\infty} z^k$. Section 2.6.4 addresses that point.

## 2.5 Powers and roots

This necessarily tedious section discusses powers and roots. It offers no surprises. Table 2.2 summarizes its definitions and results. Readers seeking more rewarding reading may prefer just to glance at the table then to skip directly to the start of the next section.

In this section, the exponents

$$j, k, m, n, p, q, r, s \in \mathbb{Z}$$

are integers, but the exponents $a$ and $b$ are arbitrary real numbers.

### 2.5.1 Notation and integral powers

The power notation

$$z^n$$

indicates the number $z$, multiplied by itself $n$ times. More formally, when the *exponent* $n$ is a nonnegative integer,[18]

$$z^n \equiv \prod_{j=1}^{n} z. \tag{2.7}$$

For example,[19]

$$\begin{aligned} z^3 &= (z)(z)(z), \\ z^2 &= (z)(z), \\ z^1 &= z, \\ z^0 &= 1. \end{aligned}$$

Notice that in general,

$$z^{n-1} = \frac{z^n}{z}.$$

This leads us to extend the definition to negative integral powers with

$$z^{-n} = \frac{1}{z^n}. \tag{2.8}$$

From the foregoing it is plain that

$$\begin{aligned} z^{m+n} &= z^m z^n, \\ z^{m-n} &= \frac{z^m}{z^n}, \end{aligned} \tag{2.9}$$

for any integral $m$ and $n$. For similar reasons,

$$z^{mn} = (z^m)^n = (z^n)^m. \tag{2.10}$$

On the other hand from multiplicative associativity and commutivity,

$$(uv)^n = u^n v^n. \tag{2.11}$$

[18]The symbol "$\equiv$" means "$=$", but it further usually indicates that the expression on its right serves to define the expression on its left. Refer to § 2.1.4.

[19]The case $0^0$ is interesting because it lacks an obvious interpretation. The specific interpretation depends on the nature and meaning of the two zeros. For interest, if $E \equiv 1/\epsilon$, then

$$\lim_{\epsilon \to 0^+} \epsilon^\epsilon = \lim_{E \to \infty} \left(\frac{1}{E}\right)^{1/E} = \lim_{E \to \infty} E^{-1/E} = \lim_{E \to \infty} e^{-(\ln E)/E} = e^0 = 1.$$

### 2.5.2 Roots

Fractional powers are not something we have defined yet, so for consistency with (2.10) we let

$$(u^{1/n})^n = u.$$

This has $u^{1/n}$ as the number which, raised to the $n$th power, yields $u$. Setting

$$v = u^{1/n},$$

it follows by successive steps that

$$\begin{aligned} v^n &= u, \\ (v^n)^{1/n} &= u^{1/n}, \\ (v^n)^{1/n} &= v. \end{aligned}$$

Taking the $u$ and $v$ formulas together, then,

$$(z^{1/n})^n = z = (z^n)^{1/n} \tag{2.12}$$

for any $z$ and integral $n$.

The number $z^{1/n}$ is called the *nth root* of $z$—or in the very common case $n = 2$, the *square root* of $z$, often written

$$\sqrt{z}.$$

When $z$ is real and nonnegative, the last notation is usually implicitly taken to mean the real, nonnegative square root. In any case, the power and root operations mutually invert one another.

What about powers expressible neither as $n$ nor as $1/n$, such as the 3/2 power? If $z$ and $w$ are numbers related by

$$w^q = z,$$

then

$$w^{pq} = z^p.$$

Taking the $q$th root,

$$w^p = (z^p)^{1/q}.$$

But $w = z^{1/q}$, so this is

$$(z^{1/q})^p = (z^p)^{1/q},$$

which says that it does not matter whether one applies the power or the root first; the result is the same. Extending (2.10) therefore, we define $z^{p/q}$ such that

$$(z^{1/q})^p = z^{p/q} = (z^p)^{1/q}. \tag{2.13}$$

Since any real number can be approximated arbitrarily closely by a ratio of integers, (2.13) implies a power definition for all real exponents.

Equation (2.13) is this subsection's main result. However, § 2.5.3 will find it useful if we can also show here that

$$(z^{1/q})^{1/s} = z^{1/qs} = (z^{1/s})^{1/q}. \tag{2.14}$$

The proof is straightforward. If

$$w \equiv z^{1/qs},$$

then raising to the $qs$ power yields

$$(w^s)^q = z.$$

Successively taking the $q$th and $s$th roots gives

$$w = (z^{1/q})^{1/s}.$$

By identical reasoning,

$$w = (z^{1/s})^{1/q}.$$

But since $w \equiv z^{1/qs}$, the last two equations imply (2.14), as we have sought.

### 2.5.3 Powers of products and powers of powers

Per (2.11),

$$(uv)^p = u^p v^p.$$

Raising this equation to the $1/q$ power, we have that

$$\begin{aligned}
(uv)^{p/q} &= [u^p v^p]^{1/q} \\
&= \left[(u^p)^{q/q}(v^p)^{q/q}\right]^{1/q} \\
&= \left[(u^{p/q})^q(v^{p/q})^q\right]^{1/q} \\
&= \left[(u^{p/q})(v^{p/q})\right]^{q/q} \\
&= u^{p/q}v^{p/q}.
\end{aligned}$$

In other words

$$(uv)^a = u^a v^a \tag{2.15}$$

for any real $a$.

On the other hand, per (2.10),

$$z^{pr} = (z^p)^r.$$

Raising this equation to the $1/qs$ power and applying (2.10), (2.13) and (2.14) to reorder the powers, we have that

$$z^{(p/q)(r/s)} = \left(z^{p/q}\right)^{r/s}.$$

By identical reasoning,

$$z^{(p/q)(r/s)} = \left(z^{r/s}\right)^{p/q}.$$

Since $p/q$ and $r/s$ can approximate any real numbers with arbitrary precision, this implies that

$$(z^a)^b = z^{ab} = (z^b)^a \tag{2.16}$$

for any real $a$ and $b$.

### 2.5.4 Sums of powers

With (2.9), (2.15) and (2.16), one can reason that

$$z^{(p/q)+(r/s)} = \left(z^{ps+rq}\right)^{1/qs} = \left(z^{ps} z^{rq}\right)^{1/qs} = z^{p/q} z^{r/s},$$

or in other words that

$$z^{a+b} = z^a z^b. \tag{2.17}$$

In the case that $a = -b$,

$$1 = z^{-b+b} = z^{-b} z^b,$$

which implies that

$$z^{-b} = \frac{1}{z^b}. \tag{2.18}$$

But then replacing $-b \leftarrow b$ in (2.17) leads to

$$z^{a-b} = z^a z^{-b},$$

which according to (2.18) is

$$z^{a-b} = \frac{z^a}{z^b}. \tag{2.19}$$

### 2.5.5 Summary and remarks

Table 2.2 on page 18 summarizes the section's definitions and results.

Looking ahead to § 2.12, § 3.11 and Ch. 5, we observe that nothing in the foregoing analysis requires the base variables $z$, $w$, $u$ and $v$ to be real numbers; if complex (§ 2.12), the formulas remain valid. Still, the analysis does imply that the various exponents $m$, $n$, $p/q$, $a$, $b$ and so on are real numbers. This restriction, we shall remove later, purposely defining the action of a complex exponent to comport with the results found here. With such a definition the results apply not only for all bases but also for all exponents, real or complex.

## 2.6 Multiplying and dividing power series

A *power series*[20] is a weighted sum of integral powers:

$$A(z) = \sum_{k=-\infty}^{\infty} a_k z^k, \tag{2.20}$$

where the several $a_k$ are arbitrary constants. This section discusses the multiplication and division of power series.

[20] Another name for the *power series* is *polynomial.* The word "polynomial" usually connotes a power series with a finite number of terms, but the two names in fact refer to essentially the same thing.

Professional mathematicians use the terms more precisely. Equation (2.20), they call a "power series" only if $a_k = 0$ for all $k < 0$—in other words, technically, not if it includes negative powers of $z$. They call it a "polynomial" only if it is a "power series" with a finite number of terms. They call (2.20) in general a *Laurent series.*

The name "Laurent series" is a name we shall meet again in § 8.14. In the meantime however we admit that the professionals have vaguely daunted us by adding to the name some pretty sophisticated connotations, to the point that we applied mathematicians (at least in the author's country) seem to feel somehow unlicensed actually to use the name. We tend to call (2.20) a "power series with negative powers," or just "a power series."

This book follows the last usage. You however can call (2.20) a *Laurent series* if you prefer (and if you pronounce it right: "lor-ON"). That is after all exactly what it is. Nevertheless if you do use the name "Laurent series," be prepared for people subjectively—for no particular reason—to expect you to establish complex radii of convergence, to sketch some annulus in the Argand plane, and/or to engage in other maybe unnecessary formalities. If that is not what you seek, then you may find it better just to call the thing by the less lofty name of "power series"—or better, if it has a finite number of terms, by the even humbler name of "polynomial."

Semantics. All these names mean about the same thing, but one is expected most

### 2.6.1 Multiplying power series

Given two power series

$$\begin{aligned} A(z) &= \sum_{k=-\infty}^{\infty} a_k z^k, \\ B(z) &= \sum_{k=-\infty}^{\infty} b_k z^k, \end{aligned} \tag{2.21}$$

the product of the two series is evidently

$$P(z) \equiv A(z)B(z) = \sum_{k=-\infty}^{\infty} \sum_{j=-\infty}^{\infty} a_j b_{k-j} z^k. \tag{2.22}$$

### 2.6.2 Dividing power series

The quotient $Q(z) = B(z)/A(z)$ of two power series is a little harder to calculate, and there are at least two ways to do it. Section 2.6.3 below will do it by matching coefficients, but this subsection does it by long division. For example,

$$\begin{aligned} \frac{2z^2 - 3z + 3}{z - 2} &= \frac{2z^2 - 4z}{z - 2} + \frac{z + 3}{z - 2} = 2z + \frac{z + 3}{z - 2} \\ &= 2z + \frac{z - 2}{z - 2} + \frac{5}{z - 2} = 2z + 1 + \frac{5}{z - 2}. \end{aligned}$$

The strategy is to take the dividend[21] $B(z)$ piece by piece, purposely choosing pieces easily divided by $A(z)$.

carefully always to give the right name in the right place. What a bother! (Someone once told the writer that the Japanese language can give different names to the same object, depending on whether the *speaker* is male or female. The power-series terminology seems to share a spirit of that kin.) If you seek just one word for the thing, the writer recommends that you call it a "power series" and then not worry too much about it until someone objects. When someone does object, you can snow him with the big word "Laurent series," instead.

The experienced scientist or engineer may notice that the above vocabulary omits the name "Taylor series." The vocabulary omits the name because that name fortunately remains unconfused in usage—it means quite specifically a power series without negative powers and tends to connote a representation of some particular function of interest—as we shall see in Ch. 8.

[21] If $Q(z)$ is a *quotient* and $R(z)$ a *remainder*, then $B(z)$ is a *dividend* (or *numerator*) and $A(z)$ a *divisor* (or *denominator*). Such are the Latin-derived names of the parts of a long division.

If you feel that you understand the example, then that is really all there is to it, and you can skip the rest of the subsection if you like. One sometimes wants to express the long division of power series more formally, however. That is what the rest of the subsection is about. (Be advised however that the cleverer technique of § 2.6.3, though less direct, is often easier and faster.)

Formally, we prepare the long division $B(z)/A(z)$ by writing

$$B(z) = A(z)Q_n(z) + R_n(z), \tag{2.23}$$

where $R_n(z)$ is a *remainder* (being the part of $B[z]$ *remaining* to be divided); and

$$\begin{aligned}
A(z) &= \sum_{k=-\infty}^{K} a_k z^k, \quad a_K \neq 0, \\
B(z) &= \sum_{k=-\infty}^{N} b_k z^k, \\
R_N(z) &= B(z), \\
Q_N(z) &= 0, \\
R_n(z) &= \sum_{k=-\infty}^{n} r_{nk} z^k, \\
Q_n(z) &= \sum_{k=n-K+1}^{N-K} q_k z^k,
\end{aligned} \tag{2.24}$$

where $K$ and $N$ identify the greatest orders $k$ of $z^k$ present in $A(z)$ and $B(z)$, respectively.

Well, that is a lot of symbology. What does it mean? The key to understanding it lies in understanding (2.23), which is not one but several equations—one equation for each value of $n$, where $n = N, N-1, N-2, \ldots$. The dividend $B(z)$ and the divisor $A(z)$ stay the same from one $n$ to the next, but the quotient $Q_n(z)$ and the remainder $R_n(z)$ change. At start, $Q_N(z) = 0$ while $R_N(z) = B(z)$, but the thrust of the long division process is to build $Q_n(z)$ up by wearing $R_n(z)$ down. The goal is to grind $R_n(z)$ away to nothing, to make it disappear as $n \to -\infty$.

As in the example, we pursue the goal by choosing from $R_n(z)$ an easily divisible piece containing the whole high-order term of $R_n(z)$. The piece we choose is $(r_{nn}/a_K)z^{n-K}A(z)$, which we add and subtract from (2.23) to

obtain

$$B(z) = A(z)\left[Q_n(z) + \frac{r_{nn}}{a_K} z^{n-K}\right] + \left[R_n(z) - \frac{r_{nn}}{a_K} z^{n-K} A(z)\right].$$

Matching this equation against the desired iterate

$$B(z) = A(z)Q_{n-1}(z) + R_{n-1}(z)$$

and observing from the definition of $Q_n(z)$ that $Q_{n-1}(z) = Q_n(z) + q_{n-K}z^{n-K}$, we find that

$$\begin{aligned} q_{n-K} &= \frac{r_{nn}}{a_K}, \\ R_{n-1}(z) &= R_n(z) - q_{n-K}z^{n-K}A(z), \end{aligned} \tag{2.25}$$

where no term remains in $R_{n-1}(z)$ higher than a $z^{n-1}$ term.

To begin the actual long division, we initialize

$$R_N(z) = B(z),$$

for which (2.23) is trivially true. Then we iterate per (2.25) as many times as desired. If an infinite number of times, then so long as $R_n(z)$ tends to vanish as $n \to -\infty$, it follows from (2.23) that

$$\frac{B(z)}{A(z)} = Q_{-\infty}(z). \tag{2.26}$$

Iterating only a finite number of times leaves a remainder,

$$\frac{B(z)}{A(z)} = Q_n(z) + \frac{R_n(z)}{A(z)}, \tag{2.27}$$

except that it may happen that $R_n(z) = 0$ for sufficiently small $n$.

Table 2.3 summarizes the long-division procedure.[22] In its $q_{n-K}$ equation, the table includes also the result of § 2.6.3 below.

It should be observed in light of Table 2.3 that if[23]

$$\begin{aligned} A(z) &= \sum_{k=K_o}^{K} a_k z^k, \\ B(z) &= \sum_{k=N_o}^{N} b_k z^k, \end{aligned}$$

[22] [59, § 3.2]

[23] The notations $K_o$, $a_k$ and $z^k$ are usually pronounced, respectively, as "$K$ naught," "$a$ sub $k$" and "$z$ to the $k$" (or, more fully, "$z$ to the $k$th power")—at least in the author's country.

Table 2.3: Dividing power series through successively smaller powers.

$$
\begin{aligned}
B(z) &= A(z)Q_n(z) + R_n(z) \\
A(z) &= \sum_{k=-\infty}^{K} a_k z^k, \quad a_K \neq 0 \\
B(z) &= \sum_{k=-\infty}^{N} b_k z^k \\
R_N(z) &= B(z) \\
Q_N(z) &= 0 \\
R_n(z) &= \sum_{k=-\infty}^{n} r_{nk} z^k \\
Q_n(z) &= \sum_{k=n-K+1}^{N-K} q_k z^k \\
q_{n-K} &= \frac{r_{nn}}{a_K} = \frac{1}{a_K}\left(b_n - \sum_{k=n-K+1}^{N-K} a_{n-k} q_k\right) \\
R_{n-1}(z) &= R_n(z) - q_{n-K} z^{n-K} A(z) \\
\frac{B(z)}{A(z)} &= Q_{-\infty}(z)
\end{aligned}
$$

then

$$R_n(z) = \sum_{k=n-(K-K_o)+1}^{n} r_{nk} z^k \quad \text{for all } n < N_o + (K - K_o). \tag{2.28}$$

That is, the remainder has order one less than the divisor has. The reason for this, of course, is that we have strategically planned the long-division iteration precisely to cause the leading term of the divisor to cancel the leading term of the remainder at each step.[24]

The long-division procedure of Table 2.3 extends the quotient $Q_n(z)$ through successively smaller powers of $z$. Often, however, one prefers to extend the quotient through successively *larger* powers of $z$, where a $z^K$ term is $A(z)$'s term of *least* order. In this case, the long division goes by the complementary rules of Table 2.4.

### 2.6.3 Dividing power series by matching coefficients

There is another, sometimes quicker way to divide power series than by the long division of § 2.6.2. One can divide them by matching coefficients.[25] If

$$Q_\infty(z) = \frac{B(z)}{A(z)}, \tag{2.29}$$

where

$$\begin{aligned} A(z) &= \sum_{k=K}^{\infty} a_k z^k, \quad a_K \neq 0, \\ B(z) &= \sum_{k=N}^{\infty} b_k z^k \end{aligned}$$

[24] If a more formal demonstration of (2.28) is wanted, then consider per (2.25) that

$$R_{m-1}(z) = R_m(z) - \frac{r_{mm}}{a_K} z^{m-K} A(z).$$

If the least-order term of $R_m(z)$ is a $z^{N_o}$ term (as clearly is the case at least for the initial remainder $R_N[z] = B[z]$), then according to the equation so also must the least-order term of $R_{m-1}(z)$ be a $z^{N_o}$ term, unless an even lower-order term be contributed by the product $z^{m-K}A(z)$. But that very product's term of least order is a $z^{m-(K-K_o)}$ term. Under these conditions, evidently the least-order term of $R_{m-1}(z)$ is a $z^{m-(K-K_o)}$ term when $m - (K - K_o) \leq N_o$; otherwise a $z^{N_o}$ term. This is better stated after the change of variable $n + 1 \leftarrow m$: the least-order term of $R_n(z)$ is a $z^{n-(K-K_o)+1}$ term when $n < N_o + (K - K_o)$; otherwise a $z^{N_o}$ term.

The greatest-order term of $R_n(z)$ is by definition a $z^n$ term. So, in summary, when $n < N_o + (K - K_o)$, the terms of $R_n(z)$ run from $z^{n-(K-K_o)+1}$ through $z^n$, which is exactly the claim (2.28) makes.

[25] [40][20, § 2.5]

Table 2.4: Dividing power series through successively larger powers.

$$
\begin{aligned}
B(z) &= A(z)Q_n(z) + R_n(z) \\
A(z) &= \sum_{k=K}^{\infty} a_k z^k, \quad a_K \neq 0 \\
B(z) &= \sum_{k=N}^{\infty} b_k z^k \\
R_N(z) &= B(z) \\
Q_N(z) &= 0 \\
R_n(z) &= \sum_{k=n}^{\infty} r_{nk} z^k \\
Q_n(z) &= \sum_{k=N-K}^{n-K-1} q_k z^k \\
q_{n-K} &= \frac{r_{nn}}{a_K} = \frac{1}{a_K}\left(b_n - \sum_{k=N-K}^{n-K-1} a_{n-k} q_k\right) \\
R_{n+1}(z) &= R_n(z) - q_{n-K} z^{n-K} A(z) \\
\frac{B(z)}{A(z)} &= Q_\infty(z)
\end{aligned}
$$

are known and

$$Q_\infty(z) = \sum_{k=N-K}^{\infty} q_k z^k$$

is to be calculated, then one can multiply (2.29) through by $A(z)$ to obtain the form

$$A(z)Q_\infty(z) = B(z).$$

Expanding the left side according to (2.22) and changing the index $n \leftarrow k$ on the right side,

$$\sum_{n=N}^{\infty} \sum_{k=N-K}^{n-K} a_{n-k} q_k z^n = \sum_{n=N}^{\infty} b_n z^n.$$

But for this to hold for all $z$, the coefficients must match for each $n$:

$$\sum_{k=N-K}^{n-K} a_{n-k} q_k = b_n, \quad n \geq N.$$

Transferring all terms but $a_K q_{n-K}$ to the equation's right side and dividing by $a_K$, we have that

$$q_{n-K} = \frac{1}{a_K} \left( b_n - \sum_{k=N-K}^{n-K-1} a_{n-k} q_k \right), \quad n \geq N. \tag{2.30}$$

Equation (2.30) computes the coefficients of $Q(z)$, each coefficient depending on the coefficients earlier computed.

The coefficient-matching technique of this subsection is easily adapted to the division of series in decreasing, rather than increasing, powers of $z$ if needed or desired. The adaptation is left as an exercise to the interested reader, but Tables 2.3 and 2.4 incorporate the technique both ways.

Admittedly, the fact that (2.30) yields a sequence of coefficients does not necessarily mean that the resulting power series $Q_\infty(z)$ converges to some definite value over a given domain. Consider for instance (2.34), which diverges when[26] $|z| > 1$, even though all its coefficients are known. At least (2.30) is correct when $Q_\infty(z)$ does converge. Even when $Q_\infty(z)$ as such does not converge, however, often what interest us are only the series' first several terms

$$Q_n(z) = \sum_{k=N-K}^{n-K-1} q_k z^k.$$

[26] See footnote 27.

In this case,

$$Q_\infty(z) = \frac{B(z)}{A(z)} = Q_n(z) + \frac{R_n(z)}{A(z)} \tag{2.31}$$

and convergence is not an issue. Solving (2.31) for $R_n(z)$,

$$R_n(z) = B(z) - A(z)Q_n(z). \tag{2.32}$$

### 2.6.4 Common power-series quotients and the geometric series

Frequently encountered power-series quotients, calculated by the long division of § 2.6.2, computed by the coefficient matching of § 2.6.3, and/or verified by multiplying, include[27]

$$\frac{1}{1 \pm z} = \begin{cases} \sum_{k=0}^{\infty} (\mp z)^k, & |z| < 1; \\ -\sum_{k=-\infty}^{-1} (\mp z)^k, & |z| > 1. \end{cases} \tag{2.33}$$

Equation (2.33) almost incidentally answers a question which has arisen in § 2.4 and which often arises in practice: to what total does the infinite *geometric series* $\sum_{k=0}^{\infty} z^k$, $|z| < 1$, sum? Answer: it sums exactly to $1/(1 - z)$. However, there is a simpler, more aesthetic, more instructive way to demonstrate the same thing, as follows. Let

$$S \equiv \sum_{k=0}^{\infty} z^k, \quad |z| < 1.$$

Multiplying by $z$ yields

$$zS \equiv \sum_{k=1}^{\infty} z^k.$$

Subtracting the latter equation from the former leaves

$$(1 - z)S = 1,$$

which, after dividing by $1 - z$, implies that

$$S \equiv \sum_{k=0}^{\infty} z^k = \frac{1}{1 - z}, \quad |z| < 1, \tag{2.34}$$

as was to be demonstrated.

[27] The notation $|z|$ represents the magnitude of $z$. For example, $|5| = 5$, but also $|-5| = 5$.

### 2.6.5 Variations on the geometric series

Besides being more aesthetic than the long division of § 2.6.2, the difference technique of § 2.6.4 permits one to extend the basic geometric series in several ways. For instance, the sum

$$S_1 \equiv \sum_{k=0}^{\infty} kz^k, \quad |z| < 1$$

(which arises in, among others, Planck's quantum blackbody radiation calculation[28]), we can compute as follows. We multiply the unknown $S_1$ by $z$, producing

$$zS_1 = \sum_{k=0}^{\infty} kz^{k+1} = \sum_{k=1}^{\infty} (k-1)z^k.$$

We then subtract $zS_1$ from $S_1$, leaving

$$(1-z)S_1 = \sum_{k=0}^{\infty} kz^k - \sum_{k=1}^{\infty} (k-1)z^k = \sum_{k=1}^{\infty} z^k = z\sum_{k=0}^{\infty} z^k = \frac{z}{1-z},$$

where we have used (2.34) to collapse the last sum. Dividing by $1-z$, we arrive at

$$S_1 \equiv \sum_{k=0}^{\infty} kz^k = \frac{z}{(1-z)^2}, \quad |z| < 1, \tag{2.35}$$

which was to be found.

Further series of the kind, such as $\sum_k k^2 z^k$, $\sum_k (k+1)(k)z^k$, $\sum_k k^3 z^k$, etc., can be calculated in like manner as the need for them arises.

## 2.7 Indeterminate constants, independent variables and dependent variables

Mathematical models use *indeterminate constants, independent variables* and *dependent variables.* The three are best illustrated by example. Consider the time $t$ a sound needs to travel from its source to a distant listener:

$$t = \frac{\Delta r}{v_{\text{sound}}},$$

[28] [44]

where $\Delta r$ is the distance from source to listener and $v_{\text{sound}}$ is the speed of sound. Here, $v_{\text{sound}}$ is an indeterminate constant (given particular atmospheric conditions, it doesn't vary), $\Delta r$ is an independent variable, and $t$ is a dependent variable. The model gives $t$ as a function of $\Delta r$; so, if you tell the model how far the listener sits from the sound source, the model returns the time needed for the sound to propagate from one to the other. Note that the abstract validity of the model does not necessarily depend on whether we actually know the right figure for $v_{\text{sound}}$ (if I tell you that sound goes at 500 m/s, but later you find out that the real figure is 331 m/s, it probably doesn't ruin the theoretical part of your analysis; you just have to recalculate numerically). Knowing the figure is not the point. The point is that conceptually there preëxists some right figure for the indeterminate constant; that sound goes at some constant speed—whatever it is—and that we can calculate the delay in terms of this.

Although there exists a definite philosophical distinction between the three kinds of quantity, nevertheless it cannot be denied that which particular quantity is an indeterminate constant, an independent variable or a dependent variable often depends upon one's immediate point of view. The same model in the example would remain valid if atmospheric conditions were changing ($v_{\text{sound}}$ would then be an independent variable) or if the model were used in designing a musical concert hall[29] to suffer a maximum acceptable sound time lag from the stage to the hall's back row ($t$ would then be an independent variable; $\Delta r$, dependent). Occasionally we go so far as deliberately to change our point of view in mid-analysis, now regarding as an independent variable what a moment ago we had regarded as an indeterminate constant, for instance (a typical case of this arises in the solution of differential equations by the method of unknown coefficients, § 9.4). Such a shift of viewpoint is fine, so long as we remember that there is a difference

[29]Math books are funny about examples like this. Such examples remind one of the kind of calculation one encounters in a childhood arithmetic textbook, as of the quantity of air contained in an astronaut's round helmet. One could calculate the quantity of water in a kitchen mixing bowl just as well, but astronauts' helmets are so much more interesting than bowls, you see.

The chance that the typical reader will ever specify the dimensions of a real musical concert hall is of course vanishingly small. However, it is the idea of the example that matters here, because the chance that the typical reader will ever specify *something* technical is quite large. Although sophisticated models with many factors and terms do indeed play a major role in engineering, the great majority of practical engineering calculations—for quick, day-to-day decisions where small sums of money and negligible risk to life are at stake—are done with models hardly more sophisticated than the one shown here. So, maybe the concert-hall example is not so unreasonable, after all.

between the three kinds of quantity and we keep track of which quantity is which kind to us at the moment.

The main reason it matters which symbol represents which of the three kinds of quantity is that in calculus, one analyzes how change in independent variables affects dependent variables as indeterminate constants remain fixed.

(Section 2.3 has introduced the dummy variable, which the present section's threefold taxonomy seems to exclude. However, in fact, most dummy variables are just independent variables—a few are dependent variables—whose scope is restricted to a particular expression. Such a dummy variable does not seem very "independent," of course; but its dependence is on the operator controlling the expression, not on some other variable within the expression. Within the expression, the dummy variable fills the role of an independent variable; without, it fills no role because logically it does not exist there. Refer to §§ 2.3 and 7.3.)

## 2.8 Exponentials and logarithms

In § 2.5 we have considered the power operation $z^a$, where (in § 2.7's language) the independent variable $z$ is the base and the indeterminate constant $a$ is the exponent. There is another way to view the power operation, however. One can view it as the *exponential* operation

$$a^z,$$

where the variable $z$ is the exponent and the constant $a$ is the base.

### 2.8.1 The logarithm

The exponential operation follows the same laws the power operation follows, but because the variable of interest is now the exponent rather than the base, the inverse operation is not the root but rather the *logarithm:*

$$\log_a(a^z) = z. \tag{2.36}$$

The logarithm $\log_a w$ answers the question, "What power must I raise $a$ to, to get $w$?"

Raising $a$ to the power of the last equation, we have that

$$a^{\log_a(a^z)} = a^z.$$

With the change of variable $w \leftarrow a^z$, this is

$$a^{\log_a w} = w. \tag{2.37}$$

Hence, the exponential and logarithmic operations mutually invert one another.

### 2.8.2 Properties of the logarithm

The basic properties of the logarithm include that

$$\begin{aligned}
\log_a uv &= \log_a u + \log_a v, && (2.38)\\
\log_a \frac{u}{v} &= \log_a u - \log_a v, && (2.39)\\
\log_a(w^z) &= z \log_a w, && (2.40)\\
w^z &= a^{z \log_a w}, && (2.41)\\
\log_b w &= \frac{\log_a w}{\log_a b}. && (2.42)
\end{aligned}$$

Of these, (2.38) follows from the steps

$$\begin{aligned}
(uv) &= (u)(v),\\
(a^{\log_a uv}) &= (a^{\log_a u})(a^{\log_a v}),\\
a^{\log_a uv} &= a^{\log_a u + \log_a v};
\end{aligned}$$

and (2.39) follows by similar reasoning. Equations (2.40) and (2.41) follow from the steps

$$\begin{aligned}
w^z = \quad (w^z) &= (w)^z,\\
w^z = a^{\log_a (w^z)} &= (a^{\log_a w})^z,\\
w^z = a^{\log_a (w^z)} &= a^{z \log_a w}.
\end{aligned}$$

Equation (2.42) follows from the steps

$$\begin{aligned}
w &= b^{\log_b w},\\
\log_a w &= \log_a (b^{\log_b w}),\\
\log_a w &= \log_b w \log_a b.
\end{aligned}$$

Among other purposes, (2.38) through (2.42) serve respectively to transform products to sums, quotients to differences, powers to products, exponentials to differently based exponentials, and logarithms to differently based logarithms. Table 2.5 repeats the equations along with (2.36) and (2.37) (which also emerge as restricted forms of eqns. 2.40 and 2.41), thus summarizing the general properties of the logarithm.

Table 2.5: General properties of the logarithm.

$$\begin{aligned}
\log_a uv &= \log_a u + \log_a v \\
\log_a \frac{u}{v} &= \log_a u - \log_a v \\
\log_a(w^z) &= z \log_a w \\
w^z &= a^{z \log_a w} \\
\log_b w &= \frac{\log_a w}{\log_a b} \\
\log_a(a^z) &= z \\
w &= a^{\log_a w}
\end{aligned}$$

## 2.9 Triangles and other polygons: simple facts

This section gives simple facts about triangles and other polygons.

### 2.9.1 The area of a triangle

The area of a *right* triangle[30] is half the area of the corresponding rectangle. This is seen by splitting a rectangle down its diagonal into a pair of right triangles of equal size. The fact that *any* triangle's area is half its base length times its height is seen by dropping a perpendicular from one point of the triangle to the opposite side (see Fig. 1.1 on page 4), dividing the triangle into two right triangles, for each of which the fact is true. In algebraic symbols,

$$A = \frac{bh}{2}, \tag{2.43}$$

where $A$ stands for area, $b$ for base length, and $h$ for perpendicular height.

### 2.9.2 The triangle inequalities

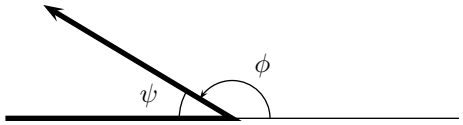
Any two sides of a triangle together are longer than the third alone, which itself is longer than the difference between the two. In symbols,

$$|a - b| < c < a + b, \tag{2.44}$$

where $a$, $b$ and $c$ are the lengths of a triangle's three sides. These are the *triangle inequalities*. The truth of the sum inequality $c < a + b$, is seen by

[30] A *right triangle* is a triangle, one of whose three angles is perfectly square.

Figure 2.2: The sum of a triangle's inner angles: turning at the corner.




sketching some triangle on a sheet of paper and asking: if $c$ is the direct route between two points and $a + b$ is an indirect route, then how can $a + b$ not be longer? Of course the sum inequality is equally good on any of the triangle's three sides, so one can write $a < c + b$ and $b < c + a$ just as well as $c < a + b$. Rearranging the $a$ and $b$ inequalities, we have that $a - b < c$ and $b - a < c$, which together say that $|a - b| < c$. The last is the difference inequality, completing (2.44)'s proof.[31]

### 2.9.3 The sum of interior angles

A triangle's three interior angles[32] sum to $2\pi/2$. One way to see the truth of this fact is to imagine a small car rolling along one of the triangle's sides. Reaching the corner, the car turns to travel along the next side, and so on round all three corners to complete a circuit, returning to the start. Since the car again faces the original direction, we reason that it has turned a total of $2\pi$: a full revolution. But the angle $\phi$ the car turns at a corner and the triangle's inner angle $\psi$ there together form the straight angle $2\pi/2$ (the sharper the inner angle, the more the car turns: see Fig. 2.2). In

[31] Section 13.9 proves the triangle inequalities more generally, though regrettably without recourse to this subsection's properly picturesque geometrical argument.

[32] Many or most readers will already know the notation $2\pi$ and its meaning as the angle of full revolution. For those who do not, the notation is introduced more properly in §§ 3.1, 3.6 and 8.11 below. Briefly, however, the symbol $2\pi$ represents a complete turn, a full circle, a spin to face the same direction as before. Hence $2\pi/4$ represents a square turn or right angle.

You may be used to the notation $360^\circ$ in place of $2\pi$, but for the reasons explained in Appendix A and in footnote 16 of Ch. 3, this book tends to avoid the former notation.

mathematical notation,

$$\begin{aligned} \phi_1 + \phi_2 + \phi_3 &= 2\pi, \\ \phi_k + \psi_k &= \frac{2\pi}{2}, \quad k = 1, 2, 3, \end{aligned}$$

where $\psi_k$ and $\phi_k$ are respectively the triangle's inner angles and the angles through which the car turns. Solving the latter equations for $\phi_k$ and substituting into the former yields

$$\psi_1 + \psi_2 + \psi_3 = \frac{2\pi}{2}, \tag{2.45}$$

which was to be demonstrated.

Extending the same technique to the case of an $n$-sided polygon, we have that

$$\begin{aligned} \sum_{k=1}^{n} \phi_k &= 2\pi, \\ \phi_k + \psi_k &= \frac{2\pi}{2}. \end{aligned}$$

Solving the latter equations for $\phi_k$ and substituting into the former yields

$$\sum_{k=1}^{n} \left( \frac{2\pi}{2} - \psi_k \right) = 2\pi,$$

or in other words

$$\sum_{k=1}^{n} \psi_k = (n-2)\frac{2\pi}{2}. \tag{2.46}$$

Equation (2.45) is then seen to be a special case of (2.46) with $n = 3$.

## 2.10 The Pythagorean theorem

Along with Euler's formula (5.12), the fundamental theorem of calculus (7.2), Cauchy's integral formula (8.29) and Fourier's equation (18.1), the Pythagorean theorem is one of the most famous results in all of mathematics. The theorem holds that

$$a^2 + b^2 = c^2, \tag{2.47}$$

where $a$, $b$ and $c$ are the lengths of the legs and diagonal of a right triangle, as in Fig. 2.3. Many proofs of the theorem are known.

Figure 2.3: A right triangle.

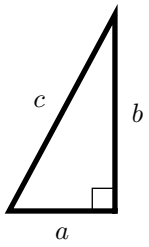
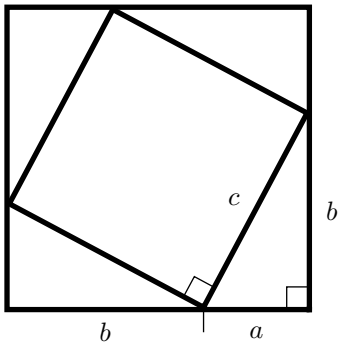



Figure 2.4: The Pythagorean theorem.



One such proof posits a square of side length $a+b$ with a tilted square of side length $c$ inscribed as in Fig. 2.4. The area of each of the four triangles in the figure is evidently $ab/2$. The area of the tilted inner square is $c^2$. The area of the large outer square is $(a+b)^2$. But the large outer square is comprised of the tilted inner square plus the four triangles, hence the area of the large outer square equals the area of the tilted inner square plus the areas of the four triangles. In mathematical symbols, this is

$$(a+b)^2 = c^2 + 4\left(\frac{ab}{2}\right),$$

which simplifies directly to (2.47).[33]

The Pythagorean theorem is readily extended to three dimensions as

$$a^2 + b^2 + h^2 = r^2, \tag{2.48}$$

where $h$ is an altitude perpendicular to both $a$ and $b$, thus also to $c$; and where $r$ is the corresponding three-dimensional diagonal: the diagonal of the right triangle whose legs are $c$ and $h$. Inasmuch as (2.47) applies to any right triangle, it follows that $c^2 + h^2 = r^2$, which equation expands directly to yield (2.48).

## 2.11 Functions

This book is not the place for a gentle introduction to the concept of the function. Briefly, however, a *function* is a mapping from one number (or vector of several numbers) to another. For example, $f(x) = x^2 - 1$ is a function which maps 1 to 0 and $-3$ to 8, among others.

One often speaks of *domains* and *ranges* when discussing functions. The *domain* of a function is the set of numbers one can put into it. The *range* of a function is the corresponding set of numbers one can get out of it. In the example, if the domain is restricted to real $x$ such that $|x| \leq 3$, then the corresponding range is $-1 \leq f(x) \leq 8$.

[33] This elegant proof is far simpler than the one famously given by the ancient geometer Euclid, yet more appealing than alternate proofs often found in print. Whether Euclid was acquainted with the simple proof given here this writer does not know, but it is possible [66, "Pythagorean theorem," 02:32, 31 March 2006] that Euclid chose his proof because it comported better with the restricted set of geometrical elements he permitted himself to work with. Be that as it may, the present writer encountered the proof this section gives somewhere years ago and has never seen it in print since, so can claim no credit for originating it. Unfortunately the citation is now long lost. A current, electronic source for the proof is [66] as cited earlier in this footnote.

The notation $f^{-1}(\cdot)$ indicates the *inverse* of the function $f(\cdot)$ such that

$$f^{-1}[f(x)] = x = f[f^{-1}(x)], \tag{2.49}$$

thus swapping the function's range with its domain. Though only a function which never maps distinct values from its domain together onto a single value in its range is strictly *invertible*—which means that the $f(x) = x^2 - 1$ of the example is not strictly invertible, since it maps both $x = -u$ and $x = u$ onto $f(x) = u^2 - 1$—it often does not pay to interpret the requirement too rigidly, for context tends to choose between available values (see for instance § 8.5). Inconsistently, inversion's notation $f^{-1}(\cdot)$ clashes with the similar-looking but different-meaning notation $f^2(\cdot) \equiv [f(\cdot)]^2$, whereas $f^{-1}(\cdot) \neq [f(\cdot)]^{-1}$: both notations are conventional and both are used in this book.

Other terms which arise when discussing functions are *root* (or *zero*), *singularity* and *pole*. A *root* (or *zero*) of a function is a domain point at which the function evaluates to zero (the example has roots at $x = \pm 1$). A *singularity* of a function is a domain point at which the function's output *diverges;* that is, where the function's output is infinite.[34] A *pole* is a singularity that behaves locally like $1/x$ (rather than, for example, like $1/\sqrt{x}$). A singularity that behaves as $1/x^N$ is a *multiple pole*, which (§ 9.6.2) can be thought of as $N$ poles. The example's function $f(x)$ has no singularities for finite $x$; however, the function $h(x) = 1/(x^2 - 1)$ has poles at $x = \pm 1$.

(Besides the root, the singularity and the pole, there is also the troublesome *branch point,* an infamous example of which is $z = 0$ in the function $g[z] = \sqrt{z}$. Branch points are important, but the book must lay a more extensive foundation before introducing them properly in § 8.5.[35])

[34] Here is one example of the book's deliberate lack of formal mathematical rigor. A more precise formalism to say that "the function's output is infinite" might be

$$\lim_{x \to x_o} |f(x)| = \infty,$$

and yet preciser formalisms than this are conceivable, and occasionally even are useful. Be that as it may, the applied mathematician tends to avoid such formalisms where there seems no immediate need for them.

[35] There is further the *essential singularity,* an example of which is $z = 0$ in $p(z) = \exp(1/z)$, but the best way to handle such unreasonable singularities is almost always to change a variable, as $w \leftarrow 1/z$, or otherwise to frame the problem such that one need not approach the singularity. Except tangentially much later when it treats asymptotic series, this book will have little to say of such singularities.

## 2.12 Complex numbers (introduction)

Section 2.5.2 has introduced square roots. What it has not done is to tell us how to regard a quantity such as $\sqrt{-1}$. Since there exists no real number $i$ such that

$$i^2 = -1 \tag{2.50}$$

and since the quantity $i$ thus defined is found to be critically important across broad domains of higher mathematics, we accept (2.50) as the definition of a fundamentally new kind of quantity: *the imaginary number.*[36]

Imaginary numbers are given their own number line, plotted at right angles to the familiar real number line as in Fig. 2.5. The sum of a real number $x$ and an imaginary number $iy$ is the *complex number*

$$z = x + iy.$$

The *conjugate* $z^*$ of this complex number is defined to be[37]

$$z^* = x - iy.$$

The *magnitude* (or *modulus,* or *absolute value*) $|z|$ of the complex number is defined to be the length $\rho$ in Fig. 2.5, which per the Pythagorean theorem

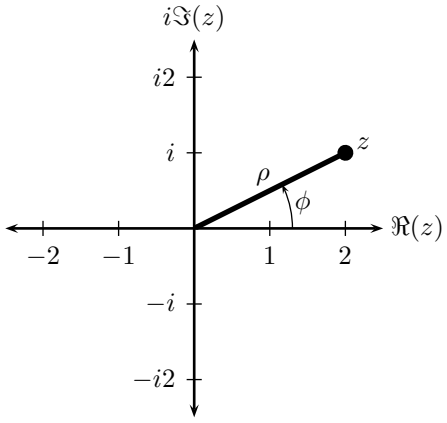
[36]The English word *imaginary* is evocative, but perhaps not of quite the right concept in this usage. Imaginary numbers are not to mathematics as, say, imaginary elfs are to the physical world. In the physical world, imaginary elfs are (presumably) not substantial objects. However, in the mathematical realm, imaginary numbers *are* substantial. The word *imaginary* in the mathematical sense is thus more of a technical term than a descriptive adjective.

The number $i$ is just a concept, of course, but then so is the number 1 (though you and I have often met one *of something*—one apple, one chair, one summer afternoon, etc.—neither of us has ever met just 1). The reason imaginary numbers are called "imaginary" probably has to do with the fact that they emerge from mathematical operations only, never directly from counting things. Notice, however, that the number 1/2 never emerges directly from counting things, either. If for some reason the *i*year were offered as a unit of time, then the period separating your fourteenth and twenty-first birthdays could have been measured as $-i7$ *i*years. Madness? No, let us not call it that; let us call it a useful formalism, rather.

The unpersuaded reader is asked to suspend judgment a while. He will soon see the use.

[37]For some inscrutable reason, in the author's country at least, professional mathematicians seem universally to write $\overline{z}$ instead of $z^*$, whereas rising engineers take the mathematicians' classes at school and then, having passed the classes, promptly start writing $z^*$ for the rest of their lives. The writer has his preference between the two notations and this book reflects it, but the curiously absolute character of the notational split is interesting as a social phenomenon.

Figure 2.5: The complex (or Argand) plane, and a complex number $z$ therein.




(§ 2.10) is such that

$$|z|^2 = x^2 + y^2. \tag{2.51}$$

The *phase* $\arg z$ of the complex number is defined to be the angle $\phi$ in the figure, which in terms of the trigonometric functions of § 3.1[38] is such that

$$\tan(\arg z) = \frac{y}{x}. \tag{2.52}$$

Specifically to extract the real and imaginary parts of a complex number, the notations

$$\begin{aligned} \Re(z) &= x, \\ \Im(z) &= y, \end{aligned} \tag{2.53}$$

are conventionally recognized (although often the symbols $\Re[\cdot]$ and $\Im[\cdot]$ are written $\mathrm{Re}[\cdot]$ and $\mathrm{Im}[\cdot]$, particularly when printed by hand).

[38] This is a forward reference. If the equation doesn't make sense to you yet for this reason, skip it for now. The important point is that $\arg z$ is the angle $\phi$ in the figure.

### 2.12.1 Multiplication and division of complex numbers in rectangular form

Several elementary properties of complex numbers are readily seen if the fact that $i^2 = -1$ is kept in mind, including that

$$z_1 z_2 = (x_1 x_2 - y_1 y_2) + i(y_1 x_2 + x_1 y_2), \tag{2.54}$$

$$\begin{aligned} \frac{z_1}{z_2} &= \frac{x_1 + iy_1}{x_2 + iy_2} = \left(\frac{x_2 - iy_2}{x_2 - iy_2}\right)\frac{x_1 + iy_1}{x_2 + iy_2} \\ &= \frac{(x_1 x_2 + y_1 y_2) + i(y_1 x_2 - x_1 y_2)}{x_2^2 + y_2^2}. \end{aligned} \tag{2.55}$$

It is a curious fact that

$$\frac{1}{i} = -i. \tag{2.56}$$

It is a useful fact that

$$z^* z = x^2 + y^2 = |z|^2 \tag{2.57}$$

(the curious fact, eqn. 2.56, is useful, too). Sometimes convenient are the forms

$$\begin{aligned} \Re(z) &= \frac{z + z^*}{2}, \\ \Im(z) &= \frac{z - z^*}{i2}, \end{aligned} \tag{2.58}$$

trivially proved.

### 2.12.2 Complex conjugation

An important property of complex numbers descends subtly from the fact that

$$i^2 = -1 = (-i)^2.$$

If one defined some number $j \equiv -i$, claiming that $j$ not $i$ were the true imaginary unit,[39] then one would find that

$$(-j)^2 = -1 = j^2,$$

and thus that all the basic properties of complex numbers in the $j$ system held just as well as they did in the $i$ system. The units $i$ and $j$ would differ indeed, but would perfectly mirror one another in every respect.

[39] [19, § I:22-5]

That is the basic idea. To establish it symbolically needs a page or so of slightly abstract algebra as follows, the goal of which will be to show that $[f(z)]^* = f(z^*)$ for some unspecified function $f(z)$ with specified properties. To begin with, if

$$z = x + iy,$$

then

$$z^* = x - iy$$

by definition. Proposing that $(z^{k-1})^* = (z^*)^{k-1}$ (which may or may not be true but for the moment we assume it), we can write

$$\begin{aligned} z^{k-1} &= s_{k-1} + it_{k-1}, \\ (z^*)^{k-1} &= s_{k-1} - it_{k-1}, \end{aligned}$$

where $s_{k-1}$ and $t_{k-1}$ are symbols introduced to represent the real and imaginary parts of $z^{k-1}$. Multiplying the former equation by $z = x + iy$ and the latter by $z^* = x - iy$, we have that

$$\begin{aligned} z^k &= (xs_{k-1} - yt_{k-1}) + i(ys_{k-1} + xt_{k-1}), \\ (z^*)^k &= (xs_{k-1} - yt_{k-1}) - i(ys_{k-1} + xt_{k-1}). \end{aligned}$$

With the definitions $s_k \equiv xs_{k-1} - yt_{k-1}$ and $t_k \equiv ys_{k-1} + xt_{k-1}$, this is written more succinctly

$$\begin{aligned} z^k &= s_k + it_k, \\ (z^*)^k &= s_k - it_k. \end{aligned}$$

In other words, if $(z^{k-1})^* = (z^*)^{k-1}$, then it necessarily follows that $(z^k)^* = (z^*)^k$. Solving the definitions of $s_k$ and $t_k$ for $s_{k-1}$ and $t_{k-1}$ yields the reverse definitions $s_{k-1} = (xs_k + yt_k)/(x^2 + y^2)$ and $t_{k-1} = (-ys_k + xt_k)/(x^2 + y^2)$. Therefore, except when $z = x + iy$ happens to be null or infinite, the implication is reversible by reverse reasoning, so by mathematical induction[40] we have that

$$(z^k)^* = (z^*)^k \tag{2.59}$$

for all integral $k$. We have also from (2.54) that

$$(z_1 z_2)^* = z_1^* z_2^* \tag{2.60}$$

[40] *Mathematical induction* is an elegant old technique for the construction of mathematical proofs. Section 8.1 elaborates on the technique and offers a more extensive example. Beyond the present book, a very good introduction to mathematical induction is found in [27].

for any complex $z_1$ and $z_2$.

Consequences of (2.59) and (2.60) include that if

$$f(z) \equiv \sum_{k=-\infty}^{\infty} (a_k + ib_k)(z - z_o)^k, \tag{2.61}$$

$$f^*(z) \equiv \sum_{k=-\infty}^{\infty} (a_k - ib_k)(z - z_o^*)^k, \tag{2.62}$$

where $a_k$ and $b_k$ are real and imaginary parts of the coefficients peculiar to the function $f(\cdot)$, then

$$[f(z)]^* = f^*(z^*). \tag{2.63}$$

In the common case where all $b_k = 0$ and $z_o = x_o$ is a real number, then $f(\cdot)$ and $f^*(\cdot)$ are the same function, so (2.63) reduces to the desired form

$$[f(z)]^* = f(z^*), \tag{2.64}$$

which says that the effect of conjugating the function's input is merely to conjugate its output.

Equation (2.64) expresses a significant, general rule of complex numbers and complex variables which is better explained in words than in mathematical symbols. The rule is this: for most equations and systems of equations used to model physical systems, *one can produce an equally valid alternate model simply by simultaneously conjugating all the complex quantities present.*[41]

### 2.12.3 Power series and analytic functions (preview)

Equation (2.61) is a general power series[42] in $z - z_o$. Such power series have broad application.[43] It happens in practice that most functions of interest

[41] [27][57]

[42] [31, § 10.8]

[43] That is a pretty impressive-sounding statement: "Such power series have broad application." However, molecules, air and words also have "broad application"; merely stating the fact does not tell us much. In fact the general power series is a sort of one-size-fits-all mathematical latex glove, which can be stretched to fit around almost any function. The interesting part is not so much in the general form (2.61) of the series as it is in the specific choice of $a_k$ and $b_k$, which this section does not discuss.

Observe that the Taylor series (which this section also does not discuss; see § 8.3) is a power series with $a_k = b_k = 0$ for $k < 0$.

in modeling physical phenomena can conveniently be constructed as power series (or sums of power series)[44] with suitable choices of $a_k$, $b_k$ and $z_o$.

The property (2.63) applies to all such functions, with (2.64) also applying to those for which $b_k = 0$ and $z_o = x_o$. The property the two equations represent is called the *conjugation property.* Basically, it says that if one replaces all the $i$ in some mathematical model with $-i$, then the resulting conjugate model is equally as valid as the original.[45]

Such functions, whether $b_k = 0$ and $z_o = x_o$ or not, are *analytic functions* (§ 8.4). In the formal mathematical definition, a function is analytic which is infinitely differentiable (Ch. 4) in the immediate domain neighborhood of interest. However, for applications a fair working definition of the analytic function might be "a function expressible as a power series." Chapter 8 elaborates. All power series are infinitely differentiable except at their poles.

There nevertheless exist one common group of functions which cannot be constructed as power series. These all have to do with the parts of complex numbers and have been introduced in this very section: the magnitude $|\cdot|$; the phase $\arg(\cdot)$; the conjugate $(\cdot)^*$; and the real and imaginary parts $\Re(\cdot)$ and $\Im(\cdot)$. These functions are not analytic and do not in general obey the conjugation property. Also not analytic are the Heaviside unit step $u(t)$ and the Dirac delta $\delta(t)$ (§ 7.7), used to model discontinuities explicitly.

We shall have more to say about analytic functions in Ch. 8. We shall have more to say about complex numbers in §§ 3.11, 4.3.3, and 4.4, and much more yet in Ch. 5.

[44] The careful reader might observe that this statement neglects *Gibbs' phenomenon,* but that curious matter will be dealt with in § 17.6.

[45] To illustrate, from the fact that $(1 + i2)(2 + i3) + (1 - i) = -3 + i6$ the conjugation property infers immediately that $(1 - i2)(2 - i3) + (1 + i) = -3 - i6$. Observe however that no such property holds for the real parts: $(-1 + i2)(-2 + i3) + (-1 - i) \neq 3 + i6$.

# Chapter 3

# Trigonometry

*Trigonometry* is the branch of mathematics which relates angles to lengths. This chapter introduces the trigonometric functions and derives their several properties.

## 3.1 Definitions

Consider the circle-inscribed right triangle of Fig. 3.1.

In considering the circle, we will find some terminology useful: the *angle* $\phi$ in the diagram is measured in *radians,* where a radian is the angle which, when centered in a *unit circle,* describes an arc of unit length.[1] Measured in radians, an angle $\phi$ intercepts an arc of curved length $\rho\phi$ on a circle of *radius* $\rho$ (that is, of distance $\rho$ from the circle's center to its perimeter). An angle in radians is a dimensionless number, so one need not write "$\phi = 2\pi/4$ radians"; it suffices to write "$\phi = 2\pi/4$." In mathematical theory, we express angles in radians.

The angle of full revolution is given the symbol $2\pi$—which thus is the circumference of a unit circle.[2] A quarter revolution, $2\pi/4$, is then the *right angle,* or square angle.

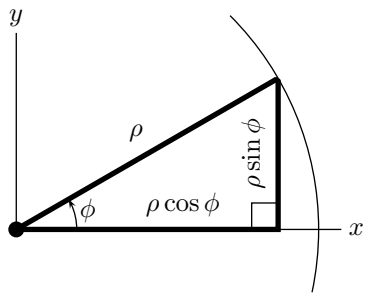
The trigonometric functions $\sin\phi$ and $\cos\phi$ (the "sine" and "cosine" of $\phi$) relate the angle $\phi$ to the lengths shown in Fig. 3.1. The tangent function is then defined as

$$\tan\phi \equiv \frac{\sin\phi}{\cos\phi}, \tag{3.1}$$

[1] The word "unit" means "one" in this context. A unit length is a length of 1 (not one centimeter or one mile, just an abstract 1). A unit circle is a circle of radius 1.

[2] Section 8.11 computes the numerical value of $2\pi$.

Figure 3.1: The sine and the cosine (shown on a circle-inscribed right triangle, with the circle centered at the triangle's point).




which is the "rise" per unit "run," or *slope,* of the triangle's diagonal.[3] Inverses of the three trigonometric functions can also be defined:

$$\begin{aligned} \arcsin(\sin\phi) &= \phi, \\ \arccos(\cos\phi) &= \phi, \\ \arctan(\tan\phi) &= \phi. \end{aligned}$$

When the last of these is written in the form

$$\arctan\left(\frac{y}{x}\right),$$

it is normally implied that $x$ and $y$ are to be interpreted as rectangular coordinates[4] and that the arctan function is to return $\phi$ in the correct quadrant $-\pi < \phi \leq \pi$ (for example, $\arctan[1/(-1)] = [+3/8][2\pi]$, whereas $\arctan[(-1)/1] = [-1/8][2\pi]$). This is similarly the usual interpretation when an equation like

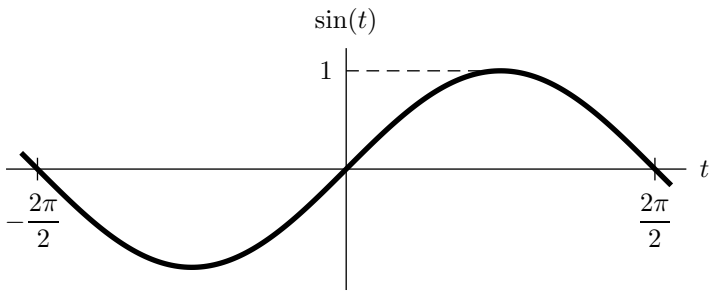
$$\tan\phi = \frac{y}{x}$$

is written.

[3]Often seen in print is the additional notation $\sec\phi \equiv 1/\cos\phi$, $\csc\phi \equiv 1/\sin\phi$ and $\cot\phi \equiv 1/\tan\phi$; respectively the "secant," "cosecant" and "cotangent." This book does not use the notation.

[4]*Rectangular coordinates* are pairs of numbers $(x, y)$ which uniquely specify points in a plane. Conventionally, the $x$ coordinate indicates distance eastward; the $y$ coordinate, northward. For instance, the coordinates $(3, -4)$ mean the point three units eastward and four units southward (that is, $-4$ units northward) from the *origin* $(0, 0)$. A third rectangular coordinate can also be added—$(x, y, z)$—where the $z$ indicates distance upward.

Figure 3.2: The sine function.




By the Pythagorean theorem (§ 2.10), it is seen generally that[5]

$$\cos^2 \phi + \sin^2 \phi = 1. \tag{3.2}$$

Fig. 3.2 plots the sine function. The shape in the plot is called a *sinusoid.*

## 3.2 Simple properties

Inspecting Fig. 3.1 and observing (3.1) and (3.2), one readily discovers the several simple trigonometric properties of Table 3.1.

## 3.3 Scalars, vectors, and vector notation

In applied mathematics, a *vector* is an amplitude of some kind coupled with a direction.[6] For example, "55 miles per hour northwestward" is a vector, as is the entity $\mathbf{u}$ depicted in Fig. 3.3. The entity $\mathbf{v}$ depicted in Fig. 3.4 is also a vector, in this case a three-dimensional one.

[5]The notation $\cos^2 \phi$ means $(\cos \phi)^2$.

[6]The same word *vector* is also used to indicate an ordered set of $N$ scalars (§ 8.16) or an $N \times 1$ matrix (Ch. 11), but those are not the uses of the word meant here. See also the introduction to Ch. 15.

Table 3.1: Simple properties of the trigonometric functions.

$$
\begin{array}{rclrcl}
\sin(-\phi) & = & -\sin\phi & \cos(-\phi) & = & +\cos\phi \\
\sin(2\pi/4-\phi) & = & +\cos\phi & \cos(2\pi/4-\phi) & = & +\sin\phi \\
\sin(2\pi/2-\phi) & = & +\sin\phi & \cos(2\pi/2-\phi) & = & -\cos\phi \\
\sin(\phi\pm 2\pi/4) & = & \pm\cos\phi & \cos(\phi\pm 2\pi/4) & = & \mp\sin\phi \\
\sin(\phi\pm 2\pi/2) & = & -\sin\phi & \cos(\phi\pm 2\pi/2) & = & -\cos\phi \\
\sin(\phi+n2\pi) & = & \sin\phi & \cos(\phi+n2\pi) & = & \cos\phi
\end{array}
$$

$$
\begin{aligned}
\tan(-\phi) &= -\tan\phi \\
\tan(2\pi/4-\phi) &= +1/\tan\phi \\
\tan(2\pi/2-\phi) &= -\tan\phi \\
\tan(\phi\pm 2\pi/4) &= -1/\tan\phi \\
\tan(\phi\pm 2\pi/2) &= +\tan\phi \\
\tan(\phi+n2\pi) &= \tan\phi
\end{aligned}
$$

$$
\begin{aligned}
\frac{\sin\phi}{\cos\phi} &= \tan\phi \\
\cos^2\phi+\sin^2\phi &= 1 \\
1+\tan^2\phi &= \frac{1}{\cos^2\phi} \\
1+\frac{1}{\tan^2\phi} &= \frac{1}{\sin^2\phi}
\end{aligned}
$$

Figure 3.3: A two-dimensional vector $\mathbf{u} = \hat{\mathbf{x}}x + \hat{\mathbf{y}}y$, shown with its rectangular components.

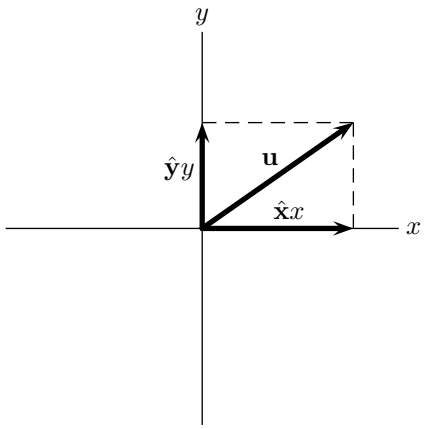
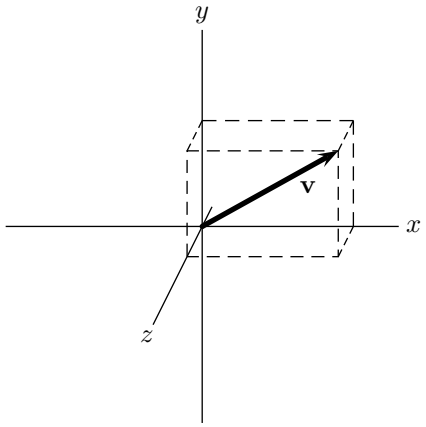



Figure 3.4: A three-dimensional vector $\mathbf{v} = \hat{\mathbf{x}}x + \hat{\mathbf{y}}y + \hat{\mathbf{z}}z$.



Many readers will already find the basic vector concept familiar, but for those who do not, a brief review: Vectors such as the

$$\begin{aligned}\mathbf{u} &= \hat{\mathbf{x}}x + \hat{\mathbf{y}}y, \\ \mathbf{v} &= \hat{\mathbf{x}}x + \hat{\mathbf{y}}y + \hat{\mathbf{z}}z\end{aligned}$$

of the figures are composed of multiples of the *unit basis vectors* $\hat{\mathbf{x}}$, $\hat{\mathbf{y}}$ and $\hat{\mathbf{z}}$, which themselves are vectors of unit length pointing in the cardinal directions their respective symbols suggest.[7] Any vector $\mathbf{a}$ can be factored into an *amplitude* $a$ and a *unit vector* $\hat{\mathbf{a}}$, as

$$\mathbf{a} = \hat{\mathbf{a}}a = \hat{\mathbf{a}}\,|\mathbf{a}|\,,$$

where the $\hat{\mathbf{a}}$ represents direction only and has unit magnitude by definition, and where the $a$ or $|\mathbf{a}|$ represents amplitude only and carries the physical units if any.[8] For example, $a = 55$ miles per hour, $\hat{\mathbf{a}} =$ northwestward. The unit vector $\hat{\mathbf{a}}$ itself can be expressed in terms of the unit basis vectors: for example, if $\hat{\mathbf{x}}$ points east and $\hat{\mathbf{y}}$ points north, then $\hat{\mathbf{a}} = -\hat{\mathbf{x}}(1/\sqrt{2}) + \hat{\mathbf{y}}(1/\sqrt{2})$, where per the Pythagorean theorem $(-1/\sqrt{2})^2 + (1/\sqrt{2})^2 = 1^2$.

A single number which is not a vector or a matrix (Ch. 11) is called a *scalar.* In the example, $a = 55$ miles per hour is a scalar. Though the scalar $a$ in the example happens to be real, scalars can be complex, too—which might surprise one, since scalars by definition lack direction and the Argand phase $\phi$ of Fig. 2.5 so strongly resembles a direction. However, phase is not an actual direction in the vector sense (the real number line

[7] Printing by hand, one customarily writes a general vector like $\mathbf{u}$ as "$\vec{u}$" or just "$\overline{u}$", and a unit vector like $\hat{\mathbf{x}}$ as "$\hat{x}$".

[8] The word "unit" here is unfortunately overloaded. As an adjective in mathematics, or in its nounal form "unity," it refers to the number one (1)—not one mile per hour, one kilogram, one Japanese yen or anything like that; just an abstract 1. The word "unit" itself as a noun however usually signifies a physical or financial reference quantity of measure, like a mile per hour, a kilogram or even a Japanese yen. There is no inherent mathematical unity to 1 mile per hour (otherwise known as 0.447 meters per second, among other names). By contrast, a "unitless 1"—a 1 with no physical unit attached—does represent mathematical unity.

Consider the ratio $r = h_1/h_o$ of your height $h_1$ to my height $h_o$. Maybe you are taller than I am and so $r = 1.05$ (not 1.05 cm or 1.05 feet, just 1.05). Now consider the ratio $h_1/h_1$ of your height to your own height. That ratio is of course unity, exactly 1.

There is nothing ephemeral in the concept of mathematical unity, nor in the concept of unitless quantities in general. The concept is quite straightforward and entirely practical. That $r > 1$ means neither more nor less than that you are taller than I am. In applications, one often puts physical quantities in ratio precisely to strip the physical units from them, comparing the ratio to unity without regard to physical units.

in the Argand plane cannot be said to run west-to-east, or anything like that). The $x$, $y$ and $z$ of Fig. 3.4 are each (possibly complex) scalars; $\mathbf{v} = \hat{\mathbf{x}}x + \hat{\mathbf{y}}y + \hat{\mathbf{z}}z$ is a vector. If $x$, $y$ and $z$ are complex, then[9]

$$\begin{aligned} |\mathbf{v}|^2 &= |x|^2 + |y|^2 + |z|^2 = x^*x + y^*y + z^*z \\ &= [\Re(x)]^2 + [\Im(x)]^2 + [\Re(y)]^2 + [\Im(y)]^2 \\ &\quad + [\Re(z)]^2 + [\Im(z)]^2 . \end{aligned} \tag{3.3}$$

A point is sometimes identified by the vector expressing its distance and direction from the origin of the coordinate system. That is, the point $(x, y)$ can be identified with the vector $\hat{\mathbf{x}}x + \hat{\mathbf{y}}y$. However, in the general case vectors are not associated with any particular origin; they represent distances and directions, not fixed positions.

Notice the relative orientation of the axes in Fig. 3.4. The axes are oriented such that if you point your flat right hand in the $x$ direction, then bend your fingers in the $y$ direction and extend your thumb, the thumb then points in the $z$ direction. This is orientation by the *right-hand rule.* A left-handed orientation is equally possible, of course, but as neither orientation has a natural advantage over the other, we arbitrarily but conventionally accept the right-handed one as standard.[10]

Sections 3.4 and 3.9 and Chs. 15 and 16 speak further of the vector.

## 3.4 Rotation

A fundamental problem in trigonometry arises when a vector

$$\mathbf{u} = \hat{\mathbf{x}}x + \hat{\mathbf{y}}y \tag{3.4}$$

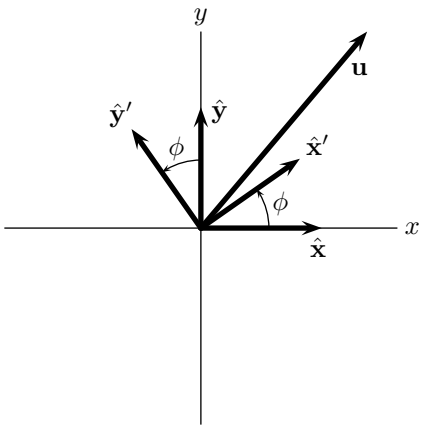
must be expressed in terms of alternate unit vectors $\hat{\mathbf{x}}'$ and $\hat{\mathbf{y}}'$, where $\hat{\mathbf{x}}'$ and $\hat{\mathbf{y}}'$ stand at right angles to one another and lie in the plane[11] of $\hat{\mathbf{x}}$ and $\hat{\mathbf{y}}$,

[9] Some books print $|\mathbf{v}|$ as $\|\mathbf{v}\|$ or even $\|\mathbf{v}\|_2$ to emphasize that it represents the real, scalar magnitude of a complex vector. The reason the last notation subscripts a numeral 2 is obscure, having to do with the professional mathematician's generalized definition of a thing he calls the "norm." This book just renders it $|\mathbf{v}|$.

[10] The writer does not know the etymology for certain, but verbal lore in American engineering has it that the name "right-handed" comes from experience with a standard right-handed wood screw or machine screw. If you hold the screwdriver in your right hand and turn the screw in the natural manner clockwise, turning the screw slot from the $x$ orientation toward the $y$, the screw advances away from you in the $z$ direction into the wood or hole. If somehow you came across a left-handed screw, you'd probably find it easier to drive that screw with the screwdriver in your left hand.

[11] A *plane,* as the reader on this tier undoubtedly knows, is a flat (but not necessarily level) surface, infinite in extent unless otherwise specified. Space is three-dimensional. A

Figure 3.5: Vector basis rotation.




but are rotated from the latter by an angle $\phi$ as depicted in Fig. 3.5.[12] In terms of the trigonometric functions of § 3.1, evidently

$$\begin{aligned} \hat{\mathbf{x}}' &= +\hat{\mathbf{x}}\cos\phi + \hat{\mathbf{y}}\sin\phi, \\ \hat{\mathbf{y}}' &= -\hat{\mathbf{x}}\sin\phi + \hat{\mathbf{y}}\cos\phi; \end{aligned} \tag{3.5}$$

and by appeal to symmetry it stands to reason that

$$\begin{aligned} \hat{\mathbf{x}} &= +\hat{\mathbf{x}}'\cos\phi - \hat{\mathbf{y}}'\sin\phi, \\ \hat{\mathbf{y}} &= +\hat{\mathbf{x}}'\sin\phi + \hat{\mathbf{y}}'\cos\phi. \end{aligned} \tag{3.6}$$

Substituting (3.6) into (3.4) yields

$$\mathbf{u} = \hat{\mathbf{x}}'(x\cos\phi + y\sin\phi) + \hat{\mathbf{y}}'(-x\sin\phi + y\cos\phi), \tag{3.7}$$

which was to be derived.

Equation (3.7) finds general application where rotations in rectangular coordinates are involved. If the question is asked, "what happens if I rotate

plane is two-dimensional. A line is one-dimensional. A point is zero-dimensional. The plane belongs to this geometrical hierarchy.

[12] The "′" mark is pronounced "prime" or "primed" (for no especially good reason of which the author is aware, but anyway, that's how it's pronounced). Mathematical writing employs the mark for a variety of purposes. Here, the mark merely distinguishes the new unit vector $\hat{\mathbf{x}}'$ from the old $\hat{\mathbf{x}}$.

not the unit basis vectors but rather the vector $\mathbf{u}$ instead?" the answer is that it amounts to the same thing, except that the sense of the rotation is reversed:

$$\mathbf{u}' = \hat{\mathbf{x}}(x\cos\phi - y\sin\phi) + \hat{\mathbf{y}}(x\sin\phi + y\cos\phi). \tag{3.8}$$

Whether it is the basis or the vector which rotates thus depends on your point of view.[13]

Much later in the book, § 15.1 will extend rotation in two dimensions to reorientation in three dimensions.

## 3.5 Trigonometric functions of sums and differences of angles

With the results of § 3.4 in hand, we now stand in a position to consider trigonometric functions of sums and differences of angles. Let

$$\hat{\mathbf{a}} \equiv \hat{\mathbf{x}}\cos\alpha + \hat{\mathbf{y}}\sin\alpha,$$
$$\hat{\mathbf{b}} \equiv \hat{\mathbf{x}}\cos\beta + \hat{\mathbf{y}}\sin\beta,$$

be vectors of unit length in the $xy$ plane, respectively at angles $\alpha$ and $\beta$ from the $x$ axis. If we wanted $\hat{\mathbf{b}}$ to coincide with $\hat{\mathbf{a}}$, we would have to rotate it by $\phi = \alpha - \beta$. According to (3.8) and the definition of $\hat{\mathbf{b}}$, if we did this we would obtain

$$\begin{aligned}\hat{\mathbf{b}}' = \hat{\mathbf{x}}&[\cos\beta\cos(\alpha-\beta) - \sin\beta\sin(\alpha-\beta)] \\ &+ \hat{\mathbf{y}}[\cos\beta\sin(\alpha-\beta) + \sin\beta\cos(\alpha-\beta)].\end{aligned}$$

Since we have deliberately chosen the angle of rotation such that $\hat{\mathbf{b}}' = \hat{\mathbf{a}}$, we can separately equate the $\hat{\mathbf{x}}$ and $\hat{\mathbf{y}}$ terms in the expressions for $\hat{\mathbf{a}}$ and $\hat{\mathbf{b}}'$ to obtain the pair of equations

$$\cos\alpha = \cos\beta\cos(\alpha-\beta) - \sin\beta\sin(\alpha-\beta),$$
$$\sin\alpha = \cos\beta\sin(\alpha-\beta) + \sin\beta\cos(\alpha-\beta).$$

Solving the last pair simultaneously[14] for $\sin(\alpha-\beta)$ and $\cos(\alpha-\beta)$ and observing that $\sin^2(\cdot) + \cos^2(\cdot) = 1$ yields

$$\begin{aligned}\sin(\alpha-\beta) &= \sin\alpha\cos\beta - \cos\alpha\sin\beta, \\ \cos(\alpha-\beta) &= \cos\alpha\cos\beta + \sin\alpha\sin\beta.\end{aligned} \tag{3.9}$$

[13] This is only true, of course, with respect to the vectors themselves. When one actually rotates a physical body, the body experiences forces during rotation which might or might not change the body internally in some relevant way.

[14] The easy way to do this is

- to subtract $\sin\beta$ times the first equation from $\cos\beta$ times the second, then to solve

With the change of variable $\beta \leftarrow -\beta$ and the observations from Table 3.1 that $\sin(-\phi) = -\sin\phi$ and $\cos(-\phi) = +\cos(\phi)$, eqns. (3.9) become

$$\begin{aligned} \sin(\alpha+\beta) &= \sin\alpha\cos\beta + \cos\alpha\sin\beta, \\ \cos(\alpha+\beta) &= \cos\alpha\cos\beta - \sin\alpha\sin\beta. \end{aligned} \tag{3.10}$$

Equations (3.9) and (3.10) are the basic formulas for trigonometric functions of sums and differences of angles.

### 3.5.1 Variations on the sums and differences

Several useful variations on (3.9) and (3.10) are achieved by combining the equations in various straightforward ways.[15] These include

$$\begin{aligned} \sin\alpha\sin\beta &= \frac{\cos(\alpha-\beta) - \cos(\alpha+\beta)}{2}, \\ \sin\alpha\cos\beta &= \frac{\sin(\alpha-\beta) + \sin(\alpha+\beta)}{2}, \\ \cos\alpha\cos\beta &= \frac{\cos(\alpha-\beta) + \cos(\alpha+\beta)}{2}. \end{aligned} \tag{3.11}$$

With the change of variables $\delta \leftarrow \alpha - \beta$ and $\gamma \leftarrow \alpha + \beta$, (3.9) and (3.10) become

$$\begin{aligned} \sin\delta &= \sin\left(\frac{\gamma+\delta}{2}\right)\cos\left(\frac{\gamma-\delta}{2}\right) - \cos\left(\frac{\gamma+\delta}{2}\right)\sin\left(\frac{\gamma-\delta}{2}\right), \\ \cos\delta &= \cos\left(\frac{\gamma+\delta}{2}\right)\cos\left(\frac{\gamma-\delta}{2}\right) + \sin\left(\frac{\gamma+\delta}{2}\right)\sin\left(\frac{\gamma-\delta}{2}\right), \\ \sin\gamma &= \sin\left(\frac{\gamma+\delta}{2}\right)\cos\left(\frac{\gamma-\delta}{2}\right) + \cos\left(\frac{\gamma+\delta}{2}\right)\sin\left(\frac{\gamma-\delta}{2}\right), \\ \cos\gamma &= \cos\left(\frac{\gamma+\delta}{2}\right)\cos\left(\frac{\gamma-\delta}{2}\right) - \sin\left(\frac{\gamma+\delta}{2}\right)\sin\left(\frac{\gamma-\delta}{2}\right). \end{aligned}$$

---

the result for $\sin(\alpha-\beta)$;

- to add $\cos\beta$ times the first equation to $\sin\beta$ times the second, then to solve the result for $\cos(\alpha-\beta)$.

This shortcut technique for solving a pair of equations simultaneously for a pair of variables is well worth mastering. In this book alone, it proves useful many times.

[15] Refer to footnote 14 above for the technique.

Combining these in various ways, we have that

$$\begin{aligned}
\sin\gamma + \sin\delta &= 2\sin\left(\frac{\gamma+\delta}{2}\right)\cos\left(\frac{\gamma-\delta}{2}\right), \\
\sin\gamma - \sin\delta &= 2\cos\left(\frac{\gamma+\delta}{2}\right)\sin\left(\frac{\gamma-\delta}{2}\right), \\
\cos\delta + \cos\gamma &= 2\cos\left(\frac{\gamma+\delta}{2}\right)\cos\left(\frac{\gamma-\delta}{2}\right), \\
\cos\delta - \cos\gamma &= 2\sin\left(\frac{\gamma+\delta}{2}\right)\sin\left(\frac{\gamma-\delta}{2}\right).
\end{aligned} \tag{3.12}$$

### 3.5.2 Trigonometric functions of double and half angles

If $\alpha = \beta$, then eqns. (3.10) become the *double-angle formulas*

$$\begin{aligned}
\sin 2\alpha &= 2\sin\alpha\cos\alpha, \\
\cos 2\alpha &= 2\cos^2\alpha - 1 = \cos^2\alpha - \sin^2\alpha = 1 - 2\sin^2\alpha.
\end{aligned} \tag{3.13}$$

Solving (3.13) for $\sin^2\alpha$ and $\cos^2\alpha$ yields the *half-angle formulas*

$$\begin{aligned}
\sin^2\alpha &= \frac{1-\cos 2\alpha}{2}, \\
\cos^2\alpha &= \frac{1+\cos 2\alpha}{2}.
\end{aligned} \tag{3.14}$$

## 3.6 Trigonometric functions of the hour angles

In general one uses the Taylor series of Ch. 8 to calculate trigonometric functions of specific angles. However, for angles which happen to be integral multiples of an *hour*—there are twenty-four or 0x18 hours in a circle, just as there are twenty-four or 0x18 hours in a day[16]—for such angles simpler expressions exist. Figure 3.6 shows the angles. Since such angles arise very frequently in practice, it seems worth our while to study them specially.

Table 3.2 tabulates the trigonometric functions of these *hour angles*. To see how the values in the table are calculated, look at the square and the

[16]Hence an hour is 15°, but you weren't going to write your angles in such inelegant conventional notation as "15°," were you? Well, if you were, you're in good company.

The author is fully aware of the barrier the unfamiliar notation poses for most first-time readers of the book. The barrier is erected neither lightly nor disrespectfully. Consider:

- There are 0x18 hours in a circle.

Figure 3.6: The 0x18 hours in a circle.

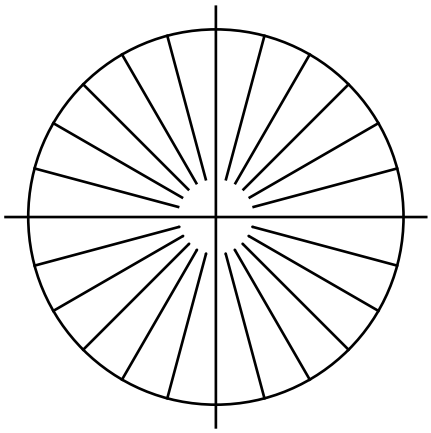
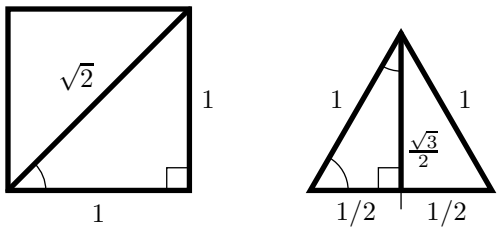


Table 3.2: Trigonometric functions of the hour angles.

| ANGLE $\phi$ [radians] | [hours] | $\sin\phi$ | $\tan\phi$ | $\cos\phi$ |
|---|---|---|---|---|
| $0$ | $0$ | $0$ | $0$ | $1$ |
| $\frac{2\pi}{0\text{x}18}$ | $1$ | $\frac{\sqrt{3}-1}{2\sqrt{2}}$ | $\frac{\sqrt{3}-1}{\sqrt{3}+1}$ | $\frac{\sqrt{3}+1}{2\sqrt{2}}$ |
| $\frac{2\pi}{0\text{xC}}$ | $2$ | $\frac{1}{2}$ | $\frac{1}{\sqrt{3}}$ | $\frac{\sqrt{3}}{2}$ |
| $\frac{2\pi}{8}$ | $3$ | $\frac{1}{\sqrt{2}}$ | $1$ | $\frac{1}{\sqrt{2}}$ |
| $\frac{2\pi}{6}$ | $4$ | $\frac{\sqrt{3}}{2}$ | $\sqrt{3}$ | $\frac{1}{2}$ |
| $\frac{(5)(2\pi)}{0\text{x}18}$ | $5$ | $\frac{\sqrt{3}+1}{2\sqrt{2}}$ | $\frac{\sqrt{3}+1}{\sqrt{3}-1}$ | $\frac{\sqrt{3}-1}{2\sqrt{2}}$ |
| $\frac{2\pi}{4}$ | $6$ | $1$ | $\infty$ | $0$ |

Figure 3.7: A square and an equilateral triangle for calculating trigonometric functions of the hour angles.




equilateral triangle[17] of Fig. 3.7. Each of the square's four angles naturally measures six hours; and since a triangle's angles always total twelve hours (§ 2.9.3), by symmetry each of the angles of the equilateral triangle in the figure measures four. Also by symmetry, the perpendicular splits the triangle's top angle into equal halves of two hours each and its bottom leg into equal segments of length 1/2 each; and the diagonal splits the square's corner into equal halves of three hours each. The Pythagorean theorem (§ 2.10) then supplies the various other lengths in the figure, after which we observe

---

- There are 360 degrees in a circle.

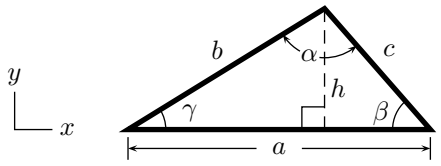
Both sentences say the same thing, don't they? But even though the "0x" hex prefix is a bit clumsy, the first sentence nevertheless says the thing rather better. The reader is urged to invest the attention and effort to master the notation.

There is a psychological trap regarding the hour. The familiar, standard clock face shows only twelve hours not twenty-four, so the angle between eleven o'clock and twelve *on the clock face* is not an hour of arc! That angle is two hours of arc. This is so because the clock face's geometry is artificial. If you have ever been to the Old Royal Observatory at Greenwich, England, you may have seen the big clock face there with all twenty-four hours on it. It'd be a bit hard to read the time from such a crowded clock face were it not so big, but anyway, the angle between hours on the Greenwich clock is indeed an honest hour of arc. [8]

The hex and hour notations are recommended mostly only for theoretical math work. It is not claimed that they offer much benefit in most technical work of the less theoretical kinds. If you wrote an engineering memo describing a survey angle as 0x1.80 hours instead of 22.5 degrees, for example, you'd probably not like the reception the memo got. Nonetheless, the improved notation fits a book of this kind so well that the author hazards it. It is hoped that after trying the notation a while, the reader will approve the choice.

[17] An *equilateral* triangle is, as the name and the figure suggest, a triangle whose three sides all have the same length.

Figure 3.8: The laws of sines and cosines.




from Fig. 3.1 that

- the sine of a non-right angle in a right triangle is the opposite leg's length divided by the diagonal's,
- the tangent is the opposite leg's length divided by the adjacent leg's, and
- the cosine is the adjacent leg's length divided by the diagonal's.

With this observation and the lengths in the figure, one can calculate the sine, tangent and cosine of angles of two, three and four hours.

The values for one and five hours are found by applying (3.9) and (3.10) against the values for two and three hours just calculated. The values for zero and six hours are, of course, seen by inspection.[18]

## 3.7 The laws of sines and cosines

Refer to the triangle of Fig. 3.8. By the definition of the sine function, one can write that

$$c \sin\beta = h = b \sin\gamma,$$

or in other words that

$$\frac{\sin\beta}{b} = \frac{\sin\gamma}{c}.$$

[18]The creative reader may notice that he can extend the table to any angle by repeated application of the various sum, difference and half-angle formulas from the preceding sections to the values already in the table. However, the Taylor series (§ 8.9) offers a cleaner, quicker way to calculate trigonometrics of non-hour angles.

But there is nothing special about $\beta$ and $\gamma$; what is true for them must be true for $\alpha$, too.[19] Hence,

$$\frac{\sin\alpha}{a} = \frac{\sin\beta}{b} = \frac{\sin\gamma}{c}. \tag{3.15}$$

This equation is known as *the law of sines.*

On the other hand, if one expresses $a$ and $b$ as vectors emanating from the point $\gamma$,[20]

$$\begin{aligned}\mathbf{a} &= \hat{\mathbf{x}}a,\\ \mathbf{b} &= \hat{\mathbf{x}}b\cos\gamma + \hat{\mathbf{y}}b\sin\gamma,\end{aligned}$$

then

$$\begin{aligned}c^2 &= |\mathbf{b}-\mathbf{a}|^2\\ &= (b\cos\gamma - a)^2 + (b\sin\gamma)^2\\ &= a^2 + (b^2)(\cos^2\gamma + \sin^2\gamma) - 2ab\cos\gamma.\end{aligned}$$

Since $\cos^2(\cdot) + \sin^2(\cdot) = 1$, this is

$$c^2 = a^2 + b^2 - 2ab\cos\gamma, \tag{3.16}$$

known as *the law of cosines.*

## 3.8 Summary of properties

Table 3.2 on page 61 has listed the values of trigonometric functions of the hour angles. Table 3.1 on page 52 has summarized simple properties of the trigonometric functions. Table 3.3 summarizes further properties, gathering them from §§ 3.4, 3.5 and 3.7.

[19] "But," it is objected, "there *is* something special about $\alpha$. The perpendicular $h$ drops from it."

True. However, the $h$ is just a utility variable to help us to manipulate the equation into the desired form; we're not interested in $h$ itself. Nothing prevents us from dropping additional perpendiculars $h_\beta$ and $h_\gamma$ from the other two corners and using those as utility variables, too, if we like. We can use any utility variables we want.

[20] Here is another example of the book's judicious relaxation of formal rigor. Of course there is no "point $\gamma$"; $\gamma$ is an angle not a point. However, the writer suspects in light of Fig. 3.8 that few readers will be confused as to which point is meant. The skillful applied mathematician does not multiply labels without need.

Table 3.3: Further properties of the trigonometric functions.

$$
\begin{aligned}
\mathbf{u} &= \hat{\mathbf{x}}'(x\cos\phi + y\sin\phi) + \hat{\mathbf{y}}'(-x\sin\phi + y\cos\phi) \\
\sin(\alpha \pm \beta) &= \sin\alpha\cos\beta \pm \cos\alpha\sin\beta \\
\cos(\alpha \pm \beta) &= \cos\alpha\cos\beta \mp \sin\alpha\sin\beta \\
\sin\alpha\sin\beta &= \frac{\cos(\alpha-\beta) - \cos(\alpha+\beta)}{2} \\
\sin\alpha\cos\beta &= \frac{\sin(\alpha-\beta) + \sin(\alpha+\beta)}{2} \\
\cos\alpha\cos\beta &= \frac{\cos(\alpha-\beta) + \cos(\alpha+\beta)}{2} \\
\sin\gamma + \sin\delta &= 2\sin\left(\frac{\gamma+\delta}{2}\right)\cos\left(\frac{\gamma-\delta}{2}\right) \\
\sin\gamma - \sin\delta &= 2\cos\left(\frac{\gamma+\delta}{2}\right)\sin\left(\frac{\gamma-\delta}{2}\right) \\
\cos\delta + \cos\gamma &= 2\cos\left(\frac{\gamma+\delta}{2}\right)\cos\left(\frac{\gamma-\delta}{2}\right) \\
\cos\delta - \cos\gamma &= 2\sin\left(\frac{\gamma+\delta}{2}\right)\sin\left(\frac{\gamma-\delta}{2}\right) \\
\sin 2\alpha &= 2\sin\alpha\cos\alpha \\
\cos 2\alpha &= 2\cos^2\alpha - 1 = \cos^2\alpha - \sin^2\alpha = 1 - 2\sin^2\alpha \\
\sin^2\alpha &= \frac{1-\cos 2\alpha}{2} \\
\cos^2\alpha &= \frac{1+\cos 2\alpha}{2} \\
\frac{\sin\gamma}{c} &= \frac{\sin\alpha}{a} = \frac{\sin\beta}{b} \\
c^2 &= a^2 + b^2 - 2ab\cos\gamma
\end{aligned}
$$

## 3.9 Cylindrical and spherical coordinates

Section 3.3 has introduced the concept of the vector

$$\mathbf{v} = \hat{\mathbf{x}}x + \hat{\mathbf{y}}y + \hat{\mathbf{z}}z.$$

The coefficients $(x, y, z)$ on the equation's right side are *coordinates*—specifically, *rectangular coordinates*—which given a specific orthonormal[21] set of unit basis vectors $[\hat{\mathbf{x}}\ \hat{\mathbf{y}}\ \hat{\mathbf{z}}]$ uniquely identify a point (see Fig. 3.4 on page 53; also, much later in the book, refer to § 15.3). Such rectangular coordinates are simple and general, and are convenient for many purposes. However, there are at least two broad classes of conceptually simple problems for which rectangular coordinates tend to be inconvenient: problems in which an axis or a point dominates. Consider for example an electric wire's magnetic field, whose intensity varies with distance from the wire (an axis); or the illumination a lamp sheds on a printed page of this book, which depends on the book's distance from the lamp (a point).

To attack a problem dominated by an axis, the *cylindrical coordinates* $(\rho; \phi, z)$ can be used instead of the rectangular coordinates $(x, y, z)$. To attack a problem dominated by a point, the *spherical coordinates* $(r; \theta; \phi)$ can be used.[22] Refer to Fig. 3.9. Such coordinates are related to one another and to the rectangular coordinates by the formulas of Table 3.4.

Cylindrical and spherical coordinates can greatly simplify the analyses of the kinds of problems they respectively fit, but they come at a price. There are no constant unit basis vectors to match them. That is,

$$\mathbf{v} = \hat{\mathbf{x}}x + \hat{\mathbf{y}}y + \hat{\mathbf{z}}z \neq \hat{\boldsymbol{\rho}}\rho + \hat{\boldsymbol{\phi}}\phi + \hat{\mathbf{z}}z \neq \hat{\mathbf{r}}r + \hat{\boldsymbol{\theta}}\theta + \hat{\boldsymbol{\phi}}\phi.$$

It doesn't work that way. Nevertheless, *variable* unit basis vectors are defined:

$$\begin{aligned}
\hat{\boldsymbol{\rho}} &\equiv +\hat{\mathbf{x}}\cos\phi + \hat{\mathbf{y}}\sin\phi, \\
\hat{\boldsymbol{\phi}} &\equiv -\hat{\mathbf{x}}\sin\phi + \hat{\mathbf{y}}\cos\phi, \\
\hat{\mathbf{r}} &\equiv +\hat{\mathbf{z}}\cos\theta + \hat{\boldsymbol{\rho}}\sin\theta, \\
\hat{\boldsymbol{\theta}} &\equiv -\hat{\mathbf{z}}\sin\theta + \hat{\boldsymbol{\rho}}\cos\theta;
\end{aligned} \tag{3.17}$$

[21] *Orthonormal* in this context means "of unit length and at right angles to the other vectors in the set." [66, "Orthonormality," 14:19, 7 May 2006]

[22] Notice that the $\phi$ is conventionally written second in cylindrical $(\rho; \phi, z)$ but third in spherical $(r; \theta; \phi)$ coordinates. This odd-seeming convention is to maintain proper right-handed coordinate rotation. (The explanation will seem clearer once Chs. 15 and 16 are read.)

Figure 3.9: A point on a sphere, in spherical $(r;\theta;\phi)$ and cylindrical $(\rho;\phi,z)$ coordinates. (The axis labels bear circumflexes in this figure only to disambiguate the $\hat{z}$ axis from the cylindrical coordinate $z$.)

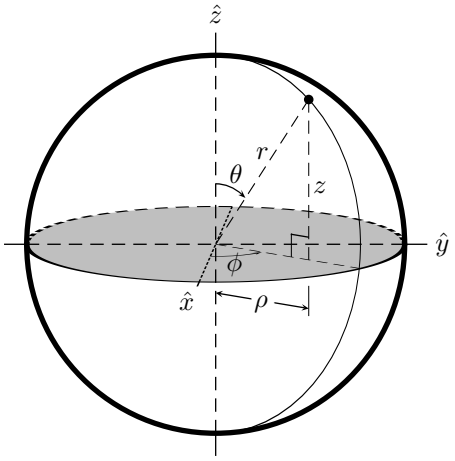



Table 3.4: Relations among the rectangular, cylindrical and spherical coordinates.

$$
\begin{aligned}
\rho^2 &= x^2 + y^2 \\
r^2 &= \rho^2 + z^2 = x^2 + y^2 + z^2 \\
\tan\theta &= \frac{\rho}{z} \\
\tan\phi &= \frac{y}{x} \\
z &= r\cos\theta \\
\rho &= r\sin\theta \\
x &= \rho\cos\phi = r\sin\theta\cos\phi \\
y &= \rho\sin\phi = r\sin\theta\sin\phi
\end{aligned}
$$

or, substituting identities from the table,

$$\begin{aligned}
\hat{\boldsymbol{\rho}} &= \frac{\hat{\mathbf{x}}x + \hat{\mathbf{y}}y}{\rho}, \\
\hat{\boldsymbol{\phi}} &= \frac{-\hat{\mathbf{x}}y + \hat{\mathbf{y}}x}{\rho}, \\
\hat{\mathbf{r}} &= \frac{\hat{\mathbf{z}}z + \hat{\boldsymbol{\rho}}\rho}{r} = \frac{\hat{\mathbf{x}}x + \hat{\mathbf{y}}y + \hat{\mathbf{z}}z}{r}, \\
\hat{\boldsymbol{\theta}} &= \frac{-\hat{\mathbf{z}}\rho + \hat{\boldsymbol{\rho}}z}{r}.
\end{aligned} \tag{3.18}$$

Such variable unit basis vectors point locally in the directions in which their respective coordinates advance.

Combining pairs of (3.17)'s equations appropriately, we have also that

$$\begin{aligned}
\hat{\mathbf{x}} &= +\hat{\boldsymbol{\rho}}\cos\phi - \hat{\boldsymbol{\phi}}\sin\phi, \\
\hat{\mathbf{y}} &= +\hat{\boldsymbol{\rho}}\sin\phi + \hat{\boldsymbol{\phi}}\cos\phi, \\
\hat{\mathbf{z}} &= +\hat{\mathbf{r}}\cos\theta - \hat{\boldsymbol{\theta}}\sin\theta, \\
\hat{\boldsymbol{\rho}} &= +\hat{\mathbf{r}}\sin\theta + \hat{\boldsymbol{\theta}}\cos\theta.
\end{aligned} \tag{3.19}$$

Convention usually orients $\hat{\mathbf{z}}$ in the direction of a problem's axis. Occasionally however a problem arises in which it is more convenient to orient $\hat{\mathbf{x}}$ or $\hat{\mathbf{y}}$ in the direction of the problem's axis (usually because $\hat{\mathbf{z}}$ has already been established in the direction of some other pertinent axis). Changing the meanings of known symbols like $\rho$, $\theta$ and $\phi$ is usually not a good idea, but you can use symbols like

$$\begin{aligned}
(\rho^x)^2 &= y^2 + z^2, & (\rho^y)^2 &= z^2 + x^2, \\
\tan\theta^x &= \frac{\rho^x}{x}, & \tan\theta^y &= \frac{\rho^y}{y}, \\
\tan\phi^x &= \frac{z}{y}, & \tan\phi^y &= \frac{x}{z},
\end{aligned} \tag{3.20}$$

instead if needed.[23]

[23] Symbols like $\rho^x$ are logical but, as far as this writer is aware, not standard. The writer is not aware of any conventionally established symbols for quantities like these, but § 15.6 at least will use the $\rho^x$-style symbology.

## 3.10 The complex triangle inequalities

If the real, two-dimensional vectors $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ represent the three sides of a triangle such that $\mathbf{a} + \mathbf{b} + \mathbf{c} = 0$, then per (2.44)

$$|\mathbf{a}| - |\mathbf{b}| \leq |\mathbf{a} + \mathbf{b}| \leq |\mathbf{a}| + |\mathbf{b}| .$$

These are just the triangle inequalities of § 2.9.2 in vector notation.[24] But if the triangle inequalities hold for real vectors in a plane, then why not equally for complex scalars? Consider the geometric interpretation of the Argand plane of Fig. 2.5 on page 43. Evidently,

$$|z_1| - |z_2| \leq |z_1 + z_2| \leq |z_1| + |z_2| \tag{3.21}$$

for any two complex numbers $z_1$ and $z_2$. Extending the sum inequality, we have that

$$\left| \sum_k z_k \right| \leq \sum_k |z_k| . \tag{3.22}$$

Naturally, (3.21) and (3.22) hold equally well for real numbers as for complex; one may find the latter formula useful for sums of real numbers, for example, when some of the numbers summed are positive and others negative.[25]

An important consequence of (3.22) is that if $\sum |z_k|$ converges, then $\sum z_k$ also converges. Such a consequence is important because mathematical derivations sometimes need the convergence of $\sum z_k$ established, which can be hard to do directly. Convergence of $\sum |z_k|$, which per (3.22) implies convergence of $\sum z_k$, is often easier to establish.

See also (9.15). Equation (3.22) will find use among other places in § 8.10.3.

## 3.11 De Moivre's theorem

Compare the Argand-plotted complex number of Fig. 2.5 (page 43) against the vector of Fig. 3.3 (page 53). Although complex numbers are scalars not vectors, the figures do suggest an analogy between complex phase and vector direction. With reference to Fig. 2.5 we can write

$$z = (\rho)(\cos\phi + i\sin\phi) = \rho \operatorname{cis}\phi, \tag{3.23}$$

[24] Reading closely, one might note that § 2.9.2 uses the "<" sign rather than the "≤," but that's all right. See § 1.2.2.

[25] Section 13.9 proves the triangle inequalities more generally.

where

$$\operatorname{cis}\phi \equiv \cos\phi + i\sin\phi. \tag{3.24}$$

If $z = x + iy$, then evidently

$$\begin{aligned} x &= \rho\cos\phi, \\ y &= \rho\sin\phi. \end{aligned} \tag{3.25}$$

Per (2.54),

$$z_1 z_2 = (x_1 x_2 - y_1 y_2) + i(y_1 x_2 + x_1 y_2).$$

Applying (3.25) to the equation yields

$$\frac{z_1 z_2}{\rho_1 \rho_2} = (\cos\phi_1 \cos\phi_2 - \sin\phi_1 \sin\phi_2) + i(\sin\phi_1 \cos\phi_2 + \cos\phi_1 \sin\phi_2).$$

But according to (3.10), this is just

$$\frac{z_1 z_2}{\rho_1 \rho_2} = \cos(\phi_1 + \phi_2) + i\sin(\phi_1 + \phi_2),$$

or in other words

$$z_1 z_2 = \rho_1 \rho_2 \operatorname{cis}(\phi_1 + \phi_2). \tag{3.26}$$

Equation (3.26) is an important result. It says that if you want to multiply complex numbers, it suffices

- to multiply their magnitudes and
- to add their phases.

It follows by parallel reasoning (or by extension) that

$$\frac{z_1}{z_2} = \frac{\rho_1}{\rho_2} \operatorname{cis}(\phi_1 - \phi_2) \tag{3.27}$$

and by extension that

$$z^a = \rho^a \operatorname{cis} a\phi. \tag{3.28}$$

Equations (3.26), (3.27) and (3.28) are known as *de Moivre's theorem.*[26,27]

[26] Also called *de Moivre's formula.* Some authors apply the name of de Moivre directly only to (3.28), or to some variation thereof; but since the three equations express essentially the same idea, if you refer to any of them as *de Moivre's theorem* then you are unlikely to be misunderstood.

[27] [57][66]

We have not shown yet, but will in § 5.4, that

$$\operatorname{cis}\phi = \exp i\phi = e^{i\phi},$$

where $\exp(\cdot)$ is the natural exponential function and $e$ is the natural logarithmic base, both defined in Ch. 5. De Moivre's theorem is most useful in this light.

Section 5.5 will revisit the derivation of de Moivre's theorem.

# Chapter 4

# The derivative

The mathematics of *calculus* concerns a complementary pair of questions:[1]

- Given some function $f(t)$, what is the function's instantaneous rate of change, or *derivative,* $f'(t)$?
- Interpreting some function $f'(t)$ as an instantaneous rate of change, what is the corresponding accretion, or *integral,* $f(t)$?

This chapter builds toward a basic understanding of the first question.

## 4.1 Infinitesimals and limits

Calculus systematically treats numbers so large and so small, they lie beyond the reach of our mundane number system.

[1] Although once grasped the concept is relatively simple, to understand this pair of questions, so briefly stated, is no trivial thing. They are the pair which eluded or confounded the most brilliant mathematical minds of the ancient world.

The greatest conceptual hurdle—the stroke of brilliance—probably lies simply in stating the pair of questions clearly. Sir Isaac Newton and G.W. Leibnitz cleared this hurdle for us in the seventeenth century, so now at least we know the right pair of questions to ask. With the pair in hand, the calculus beginner's first task is quantitatively to understand the pair's interrelationship, generality and significance. Such an understanding constitutes the basic calculus concept.

It cannot be the role of a book like this one to lead the beginner gently toward an apprehension of the basic calculus concept. Once grasped, the concept is simple and briefly stated. In this book we necessarily state the concept briefly, then move along. Many instructional textbooks—[27] is a worthy example—have been written to lead the beginner gently. Although a sufficiently talented, dedicated beginner could perhaps obtain the basic calculus concept directly here, he would probably find it quicker and more pleasant to begin with a book like the one referenced.

### 4.1.1 The infinitesimal

A number $\epsilon$ is an *infinitesimal* if it is so small that

$$0 < |\epsilon| < a$$

for all possible mundane positive numbers $a$.

This is somewhat a difficult concept, so if it is not immediately clear then let us approach the matter colloquially. Let me propose to you that I have an infinitesimal.

"How big is your infinitesimal?" you ask.

"Very, very small," I reply.

"How small?"

"Very small."

"Smaller than 0x0.01?"

"Smaller than what?"

"Than $2^{-8}$. You said that we should use hexadecimal notation in this book, remember?"

"Sorry. Yes, right, smaller than 0x0.01."

"What about 0x0.0001? Is it smaller than that?"

"Much smaller."

"Smaller than $0\text{x}0.0000\,0000\,0000\,0001$?"

"Smaller."

"Smaller than $2^{-0\text{x}1\,0000\,0000\,0000\,0000}$?"

"Now *that* is an impressively small number. Nevertheless, my infinitesimal is smaller still."

"Zero, then."

"Oh, no. Bigger than that. My infinitesimal is definitely bigger than zero."

This is the idea of the infinitesimal. It is a definite number of a certain nonzero magnitude, but its smallness conceptually lies beyond the reach of our mundane number system.

If $\epsilon$ is an infinitesimal, then $1/\epsilon$ can be regarded as an *infinity:* a very large number much larger than any mundane number one can name.

The principal advantage of using symbols like $\epsilon$ rather than 0 for infinitesimals is in that it permits us conveniently to compare one infinitesimal against another, to add them together, to divide them, etc. For instance, if $\delta = 3\epsilon$ is another infinitesimal, then the quotient $\delta/\epsilon$ is not some unfathomable 0/0; rather it is $\delta/\epsilon = 3$. In physical applications, the infinitesimals are often not true mathematical infinitesimals but rather relatively very small quantities such as the mass of a wood screw compared to the mass

of a wooden house frame, or the audio power of your voice compared to that of a jet engine. The additional cost of inviting one more guest to the wedding may or may not be infinitesimal, depending on your point of view. The key point is that the infinitesimal quantity be negligible by comparison, whatever "negligible" means in the context.[2]

The second-order infinitesimal $\epsilon^2$ is so small on the scale of the common, first-order infinitesimal $\epsilon$ that the even latter cannot measure it. The $\epsilon^2$ is an infinitesimal to the infinitesimals. Third- and higher-order infinitesimals are likewise possible.

The notation $u \ll v$, or $v \gg u$, indicates that $u$ is much less than $v$, typically such that one can regard the quantity $u/v$ to be an infinitesimal. In fact, one common way to specify that $\epsilon$ be infinitesimal is to write that $\epsilon \ll 1$.

### 4.1.2 Limits

The notation $\lim_{z \to z_o}$ indicates that $z$ draws as near to $z_o$ as it possibly can. When written $\lim_{z \to z_o^+}$, the implication is that $z$ draws toward $z_o$ from the positive side such that $z > z_o$. Similarly, when written $\lim_{z \to z_o^-}$, the implication is that $z$ draws toward $z_o$ from the negative side.

The reason for the notation is to provide a way to handle expressions like

$$\frac{3z}{2z}$$

as $z$ vanishes:

$$\lim_{z \to 0} \frac{3z}{2z} = \frac{3}{2}.$$

The symbol "$\lim_Q$" is short for "in the limit as $Q$."

Notice that lim is not a function like log or sin. It is just a reminder that a quantity approaches some value, used when saying that the quantity

[2] Among scientists and engineers who study wave phenomena, there is an old rule of thumb that sinusoidal waveforms be discretized not less finely than ten points per wavelength. In keeping with this book's adecimal theme (Appendix A) and the concept of the hour of arc (§ 3.6), we should probably render the rule as *twelve* points per wavelength here. In any case, even very roughly speaking, a quantity greater then 1/0xC of the principal to which it compares probably cannot rightly be regarded as infinitesimal. On the other hand, a quantity less than 1/0x10000 of the principal is indeed infinitesimal for most practical purposes (but not all: for example, positions of spacecraft and concentrations of chemical impurities must sometimes be accounted more precisely). For quantities between 1/0xC and 1/0x10000, it depends on the accuracy one seeks.

*equaled* the value would be confusing. Consider that to say

$$\lim_{z \to 2^-} (z + 2) = 4$$

is just a fancy way of saying that $2 + 2 = 4$. The lim notation is convenient to use sometimes, but it is not magical. Don't let it confuse you.

## 4.2 Combinatorics

In its general form, the problem of selecting $k$ specific items out of a set of $n$ available items belongs to probability theory (Ch. 20). In its basic form, however, the same problem also applies to the handling of polynomials or power series. This section treats the problem in its basic form.[3]

### 4.2.1 Combinations and permutations

Consider the following scenario. I have several small wooden blocks of various shapes and sizes, painted different colors so that you can clearly tell each block from the others. If I offer you the blocks and you are free to take all, some or none of them at your option, if you can take whichever blocks you want, then how many distinct choices of blocks do you have? Answer: you have $2^n$ choices, because you can accept or reject the first block, then accept or reject the second, then the third, and so on.

Now, suppose that what you want is exactly $k$ blocks, neither more nor fewer. Desiring exactly $k$ blocks, you select your favorite block first: there are $n$ options for this. Then you select your second favorite: for this, there are $n - 1$ options (why not $n$ options? because you have already taken one block from me; I have only $n - 1$ blocks left). Then you select your third favorite—for this there are $n - 2$ options—and so on until you have $k$ blocks. There are evidently

$$P \left( \begin{array}{c} n \\ k \end{array} \right) \equiv n!/(n - k)! \tag{4.1}$$

ordered ways, or *permutations,* available for you to select exactly $k$ blocks.

However, some of these distinct permutations put exactly the same *combination* of blocks in your hand; for instance, the permutations red-green-blue and green-red-blue constitute the same combination, whereas red-white-blue is a different combination entirely. For a single combination

[3] [27]

of $k$ blocks (red, green, blue), evidently $k!$ permutations are possible (red-green-blue, red-blue-green, green-red-blue, green-blue-red, blue-red-green, blue-green-red). Hence dividing the number of permutations (4.1) by $k!$ yields the number of combinations

$$\left( \begin{array}{c} n \\ k \end{array} \right) \equiv \frac{n!/(n-k)!}{k!}. \tag{4.2}$$

Properties of the number $\left( \begin{array}{c} n \\ k \end{array} \right)$ of combinations include that

$$\left( \begin{array}{c} n \\ n-k \end{array} \right) = \left( \begin{array}{c} n \\ k \end{array} \right), \tag{4.3}$$

$$\sum_{k=0}^{n} \left( \begin{array}{c} n \\ k \end{array} \right) = 2^n, \tag{4.4}$$

$$\left( \begin{array}{c} n-1 \\ k-1 \end{array} \right) + \left( \begin{array}{c} n-1 \\ k \end{array} \right) = \left( \begin{array}{c} n \\ k \end{array} \right), \tag{4.5}$$

$$\left( \begin{array}{c} n \\ k \end{array} \right) = \frac{n-k+1}{k} \left( \begin{array}{c} n \\ k-1 \end{array} \right) \tag{4.6}$$

$$= \frac{k+1}{n-k} \left( \begin{array}{c} n \\ k+1 \end{array} \right) \tag{4.7}$$

$$= \frac{n}{k} \left( \begin{array}{c} n-1 \\ k-1 \end{array} \right) \tag{4.8}$$

$$= \frac{n}{n-k} \left( \begin{array}{c} n-1 \\ k \end{array} \right). \tag{4.9}$$

Equation (4.3) results from changing the variable $k \leftarrow n-k$ in (4.2). Equation (4.4) comes directly from the observation (made at the head of this section) that $2^n$ total combinations are possible if any $k$ is allowed. Equation (4.5) is seen when an $n$th block—let us say that it is a black block—is added to an existing set of $n-1$ blocks; to choose $k$ blocks then, you can either choose $k$ from the original set, or the black block plus $k-1$ from the original set. Equations (4.6) through (4.9) come directly from the definition (4.2); they relate combinatoric coefficients to their neighbors in Pascal's triangle (§ 4.2.2).

Because one can choose neither fewer than zero nor more than $n$ from $n$ blocks,

$$\left( \begin{array}{c} n \\ k \end{array} \right) = 0 \quad \text{unless } 0 \le k \le n. \tag{4.10}$$

For $\left( \begin{array}{c} n \\ k \end{array} \right)$ when $n < 0$, there is no obvious definition.

Figure 4.1: The plan for Pascal's triangle.

$$\begin{array}{ccccccccc}
 & & & \binom{0}{0} & & & & & \\
 & & \binom{1}{0} & & \binom{1}{1} & & & & \\
 & \binom{2}{0} & & \binom{2}{1} & & \binom{2}{2} & & & \\
\binom{3}{0} & & \binom{3}{1} & & \binom{3}{2} & & \binom{3}{3} & & \\
\binom{4}{0} & & \binom{4}{1} & & \binom{4}{2} & & \binom{4}{3} & & \binom{4}{4} \\
\vdots & & & & & & & &
\end{array}$$

### 4.2.2 Pascal's triangle

Consider the triangular layout in Fig. 4.1 of the various possible $\binom{n}{k}$. Evaluated, this yields Fig. 4.2, *Pascal's triangle.* Notice how each entry in the triangle is the sum of the two entries immediately above, as (4.5) predicts. (In fact this is the easy way to fill Pascal's triangle out: for each entry, just add the two entries above.)

## 4.3 The binomial theorem

This section presents the binomial theorem and one of its significant consequences.

### 4.3.1 Expanding the binomial

The *binomial theorem* holds that[4]

$$(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^{n-k} b^k. \tag{4.11}$$

[4]The author is given to understand that, by an heroic derivational effort, (4.11) can be extended to nonintegral $n$. However, since applied mathematics does not usually concern itself with hard theorems of little known practical use, the extension as such is not covered in this book. What is covered—in Table 8.1—is the Taylor series for $(1+z)^{a-1}$ for complex $z$ and complex $a$, which amounts to much the same thing.

Figure 4.2: Pascal's triangle.

$$
\begin{array}{c}
1 \\
1\ \ 1 \\
1\ \ 2\ \ 1 \\
1\ \ 3\ \ 3\ \ 1 \\
1\ \ 4\ \ 6\ \ 4\ \ 1 \\
1\ \ 5\ \ \mathrm{A}\ \ \mathrm{A}\ \ 5\ \ 1 \\
1\ \ 6\ \ \mathrm{F}\ \ 14\ \ \mathrm{F}\ \ 6\ \ 1 \\
1\ \ 7\ \ 15\ \ 23\ \ 23\ \ 15\ \ 7\ \ 1 \\
\vdots
\end{array}
$$

In the common case that $a = 1$, $b = \epsilon$, $|\epsilon| \ll 1$, this is

$$(1 + \epsilon)^n = \sum_{k=0}^{n} \left( \begin{array}{c} n \\ k \end{array} \right) \epsilon^k \tag{4.12}$$

(actually this holds for any $\epsilon$, small or large; but the typical case of interest has $|\epsilon| \ll 1$). In either form, the binomial theorem is a direct consequence of the combinatorics of § 4.2. Since

$$(a + b)^n = (a + b)(a + b) \cdots (a + b)(a + b),$$

each $(a+b)$ factor corresponds to one of the "wooden blocks," where $a$ means rejecting the block and $b$, accepting it.

### 4.3.2 Powers of numbers near unity

Since $\left( \begin{array}{c} n \\ 0 \end{array} \right) = 1$ and $\left( \begin{array}{c} n \\ 1 \end{array} \right) = n$, it follows from (4.12) for

$$(m, n) \in \mathbb{Z},\ m > 0,\ n \geq 0,\ |\delta| \ll 1,\ |\epsilon| \ll 1,\ |\epsilon_o| \ll 1,$$

that[5]

$$1 + m\epsilon_o \approx (1 + \epsilon_o)^m$$

[5]The symbol $\approx$ means "approximately equals."

to excellent precision. Furthermore, raising the equation to the $1/m$ power then changing $\delta \leftarrow m\epsilon_o$, we have that

$$(1+\delta)^{1/m} \approx 1 + \frac{\delta}{m}.$$

Changing $1+\delta \leftarrow (1+\epsilon)^n$ and observing from the $(1+\epsilon_o)^m$ equation above that this implies that $\delta \approx n\epsilon$, we have that

$$(1+\epsilon)^{n/m} \approx 1 + \frac{n}{m}\epsilon.$$

Inverting this equation yields

$$(1+\epsilon)^{-n/m} \approx \frac{1}{1+(n/m)\epsilon} = \frac{[1-(n/m)\epsilon]}{[1-(n/m)\epsilon][1+(n/m)\epsilon]} \approx 1 - \frac{n}{m}\epsilon.$$

Taken together, the last two equations imply that

$$(1+\epsilon)^x \approx 1 + x\epsilon \tag{4.13}$$

for any real $x$.

The writer knows of no conventional name[6] for (4.13), but named or unnamed it is an important equation. The equation offers a simple, accurate way of approximating any real power of numbers in the near neighborhood of 1.

### 4.3.3 Complex powers of numbers near unity

Equation (4.13) is fine as far as it goes, but its very form suggests the question: what if $\epsilon$ or $x$, or both, are complex? Changing the symbol $z \leftarrow x$ and observing that the infinitesimal $\epsilon$ may also be complex, one wants to know whether

$$(1+\epsilon)^z \approx 1 + z\epsilon \tag{4.14}$$

still holds. No work we have yet done in the book answers the question, because although a complex infinitesimal $\epsilon$ poses no particular problem, the action of a complex power $z$ remains undefined. Still, for consistency's sake, one would like (4.14) to hold. In fact nothing prevents us from defining the action of a complex power such that (4.14) does hold, which we now do, logically extending the known result (4.13) into the new domain.

[6] Actually, "the first-order Taylor expansion" is a conventional name for it, but so unwieldy a name does not fit the present context. Ch. 8 will introduce the Taylor expansion as such.

Section 5.4 will investigate the extremely interesting effects which arise when $\Re(\epsilon) = 0$ and the power $z$ in (4.14) grows large, but for the moment we shall use the equation in a more ordinary manner to develop the concept and basic application of the derivative, as follows.

## 4.4 The derivative

Having laid down (4.14), we now stand in a position properly to introduce the chapter's subject, the derivative. What is the derivative? The *derivative* is the instantaneous rate or slope of a function. In mathematical symbols and for the moment using real numbers,

$$f'(t) \equiv \lim_{\epsilon \to 0^+} \frac{f(t + \epsilon/2) - f(t - \epsilon/2)}{\epsilon}. \tag{4.15}$$

Alternately, one can define the same derivative in the unbalanced form

$$f'(t) = \lim_{\epsilon \to 0^+} \frac{f(t + \epsilon) - f(t)}{\epsilon},$$

but this book generally prefers the more elegant balanced form (4.15), which we will now use in developing the derivative's several properties through the rest of the chapter.[7]

### 4.4.1 The derivative of the power series

In the very common case that $f(t)$ is the power series

$$f(t) = \sum_{k=-\infty}^{\infty} c_k t^k, \tag{4.16}$$

where the $c_k$ are in general complex coefficients, (4.15) says that

$$\begin{aligned} f'(t) &= \sum_{k=-\infty}^{\infty} \lim_{\epsilon \to 0^+} \frac{(c_k)(t + \epsilon/2)^k - (c_k)(t - \epsilon/2)^k}{\epsilon} \\ &= \sum_{k=-\infty}^{\infty} \lim_{\epsilon \to 0^+} c_k t^k \frac{(1 + \epsilon/2t)^k - (1 - \epsilon/2t)^k}{\epsilon}. \end{aligned}$$

[7] From this section through § 4.7, the mathematical notation grows a little thick. There is no helping this. The reader is advised to tread through these sections line by stubborn line, in the good trust that the math thus gained will prove both interesting and useful.

Applying (4.14), this is

$$f'(t) = \sum_{k=-\infty}^{\infty} \lim_{\epsilon \to 0^+} c_k t^k \frac{(1 + k\epsilon/2t) - (1 - k\epsilon/2t)}{\epsilon},$$

which simplifies to

$$f'(t) = \sum_{k=-\infty}^{\infty} c_k k t^{k-1}. \tag{4.17}$$

Equation (4.17) gives the general derivative of the power series.[8]

### 4.4.2 The Leibnitz notation

The $f'(t)$ notation used above for the derivative is due to Sir Isaac Newton, and is easier to start with. Usually better on the whole, however, is G.W. Leibnitz's notation[9]

$$\begin{aligned} dt &= \epsilon, \\ df &= f(t + dt/2) - f(t - dt/2), \end{aligned}$$

such that per (4.15),

$$f'(t) = \frac{df}{dt}. \tag{4.18}$$

Here $dt$ is the infinitesimal, and $df$ is a dependent infinitesimal whose size *relative to* $dt$ depends on the independent variable $t$. For the independent infinitesimal $dt$, conceptually, one can choose any infinitesimal size $\epsilon$. Usually the exact choice of size does not matter, but occasionally when there are two independent variables it helps the analysis to adjust the size of one of the independent infinitesimals with respect to the other.

The meaning of the symbol $d$ unfortunately depends on the context. In (4.18), the meaning is clear enough: $d(\cdot)$ signifies how much $(\cdot)$ changes

[8] Equation (4.17) admittedly has not explicitly considered what happens when the real $t$ becomes the complex $z$, but § 4.4.3 will remedy the oversight.

[9] This subsection is likely to confuse many readers the first time they read it. The reason is that Leibnitz elements like $dt$ and $\partial f$ usually tend to appear in practice in certain specific relations to one another, like $\partial f / \partial z$. As a result, many users of applied mathematics have never developed a clear understanding as to precisely what the individual symbols mean. Often they have developed positive misunderstandings. Because there is significant practical benefit in learning how to handle the Leibnitz notation correctly—particularly in applied complex variable theory—this subsection seeks to present each Leibnitz element in its correct light.

when the independent variable $t$ increments by $dt$.[10] Notice, however, that the notation $dt$ itself has two distinct meanings:[11]

- the independent infinitesimal $dt = \epsilon$; and

[10] If you do not fully understand this sentence, reread it carefully with reference to (4.15) and (4.18) until you do; it's important.

[11] This is difficult, yet the author can think of no clearer, more concise way to state it. The quantities $dt$ and $df$ represent coordinated infinitesimal changes in $t$ and $f$ respectively, so there is usually no trouble with treating $dt$ and $df$ as though they were the same kind of thing. However, at the fundamental level they really aren't.

If $t$ is an independent variable, then $dt$ is just an infinitesimal of some kind, whose specific size could be a function of $t$ but more likely is just a constant. If a constant, then $dt$ does not fundamentally have anything to do with $t$ as such. In fact, if $s$ and $t$ are both independent variables, then we can (and in complex analysis sometimes do) say that $ds = dt = \epsilon$, after which nothing prevents us from using the symbols $ds$ and $dt$ interchangeably. Maybe it would be clearer in some cases to write $\epsilon$ instead of $dt$, but the latter is how it is conventionally written.

By contrast, if $f$ is a dependent variable, then $df$ or $d(f)$ is the amount by which $f$ changes as $t$ changes by $dt$. The $df$ is infinitesimal but not constant; it is a function of $t$. Maybe it would be clearer in some cases to write $d_t f$ instead of $df$, but for most cases the former notation is unnecessarily cluttered; the latter is how it is conventionally written.

Now, most of the time, what we are interested in is not $dt$ or $df$ as such, but rather the ratio $df/dt$ or the sum $\sum_k f(k\,dt)\,dt = \int f(t)\,dt$. For this reason, we do not usually worry about which of $df$ and $dt$ is the independent infinitesimal, nor do we usually worry about the precise value of $dt$. This leads one to forget that $dt$ does indeed have a precise value. What confuses is when one changes perspective in mid-analysis, now regarding $f$ as the independent variable. Changing perspective is allowed and perfectly proper, but one must take care: the $dt$ and $df$ after the change are not the same as the $dt$ and $df$ before the change. However, the ratio $df/dt$ remains the same in any case.

Sometimes when writing a differential equation like the potential-kinetic energy equation $ma\,dx = mv\,dv$, we do not necessarily have either $v$ or $x$ in mind as the independent variable. This is fine. The important point is that $dv$ and $dx$ be coordinated so that the ratio $dv/dx$ has a definite value no matter which of the two be regarded as independent, or whether the independent be some third variable (like $t$) not in the equation.

One can avoid the confusion simply by keeping the $dv/dx$ or $df/dt$ always in ratio, never treating the infinitesimals individually. Many applied mathematicians do precisely that. That is okay as far as it goes, but it really denies the entire point of the Leibnitz notation. One might as well just stay with the Newton notation in that case. Instead, this writer recommends that you learn the Leibnitz notation properly, developing the ability to treat the infinitesimals individually.

Because the book is a book of applied mathematics, this footnote does not attempt to say everything there is to say about infinitesimals. For instance, it has not yet pointed out (but does so now) that even if $s$ and $t$ are equally independent variables, one can have $dt = \epsilon(t)$, $ds = \delta(s, t)$, such that $dt$ has prior independence to $ds$. The point is not to fathom all the possible implications from the start; you can do that as the need arises. The point is to develop a clear picture in your mind of what a Leibnitz infinitesimal really is. Once you have the picture, you can go from there.

- $d(t)$, which is how much $(t)$ changes as $t$ increments by $dt$.

At first glance, the distinction between $dt$ and $d(t)$ seems a distinction without a difference; and for most practical cases of interest, so indeed it is. However, when switching perspective in mid-analysis as to which variables are dependent and which are independent, or when changing multiple independent complex variables simultaneously, the math can get a little tricky. In such cases, it may be wise to use the symbol $dt$ to mean $d(t)$ only, introducing some unambiguous symbol like $\epsilon$ to represent the independent infinitesimal. In any case you should appreciate the conceptual difference between $dt = \epsilon$ and $d(t)$, both of which nonetheless normally are written $dt$.

Where two or more independent variables are at work in the same equation, it is conventional to use the symbol $\partial$ instead of $d$, as a reminder that the reader needs to pay attention to which $\partial$ tracks which independent variable.[12] A derivative $\partial f/\partial t$ or $\partial f/\partial s$ in this case is sometimes called by the slightly misleading name of *partial derivative*. (If needed or desired, one can write $\partial_t f$ when tracking $t$, $\partial_s f$ when tracking $s$, etc. Use discretion, though. Such notation appears only rarely in the literature, so your audience might not understand it when you write it.) Conventional shorthand for $d(df)$ is $d^2 f$; for $(dt)^2$, $dt^2$; so

$$\frac{d(df/dt)}{dt} = \frac{d^2 f}{dt^2}$$

is a derivative of a derivative, or *second derivative*. By extension, the notation

$$\frac{d^k f}{dt^k}$$

represents the $k$th derivative.

### 4.4.3 The derivative of a function of a complex variable

For (4.15) to be robust, written here in the slightly more general form

$$\frac{df}{dz} = \lim_{\epsilon \to 0} \frac{f(z + \epsilon/2) - f(z - \epsilon/2)}{\epsilon}, \tag{4.19}$$

one should like it to evaluate the same in the limit regardless of the complex phase of $\epsilon$. That is, if $\delta$ is a positive real infinitesimal, then it should be equally valid to let $\epsilon = \delta$, $\epsilon = -\delta$, $\epsilon = i\delta$, $\epsilon = -i\delta$, $\epsilon = (4 - i3)\delta$ or any other infinitesimal value, so long as $0 < |\epsilon| \ll 1$. One should like the

[12] The writer confesses that he remains unsure why this minor distinction merits the separate symbol $\partial$, but he accepts the notation as conventional nevertheless.

derivative (4.19) to come out the same regardless of the Argand direction from which $\epsilon$ approaches 0 (see Fig. 2.5). In fact for the sake of robustness, one normally demands that derivatives do come out the same regardless of the Argand direction; and (4.19) rather than (4.15) is the definition we normally use for the derivative for this reason. Where the limit (4.19) is sensitive to the Argand direction or complex phase of $\epsilon$, there we normally say that the derivative does not exist.

Where the derivative (4.19) does exist—where the derivative is finite and insensitive to Argand direction—there we say that the function $f(z)$ is *differentiable.*[13]

Excepting the nonanalytic parts of complex numbers ($|\cdot|$, $\arg[\cdot]$, $[\cdot]^*$, $\Re[\cdot]$ and $\Im[\cdot]$; see § 2.12.3), plus the Heaviside unit step $u(t)$ and the Dirac delta $\delta(t)$ (§ 7.7), most functions encountered in applications do meet the criterion (4.19) except at isolated nonanalytic points (like $z = 0$ in $h[z] = 1/z$ or $g[z] = \sqrt{z}$). Meeting the criterion, such functions are fully differentiable except at their poles (where the derivative goes infinite in any case) and other nonanalytic points. Particularly, the key formula (4.14), written here as

$$(1 + \epsilon)^w \approx 1 + w\epsilon,$$

works without modification when $\epsilon$ is complex; so the derivative (4.17) of the general power series,

$$\frac{d}{dz} \sum_{k=-\infty}^{\infty} c_k z^k = \sum_{k=-\infty}^{\infty} c_k k z^{k-1} \tag{4.20}$$

[13]The unbalanced definition of the derivative from § 4.4, whose complex form is

$$\frac{df}{dz} = \lim_{\epsilon \to 0} \frac{f(z + \epsilon) - f(z)}{\epsilon},$$

does not always serve applications as well as does the balanced definition (4.19) this book prefers. Professional mathematicians have different needs, though. They seem to prefer the unbalanced nonetheless.

In the professionals' favor, one acknowledges that the balanced definition strictly misjudges the modulus function $f(z) = |z|$ to be differentiable solely at the point $z = 0$, whereas that the unbalanced definition, probably more sensibly, judges the modulus to be differentiable nowhere—though the writer is familiar with no significant applied-mathematical implication of the distinction. (Would it coördinate the two definitions to insist that a derivative exist not only at a point but everywhere in the point's immediate, complex neighborhood? The writer does not know. It is a question for the professionals.) Scientists and engineers tend to prefer the balanced definition among other reasons because it more reliably approximates the derivative of a function for which only discrete samples are available [19, §§ I:9.6 and I:9.7]. Moreover, for this writer at least the balanced definition just better captures the subjective sense of the thing.

holds equally well for complex $z$ as for real.

### 4.4.4 The derivative of $z^a$

Inspection of § 4.4.1's logic in light of (4.14) reveals that nothing prevents us from replacing the real $t$, real $\epsilon$ and integral $k$ of that section with arbitrary complex $z$, $\epsilon$ and $a$. That is,

$$\begin{aligned}\frac{d(z^a)}{dz} &= \lim_{\epsilon\to 0}\frac{(z+\epsilon/2)^a-(z-\epsilon/2)^a}{\epsilon}\\ &= \lim_{\epsilon\to 0} z^a\frac{(1+\epsilon/2z)^a-(1-\epsilon/2z)^a}{\epsilon}\\ &= \lim_{\epsilon\to 0} z^a\frac{(1+a\epsilon/2z)-(1-a\epsilon/2z)}{\epsilon},\end{aligned}$$

which simplifies to

$$\frac{d(z^a)}{dz} = az^{a-1} \tag{4.21}$$

for any complex $z$ and $a$.

How exactly to evaluate $z^a$ or $z^{a-1}$ when $a$ is complex is another matter, treated in § 5.4 and its (5.13); but in any case you can use (4.21) for real $a$ right now.

### 4.4.5 The logarithmic derivative

Sometimes one is more interested in knowing the rate of $f(t)$ *relative to the value of* $f(t)$ than in knowing the absolute rate itself. For example, if you inform me that you earn \$ 1000 a year on a bond you hold, then I may commend you vaguely for your thrift but otherwise the information does not tell me much. However, if you inform me instead that you earn 10 percent a year on the same bond, then I might want to invest. The latter figure is a relative rate, or *logarithmic derivative,*

$$\frac{df/dt}{f(t)} = \frac{d}{dt}\ln f(t). \tag{4.22}$$

The investment principal grows at the absolute rate $df/dt$, but the bond's interest rate is $(df/dt)/f(t)$.

The natural logarithmic notation $\ln f(t)$ may not mean much to you yet, as we'll not introduce it formally until § 5.2, so you can ignore the right side of (4.22) for the moment; but the equation's left side at least should make sense to you. It expresses the significant concept of a relative rate, like 10 percent annual interest on a bond.

## 4.5 Basic manipulation of the derivative

This section introduces the derivative chain and product rules.

### 4.5.1 The derivative chain rule

If $f$ is a function of $w$, which itself is a function of $z$, then[14]

$$\frac{df}{dz} = \left(\frac{df}{dw}\right)\left(\frac{dw}{dz}\right). \tag{4.23}$$

Equation (4.23) is the *derivative chain rule.*[15]

### 4.5.2 The derivative product rule

In general per (4.19),

$$d\left[\prod_j f_j(z)\right] = \prod_j f_j\left(z + \frac{dz}{2}\right) - \prod_j f_j\left(z - \frac{dz}{2}\right).$$

But to first order,

$$f_j\left(z \pm \frac{dz}{2}\right) \approx f_j(z) \pm \left(\frac{df_j}{dz}\right)\left(\frac{dz}{2}\right) = f_j(z) \pm \frac{df_j}{2};$$

[14]For example, one can rewrite

$$f(z) = \sqrt{3z^2 - 1}$$

in the form

$$\begin{aligned} f(w) &= w^{1/2}, \\ w(z) &= 3z^2 - 1. \end{aligned}$$

Then

$$\begin{aligned} \frac{df}{dw} &= \frac{1}{2w^{1/2}} = \frac{1}{2\sqrt{3z^2 - 1}}, \\ \frac{dw}{dz} &= 6z, \end{aligned}$$

so by (4.23),

$$\frac{df}{dz} = \left(\frac{df}{dw}\right)\left(\frac{dw}{dz}\right) = \frac{6z}{2\sqrt{3z^2 - 1}} = \frac{3z}{\sqrt{3z^2 - 1}}.$$

[15]It bears emphasizing to readers who may inadvertently have picked up unhelpful ideas about the Leibnitz notation in the past: the $dw$ factor in the denominator cancels the $dw$ factor in the numerator; a thing divided by itself is 1. That's it. There is nothing more to the proof of the derivative chain rule than that.

so, in the limit,

$$d\left[\prod_j f_j(z)\right] = \prod_j \left(f_j(z) + \frac{df_j}{2}\right) - \prod_j \left(f_j(z) - \frac{df_j}{2}\right).$$

Since the product of two or more $df_j$ is negligible compared to the first-order infinitesimals to which they are added here, this simplifies to

$$d\left[\prod_j f_j(z)\right] = \left[\prod_j f_j(z)\right]\left[\sum_k \frac{df_k}{2f_k(z)}\right] - \left[\prod_j f_j(z)\right]\left[\sum_k \frac{-df_k}{2f_k(z)}\right],$$

or in other words

$$d\prod_j f_j = \left[\prod_j f_j\right]\left[\sum_k \frac{df_k}{f_k}\right]. \tag{4.24}$$

In the common case of only two $f_j$, this comes to

$$d(f_1 f_2) = f_2\, df_1 + f_1\, df_2. \tag{4.25}$$

On the other hand, if $f_1(z) = f(z)$ and $f_2(z) = 1/g(z)$, then by the derivative chain rule (4.23), $df_2 = -dg/g^2$; so,

$$d\left(\frac{f}{g}\right) = \frac{g\, df - f\, dg}{g^2}. \tag{4.26}$$

Equation (4.24) is the *derivative product rule.*

After studying the complex exponential in Ch. 5, we shall stand in a position to write (4.24) in the slightly specialized but often useful form[16]

$$\begin{aligned} d&\left[\prod_j g_j^{a_j} \prod_j e^{b_j h_j} \prod_j \ln c_j p_j\right] \\ &= \left[\prod_j g_j^{a_j} \prod_j e^{b_j h_j} \prod_j \ln c_j p_j\right] \\ &\quad \times \left[\sum_k a_k \frac{dg_k}{g_k} + \sum_k b_k\, dh_k + \sum_k \frac{dp_k}{p_k \ln c_k p_k}\right]. \end{aligned} \tag{4.27}$$

[16]This paragraph is extra. You can skip it for now if you prefer.

where the $a_k$, $b_k$ and $c_k$ are arbitrary complex coefficients and the $g_k$, $h_k$ and $p_k$ are arbitrary functions.[17]

### 4.5.3 A derivative product pattern

According to (4.25) and (4.21), the derivative of the product $z^a f(z)$ with respect to its independent variable $z$ is

$$\frac{d}{dz}[z^a f(z)] = z^a \frac{df}{dz} + a z^{a-1} f(z).$$

Swapping the equation's left and right sides then dividing through by $z^a$ yields

$$\frac{df}{dz} + a\frac{f}{z} = \frac{d(z^a f)}{z^a\, dz}, \tag{4.28}$$

a pattern worth committing to memory, emerging among other places in § 16.9.

## 4.6 Extrema and higher derivatives

One problem which arises very frequently in applied mathematics is the problem of finding a local *extremum*—that is, a local minimum or maximum—of a real-valued function $f(x)$. Refer to Fig. 4.3. The almost distinctive characteristic of the extremum $f(x_o)$ is that[18]

$$\left.\frac{df}{dx}\right|_{x=x_o} = 0. \tag{4.29}$$

At the extremum, the slope is zero. The curve momentarily runs level there. One solves (4.29) to find the extremum.

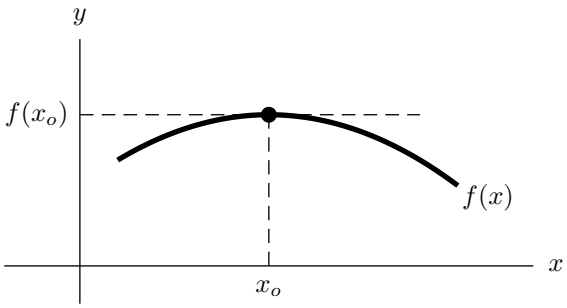
Whether the extremum be a minimum or a maximum depends on whether the curve turn from a downward slope to an upward, or from an upward slope to a downward, respectively. If from downward to upward, then the derivative of the slope is evidently positive; if from upward to downward,

[17] The subsection is sufficiently abstract that it is a little hard to understand unless one already knows what it means. An example may help:

$$d\left[\frac{u^2v^3}{z}e^{-5t}\ln 7s\right] = \left[\frac{u^2v^3}{z}e^{-5t}\ln 7s\right]\left[2\frac{du}{u} + 3\frac{dv}{v} - \frac{dz}{z} - 5\,dt + \frac{ds}{s\ln 7s}\right].$$

[18] The notation $P|_Q$ means "$P$ when $Q$," "$P$, given $Q$," or "$P$ evaluated at $Q$." Sometimes it is alternately written $P|Q$ or $[P]_Q$.

Figure 4.3: A local extremum.




then negative. But the derivative of the slope is just the derivative of the derivative, or second derivative. Hence if $df/dx = 0$ at $x = x_o$, then

$$\begin{aligned} \left.\frac{d^2f}{dx^2}\right|_{x=x_o} &> 0 \text{ implies a local minimum at } x_o; \\ \left.\frac{d^2f}{dx^2}\right|_{x=x_o} &< 0 \text{ implies a local maximum at } x_o. \end{aligned}$$

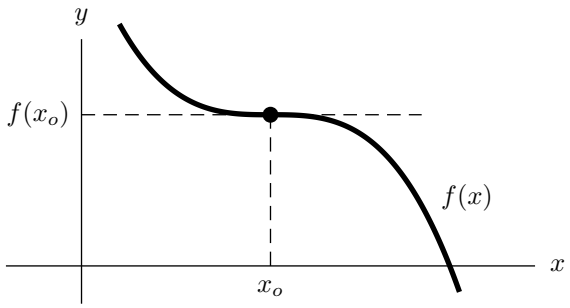
Regarding the case

$$\left.\frac{d^2f}{dx^2}\right|_{x=x_o} = 0,$$

this might be either a minimum or a maximum but more probably is neither, being rather a *level inflection point* as depicted in Fig. 4.4.[19] (In general the term *inflection point* signifies a point at which the second derivative is zero. The inflection point of Fig. 4.4 is *level* because its first derivative is zero, too.)

[19] Of course if the first and second derivatives are zero not just at $x = x_o$ but everywhere, then $f(x) = y_o$ is just a level straight line, but you knew that already. Whether one chooses to call some random point on a level straight line an inflection point or an extremum, or both or neither, would be a matter of definition, best established not by prescription but rather by the needs of the model at hand.

Figure 4.4: A level inflection.




## 4.7 L'Hôpital's rule

If $z = z_o$ is a root of both $f(z)$ and $g(z)$, or alternately if $z = z_o$ is a pole of both functions—that is, if both functions go to zero or infinity together at $z = z_o$—then *l'Hôpital's rule* holds that

$$\lim_{z \to z_o} \frac{f(z)}{g(z)} = \left. \frac{df/dz}{dg/dz} \right|_{z=z_o}. \tag{4.30}$$

In the case where $z = z_o$ is a root, l'Hôpital's rule is proved by reasoning[20]

$$\begin{aligned} \lim_{z \to z_o} \frac{f(z)}{g(z)} &= \lim_{z \to z_o} \frac{f(z) - 0}{g(z) - 0} \\ &= \lim_{z \to z_o} \frac{f(z) - f(z_o)}{g(z) - g(z_o)} = \lim_{z \to z_o} \frac{df}{dg} = \lim_{z \to z_o} \frac{df/dz}{dg/dz}. \end{aligned}$$

In the case where $z = z_o$ is a pole, new functions $F(z) \equiv 1/f(z)$ and $G(z) \equiv 1/g(z)$ of which $z = z_o$ is a root are defined, with which

$$\lim_{z \to z_o} \frac{f(z)}{g(z)} = \lim_{z \to z_o} \frac{G(z)}{F(z)} = \lim_{z \to z_o} \frac{dG}{dF} = \lim_{z \to z_o} \frac{-dg/g^2}{-df/f^2},$$

where we have used the fact from (4.21) that $d(1/u) = -du/u^2$ for any $u$. Canceling the minus signs and multiplying by $g^2/f^2$, we have that

$$\lim_{z \to z_o} \frac{g(z)}{f(z)} = \lim_{z \to z_o} \frac{dg}{df}.$$

[20] Partly with reference to [66, "L'Hopital's rule," 03:40, 5 April 2006].

Inverting,

$$\lim_{z\to z_o} \frac{f(z)}{g(z)} = \lim_{z\to z_o} \frac{df}{dg} = \lim_{z\to z_o} \frac{df/dz}{dg/dz}.$$

And if $z_o$ itself is infinite? Then, whether it represents a root or a pole, we define the new variable $Z = 1/z$ and the new functions $\Phi(Z) = f(1/Z) = f(z)$ and $\Gamma(Z) = g(1/Z) = g(z)$, with which we apply l'Hôpital's rule for $Z \to 0$ to obtain

$$\begin{aligned}\lim_{z\to\infty} \frac{f(z)}{g(z)} &= \lim_{Z\to 0} \frac{\Phi(Z)}{\Gamma(Z)} = \lim_{Z\to 0} \frac{d\Phi/dZ}{d\Gamma/dZ} = \lim_{Z\to 0} \frac{df/dZ}{dg/dZ} \\ &= \lim_{\substack{z\to\infty,\\ Z\to 0}} \frac{(df/dz)(dz/dZ)}{(dg/dz)(dz/dZ)} = \lim_{z\to\infty} \frac{(df/dz)(-z^2)}{(dg/dz)(-z^2)} = \lim_{z\to\infty} \frac{df/dz}{dg/dz}.\end{aligned}$$

Nothing in the derivation requires that $z$ or $z_o$ be real. Nothing prevents one from applying l'Hôpital's rule recursively, should the occasion arise.[21]

L'Hôpital's rule is used in evaluating indeterminate forms of the kinds $0/0$ and $\infty/\infty$, plus related forms like $(0)(\infty)$ which can be recast into either of the two main forms. Good examples of the use require math from Ch. 5 and later, but if we may borrow from (5.8) the natural logarithmic function and its derivative,[22]

$$\frac{d}{dx} \ln x = \frac{1}{x},$$

then a typical l'Hôpital example is[23]

$$\lim_{x\to\infty} \frac{\ln x}{\sqrt{x}} = \lim_{x\to\infty} \frac{1/x}{1/2\sqrt{x}} = \lim_{x\to\infty} \frac{2}{\sqrt{x}} = 0.$$

The example incidentally shows that natural logarithms grow slower than square roots, an instance of a more general principle we shall meet in § 5.3.

Section 5.3 will put l'Hôpital's rule to work.

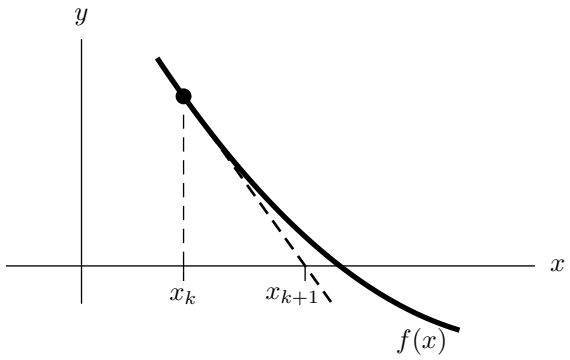
[21] Consider for example the ratio $\lim_{x\to 0}(x^3 + x)^2/x^2$, which is $0/0$. The easier way to resolve this particular ratio would naturally be to cancel a factor of $x^2$ from it; but just to make the point let us apply l'Hôpital's rule instead, reducing the ratio to $\lim_{x\to 0} 2(x^3 + x)(3x^2 + 1)/2x$, which is still $0/0$. Applying l'Hôpital's rule again to the result yields $\lim_{x\to 0} 2[(3x^2+1)^2+(x^3+x)(6x)]/2 = 2/2 = 1$. Where expressions involving trigonometric or special functions (Chs. 3, 5 and [not yet written]) appear in ratio, a recursive application of l'Hôpital's rule can be just the thing one needs.

Observe that one must stop applying l'Hôpital's rule once the ratio is no longer $0/0$ or $\infty/\infty$. In the example, applying the rule a third time would have ruined the result.

[22] This paragraph is optional reading for the moment. You can read Ch. 5 first, then come back here and read the paragraph if you prefer.

[23] [55, § 10-2]

Figure 4.5: The Newton-Raphson iteration.




## 4.8 The Newton-Raphson iteration

The *Newton-Raphson iteration* is a powerful, fast converging, broadly applicable method for finding roots numerically. Given a function $f(z)$ of which the root is desired, the Newton-Raphson iteration is

$$z_{k+1} = z - \left.\frac{f(z)}{\frac{d}{dz}f(z)}\right|_{z=z_k}. \tag{4.31}$$

One begins the iteration by guessing the root and calling the guess $z_0$. Then $z_1$, $z_2$, $z_3$, etc., calculated in turn by the iteration (4.31), give successively better estimates of the true root $z_\infty$.

To understand the Newton-Raphson iteration, consider the function $y = f(x)$ of Fig 4.5. The iteration approximates the curve $f(x)$ by its tangent line[24] (shown as the dashed line in the figure):

$$\tilde{f}_k(x) = f(x_k) + \left[\frac{d}{dx}f(x)\right]_{x=x_k}(x - x_k).$$

[24] A *tangent* line, also just called a *tangent,* is the line which most nearly approximates a curve at a given point. The tangent touches the curve at the point, and in the neighborhood of the point it goes in the same direction the curve goes. The dashed line in Fig. 4.5 is a good example of a tangent line.

The relationship between the tangent line and the trigonometric tangent function of Ch. 3 is slightly obscure, maybe more of linguistic interest than of mathematical. The trigonometric tangent function is named from a variation on Fig. 3.1 in which the triangle's bottom leg is extended to unit length, leaving the rightward leg tangent to the circle.

It then approximates the root $x_{k+1}$ as the point at which $\tilde{f}_k(x_{k+1}) = 0$:

$$\tilde{f}_k(x_{k+1}) = 0 = f(x_k) + \left[\frac{d}{dx}f(x)\right]_{x=x_k}(x_{k+1} - x_k).$$

Solving for $x_{k+1}$, we have that

$$x_{k+1} = x - \left.\frac{f(x)}{\frac{d}{dx}f(x)}\right|_{x=x_k},$$

which is (4.31) with $x \leftarrow z$.

Although the illustration uses real numbers, nothing forbids complex $z$ and $f(z)$. The Newton-Raphson iteration works just as well for these.

The principal limitation of the Newton-Raphson arises when the function has more than one root, as most interesting functions do. The iteration often converges on the root nearest the initial guess $z_o$ but does not always, and in any case there is no guarantee that the root it finds is the one you wanted. The most straightforward way to beat this problem is to find *all* the roots: first you find some root $\alpha$, then you remove that root (without affecting any of the other roots) by dividing $f(z)/(z-\alpha)$, then you find the next root by iterating on the new function $f(z)/(z-\alpha)$, and so on until you have found all the roots. If this procedure is not practical (perhaps because the function has a large or infinite number of roots), then you should probably take care to make a sufficiently accurate initial guess if you can.

A second limitation of the Newton-Raphson is that, if you happen to guess $z_0$ especially unfortunately, then the iteration might never converge at all. For example, the roots of $f(z) = z^2 + 2$ are $z = \pm i\sqrt{2}$, but if you guess that $z_0 = 1$ then the iteration has no way to leave the real number line, so it never converges[25] (and if you guess that $z_0 = \sqrt{2}$—well, try it with your pencil and see what $z_2$ comes out to be). You can fix the problem with a different, possibly complex initial guess.

A third limitation arises where there is a multiple root. In this case, the Newton-Raphson normally still converges, but relatively slowly. For instance, the Newton-Raphson converges relatively slowly on the triple root of $f(z) = z^3$. However, even the relatively slow convergence is still pretty fast and is usually adequate, even for calculations by hand.

Usually in practice, the Newton-Raphson iteration works very well. For most functions, once the Newton-Raphson finds the root's neighborhood, it

[25] It is entertaining to try this on a computer. Then try again with $z_0 = 1 + i2^{-0\text{x}10}$.

converges on the actual root remarkably quickly. Figure 4.5 shows why: in the neighborhood, the curve hardly departs from the straight line.

The Newton-Raphson iteration is a champion square root calculator, incidentally. Consider

$$f(x) = x^2 - p,$$

whose roots are

$$x = \pm\sqrt{p}.$$

Per (4.31), the Newton-Raphson iteration for this is

$$x_{k+1} = \frac{1}{2}\left[x_k + \frac{p}{x_k}\right]. \tag{4.32}$$

If you start by guessing

$$x_0 = 1$$

and iterate several times, the iteration (4.32) converges on $x_\infty = \sqrt{p}$ fast. To calculate the $n$th root $x = p^{1/n}$, let

$$f(x) = x^n - p$$

and iterate[26,27]

$$x_{k+1} = \frac{1}{n}\left[(n-1)x_k + \frac{p}{x_k^{n-1}}\right]. \tag{4.33}$$

Section 13.7 generalizes the Newton-Raphson iteration to handle vector-valued functions.

This concludes the chapter. Chapter 8, treating the Taylor series, will continue the general discussion of the derivative.

[26] Equations (4.32) and (4.33) work not only for real $p$ but also usually for complex. Given $x_0 = 1$, however, they converge reliably and orderly only for real, nonnegative $p$. (To see why, sketch $f[x]$ in the fashion of Fig. 4.5.)

If reliable, orderly convergence is needed for complex $p = u + iv = \sigma \operatorname{cis}\psi$, $\sigma \geq 0$, you can decompose $p^{1/n}$ per de Moivre's theorem (3.28) as $p^{1/n} = \sigma^{1/n}\operatorname{cis}(\psi/n)$, in which $\operatorname{cis}(\psi/n) = \cos(\psi/n) + i\sin(\psi/n)$ is calculated by the Taylor series of Table 8.1. Then $\sigma$ is real and nonnegative, upon which (4.33) reliably, orderly computes $\sigma^{1/n}$.

The Newton-Raphson iteration however excels as a *practical* root-finding technique, so it often pays to be a little less theoretically rigid in applying it. If so, then don't bother to decompose; seek $p^{1/n}$ directly, using complex $z_k$ in place of the real $x_k$. In the uncommon event that the direct iteration does not seem to converge, start over again with some randomly chosen complex $z_0$. This saves effort and usually works.

[27] [55, § 4-9][45, § 6.1.1][65]

# Chapter 5

# The complex exponential

The complex natural exponential is ubiquitous in higher mathematics. There seems hardly a corner of calculus, basic or advanced, in which the complex exponential does not strongly impress itself and frequently arise. Because the complex natural exponential emerges (at least pedagogically) out of the real natural exponential, this chapter introduces first the real natural exponential and its inverse, the real natural logarithm; and then proceeds to show how the two can operate on complex arguments. It derives the exponential and logarithmic functions' basic properties and explains their close relationship to the trigonometrics. It works out the functions' derivatives and the derivatives of the basic members of the trigonometric and inverse trigonometric families to which they respectively belong.

## 5.1 The real exponential

Consider the factor

$$(1 + \epsilon)^N.$$

This is the overall factor by which a quantity grows after $N$ iterative rounds of multiplication by $(1 + \epsilon)$. What happens when $\epsilon$ is very small but $N$ is very large? The really interesting question is, what happens in the limit, as $\epsilon \to 0$ and $N \to \infty$, while $x = \epsilon N$ remains a finite number? The answer is that the factor becomes

$$\exp x \equiv \lim_{\epsilon \to 0} (1 + \epsilon)^{x/\epsilon}. \tag{5.1}$$

Equation (5.1) defines the *natural exponential function*—commonly, more briefly named the *exponential function.* Another way to write the same

definition is

$$\exp x = e^x, \tag{5.2}$$
$$e \equiv \lim_{\epsilon \to 0} (1+\epsilon)^{1/\epsilon}. \tag{5.3}$$

Whichever form we write it in, the question remains as to whether the limit actually exists; that is, whether $0 < e < \infty$; whether in fact we can put some concrete bound on $e$. To show that we can,[1] we observe per (4.19) and (4.13) that the derivative of the exponential function is

$$\begin{aligned}
\frac{d}{dx}\exp x &= \lim_{\delta \to 0} \frac{\exp(x+\delta/2) - \exp(x-\delta/2)}{\delta} \\
&= \lim_{\delta,\epsilon \to 0} \frac{(1+\epsilon)^{(x+\delta/2)/\epsilon} - (1+\epsilon)^{(x-\delta/2)/\epsilon}}{\delta} \\
&= \lim_{\delta,\epsilon \to 0} (1+\epsilon)^{x/\epsilon} \frac{(1+\epsilon)^{+\delta/2\epsilon} - (1+\epsilon)^{-\delta/2\epsilon}}{\delta} \\
&= \lim_{\delta,\epsilon \to 0} (1+\epsilon)^{x/\epsilon} \frac{(1+\delta/2) - (1-\delta/2)}{\delta} \\
&= \lim_{\epsilon \to 0} (1+\epsilon)^{x/\epsilon},
\end{aligned}$$

which is to say that

$$\frac{d}{dx}\exp x = \exp x. \tag{5.4}$$

This is a curious, important result: the derivative of the exponential function is the exponential function itself; the slope and height of the exponential function are everywhere equal. For the moment, however, what interests us is that

$$\frac{d}{dx}\exp 0 = \exp 0 = \lim_{\epsilon \to 0} (1+\epsilon)^0 = 1,$$

which says that the slope and height of the exponential function are both unity at $x = 0$, implying that the straight line which best approximates the exponential function in that neighborhood—the *tangent line,* which just grazes the curve—is

$$y(x) = 1 + x.$$

With the tangent line $y(x)$ found, the next step toward putting a concrete bound on $e$ is to show that $y(x) \leq \exp x$ for all real $x$, that the curve runs

[1] Excepting (5.4), the author would prefer to omit much of the rest of this section, but even at the applied level cannot think of a logically permissible way to do it. It seems nonobvious that the limit $\lim_{\epsilon \to 0}(1+\epsilon)^{1/\epsilon}$ actually does exist. The rest of this section shows why it does.

nowhere below the line. To show this, we observe per (5.1) that the essential action of the exponential function is to multiply repeatedly by $1 + \epsilon$ as $x$ increases, to divide repeatedly by $1 + \epsilon$ as $x$ decreases. Since $1 + \epsilon > 1$, this action means for real $x$ that

$$\exp x_1 \leq \exp x_2 \quad \text{if} \quad x_1 \leq x_2.$$

However, a positive number remains positive no matter how many times one multiplies or divides it by $1 + \epsilon$, so the same action also means that

$$0 \leq \exp x$$

for all real $x$. In light of (5.4), the last two equations imply further that

$$\begin{aligned} \frac{d}{dx} \exp x_1 &\leq \frac{d}{dx} \exp x_2 \quad \text{if } x_1 \leq x_2, \\ 0 &\leq \frac{d}{dx} \exp x. \end{aligned}$$

But we have purposely defined the tangent line $y(x) = 1 + x$ such that

$$\begin{aligned} \exp 0 &= y(0) = 1, \\ \frac{d}{dx} \exp 0 &= \frac{d}{dx} y(0) = 1; \end{aligned}$$

that is, such that the line just grazes the curve of $\exp x$ at $x = 0$. Rightward, at $x > 0$, evidently the curve's slope only increases, bending upward away from the line. Leftward, at $x < 0$, evidently the curve's slope only decreases, again bending upward away from the line. Either way, the curve never crosses below the line for real $x$. In symbols,

$$y(x) \leq \exp x.$$

Figure 5.1 depicts.

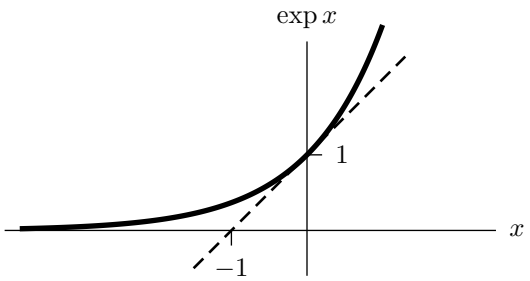
Evaluating the last inequality at $x = -1/2$ and $x = 1$, we have that

$$\begin{aligned} \frac{1}{2} &\leq \exp\left(-\frac{1}{2}\right), \\ 2 &\leq \exp(1). \end{aligned}$$

But per (5.2) $\exp x = e^x$, so

$$\begin{aligned} \frac{1}{2} &\leq e^{-1/2}, \\ 2 &\leq e^1, \end{aligned}$$

Figure 5.1: The natural exponential.




or in other words,

$$2 \leq e \leq 4, \tag{5.5}$$

which in consideration of (5.2) puts the desired bound on the exponential function. The limit does exist.

Dividing (5.4) by $\exp x$ yields the *logarithmic derivative* (§ 4.4.5)

$$\frac{d(\exp x)}{(\exp x)\,dx} = 1, \tag{5.6}$$

a form which expresses or captures the deep curiosity of the natural exponential maybe even better than does (5.4).

By the Taylor series of Table 8.1, the value

$$e \approx \text{0x2.B7E1}$$

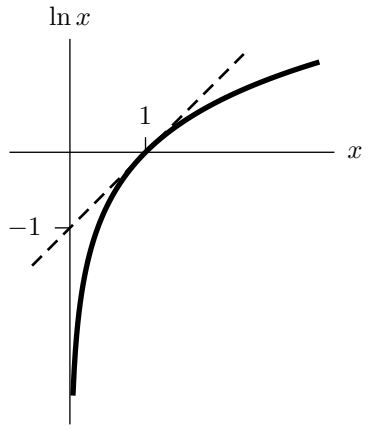
can readily be calculated, but the derivation of that series does not come until Ch. 8.

## 5.2 The natural logarithm

In the general exponential expression $b^x$, one can choose any base $b$; for example, $b = 2$ is an interesting choice. As we shall see in § 5.4, however, it turns out that $b = e$, where $e$ is the constant introduced in (5.3), is the most interesting choice of all. For this reason among others, the base-$e$ logarithm is similarly interesting, such that we define for it the special notation

$$\ln(\cdot) = \log_e(\cdot),$$

Figure 5.2: The natural logarithm.




and call it the *natural logarithm.* Just as for any other base $b$, so also for base $b = e$; thus the natural logarithm inverts the natural exponential and vice versa:

$$\begin{aligned} \ln \exp x &= \ln e^x &&= x, \\ \exp \ln x &= e^{\ln x} &&= x. \end{aligned} \tag{5.7}$$

Figure 5.2 plots the natural logarithm.

If

$$y = \ln x,$$

then

$$x = \exp y,$$

and per (5.4),

$$\frac{dx}{dy} = \exp y.$$

But this means that

$$\frac{dx}{dy} = x,$$

the inverse of which is

$$\frac{dy}{dx} = \frac{1}{x}.$$

In other words,

$$\frac{d}{dx} \ln x = \frac{1}{x}. \tag{5.8}$$

Like many of the equations in these early chapters, here is another rather significant result.[2]

One can specialize Table 2.5's logarithmic base-conversion identity to read

$$\log_b w = \frac{\ln w}{\ln b}. \tag{5.9}$$

This equation converts any logarithm to a natural logarithm. Base $b = 2$ logarithms are interesting, so we note here that

$$\ln 2 = -\ln \frac{1}{2} \approx 0\text{x}0.\text{B}172,$$

which Ch. 8 and its Table 8.1 will show how to calculate.

## 5.3 Fast and slow functions

The exponential $\exp x$ is a *fast function.* The logarithm $\ln x$ is a *slow function.* These functions grow, diverge or decay respectively faster and slower than $x^a$.

Such claims are proved by l'Hôpital's rule (4.30). Applying the rule, we have that

$$\begin{aligned} \lim_{x\to\infty} \frac{\ln x}{x^a} &= \lim_{x\to\infty} \frac{-1}{ax^a} = \begin{cases} 0 & \text{if } a > 0, \\ +\infty & \text{if } a \le 0, \end{cases} \\ \lim_{x\to 0} \frac{\ln x}{x^a} &= \lim_{x\to 0} \frac{-1}{ax^a} = \begin{cases} -\infty & \text{if } a \ge 0, \\ 0 & \text{if } a < 0, \end{cases} \end{aligned} \tag{5.10}$$

which reveals the logarithm to be a slow function. Since the $\exp(\cdot)$ and $\ln(\cdot)$ functions are mutual inverses, we can leverage (5.10) to show also that

$$\begin{aligned} \lim_{x\to\infty} \frac{\exp(\pm x)}{x^a} &= \lim_{x\to\infty} \exp\left[\ln \frac{\exp(\pm x)}{x^a}\right] \\ &= \lim_{x\to\infty} \exp\left[\pm x - a \ln x\right] \\ &= \lim_{x\to\infty} \exp\left[(x)\left(\pm 1 - a\frac{\ln x}{x}\right)\right] \\ &= \lim_{x\to\infty} \exp\left[(x)\left(\pm 1 - 0\right)\right] \\ &= \lim_{x\to\infty} \exp\left[\pm x\right]. \end{aligned}$$

[2]Besides the result itself, the technique which leads to the result is also interesting and is worth mastering. We will use the technique more than once in this book.

That is,

$$\begin{aligned}\lim_{x\to\infty}\frac{\exp(+x)}{x^a}&=\infty,\\ \lim_{x\to\infty}\frac{\exp(-x)}{x^a}&=0,\end{aligned}\tag{5.11}$$

which reveals the exponential to be a fast function. Exponentials grow or decay faster than powers; logarithms diverge slower.

Such conclusions are extended to bases other than the natural base $e$ simply by observing that $\log_b x = \ln x/\ln b$ and that $b^x = \exp(x\ln b)$. Thus exponentials generally are fast and logarithms generally are slow, regardless of the base.[3]

It is interesting and worthwhile to contrast the sequence

$$\ldots, -\frac{3!}{x^4}, \frac{2!}{x^3}, -\frac{1!}{x^2}, \frac{0!}{x^1}, \frac{x^0}{0!}, \frac{x^1}{1!}, \frac{x^2}{2!}, \frac{x^3}{3!}, \frac{x^4}{4!}, \ldots$$

against the sequence

$$\ldots, -\frac{3!}{x^4}, \frac{2!}{x^3}, -\frac{1!}{x^2}, \frac{0!}{x^1}, \ln x, \frac{x^1}{1!}, \frac{x^2}{2!}, \frac{x^3}{3!}, \frac{x^4}{4!}, \ldots$$

As $x \to +\infty$, each sequence increases in magnitude going rightward. Also, each term in each sequence is the derivative with respect to $x$ of the term to its right—except left of the middle element in the first sequence and right of the middle element in the second. The exception is peculiar. What is going on here?

The answer is that $x^0$ (which is just a constant) and $\ln x$ *both are of zeroth order in* $x$. This seems strange at first because $\ln x$ diverges as $x \to \infty$ whereas $x^0$ does not, but the divergence of the former is extremely slow—so slow, in fact, that per (5.10) $\lim_{x\to\infty}(\ln x)/x^\epsilon = 0$ for any positive $\epsilon$ no matter how small.[4] Figure 5.2 has plotted $\ln x$ only for $x \sim 1$, but beyond the figure's window the curve (whose slope is $1/x$) flattens rapidly rightward, to the extent that it locally resembles the plot of a constant value; and indeed one can write

$$x^0 = \lim_{u\to\infty}\frac{\ln(x+u)}{\ln u},$$

[3] There are of course some degenerate edge cases like $b = 0$ and $b = 1$. The reader can detail these as the need arises.

[4] One does not grasp how truly slow the divergence is until one calculates a few concrete values. Consider for instance how far out $x$ must run to make $\ln x = 0\text{x}100$. It's a long, long way. The natural logarithm does indeed eventually diverge to infinity, in the literal sense that there is no height it does not eventually reach, but it certainly does not hurry. As we have seen, it takes practically forever just to reach 0x100.

which casts $x^0$ as a logarithm shifted and scaled. Admittedly, one ought not strain such logic too far, because $\ln x$ is not in fact a constant, but the point nevertheless remains that $x^0$ and $\ln x$ often play analogous roles in mathematics. The logarithm can in some situations profitably be thought of as a "diverging constant" of sorts.

Less strange-seeming perhaps is the consequence of (5.11) that $\exp x$ is of infinite order in $x$, that $x^\infty$ and $\exp x$ play analogous roles.

It befits an applied mathematician subjectively to internalize (5.10) and (5.11), to remember that $\ln x$ resembles $x^0$ and that $\exp x$ resembles $x^\infty$. A qualitative sense that logarithms are slow and exponentials, fast, helps one to grasp mentally the essential features of many mathematical models one encounters in practice.

Now leaving aside fast and slow functions for the moment, we turn our attention in the next section to the highly important matter of the exponential of a complex argument.

## 5.4 Euler's formula

The result of § 5.1 leads to one of the central questions in all of mathematics. How can one evaluate

$$\exp i\theta = \lim_{\epsilon \to 0} (1 + \epsilon)^{i\theta/\epsilon},$$

where $i^2 = -1$ is the imaginary unit introduced in § 2.12?

To begin, one can take advantage of (4.14) to write the last equation in the form

$$\exp i\theta = \lim_{\epsilon \to 0} (1 + i\epsilon)^{\theta/\epsilon},$$

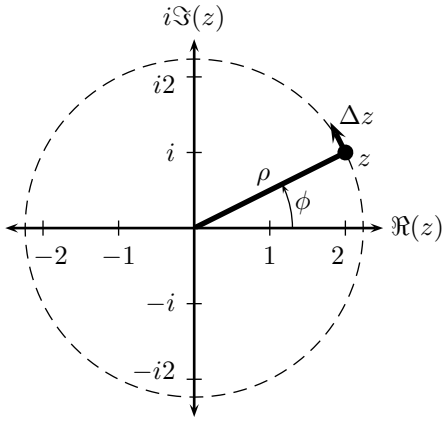
but from here it is not obvious where to go. The book's development up to the present point gives no obvious direction. In fact it appears that the interpretation of $\exp i\theta$ remains for us to define, if we can find a way to define it which fits sensibly with our existing notions of the real exponential. So, if we don't quite know where to go with this yet, what do we know?

One thing we know is that if $\theta = \epsilon$, then

$$\exp(i\epsilon) = (1 + i\epsilon)^{\epsilon/\epsilon} = 1 + i\epsilon.$$

But per § 5.1, the essential operation of the exponential function is to multiply repeatedly by some factor, the factor being not quite exactly unity and,

Figure 5.3: The complex exponential and Euler's formula.




in this case, being $1 + i\epsilon$. With such thoughts in mind, let us multiply a complex number $z = x + iy$ by $1 + i\epsilon$, obtaining

$$(1 + i\epsilon)(x + iy) = (x - \epsilon y) + i(y + \epsilon x).$$

The resulting change in $z$ is

$$\Delta z = (1 + i\epsilon)(x + iy) - (x + iy) = (\epsilon)(-y + ix),$$

in which it is seen that

$$\begin{aligned} |\Delta z| &= (\epsilon)\sqrt{y^2 + x^2} &&= \epsilon\rho, \\ \arg(\Delta z) &= \arctan\frac{x}{-y} &&= \phi + \frac{2\pi}{4}. \end{aligned}$$

The $\Delta z$, $\rho = |z|$ and $\phi = \arg z$ are as shown in Fig. 5.3. Whether in the figure or in the equations, *the change $\Delta z$ is evidently proportional to the magnitude of $z$, but at a right angle to $z$'s radial arm in the complex plane.*

To travel about a circle wants motion always perpendicular to the circle's radial arm, which happens to be just the kind of motion $\Delta z$ represents. Referring to the figure and the last equations, we have then that

$$\begin{aligned} \Delta\rho &\equiv |z + \Delta z| - |z| &&= 0, \\ \Delta\phi &\equiv \arg(z + \Delta z) - \arg z = \frac{|\Delta z|}{\rho} = \frac{\epsilon\rho}{\rho} &&= \epsilon, \end{aligned}$$

which results evidently are valid for infinitesimal $\epsilon \to 0$ and, importantly, stand independently of the value of $\rho$. (But does $\rho$ not grow at least a little, as the last equations almost seem to suggest? The answer is no; or, if you prefer, the answer is that $\Delta\rho \approx \{[\sqrt{1+\epsilon^2}] - 1\}\rho \approx \epsilon^2\rho/2 \approx 0$, a second-order infinitesimal inconsequential on the scale of $\epsilon$ or $\epsilon\rho$, utterly vanishing by comparison in the limit $\epsilon \to 0$.) With such results in hand, now let us recall from earlier in the section that—as we have asserted or defined—

$$\exp i\theta = \lim_{\epsilon \to 0} (1 + i\epsilon)^{\theta/\epsilon},$$

and that this remains so for arbitrary real $\theta$. Yet what does such an equation do, mechanically, but to compute $\exp i\theta$ by multiplying 1 by $1+i\epsilon$ repeatedly, $\theta/\epsilon$ times? The plain answer is that such an equation does precisely this and nothing else. We have recently seen how each multiplication of the kind the equation suggests increments the phase $\phi$ by $\Delta\phi = \epsilon$ while not changing the magnitude $\rho$. Since the phase $\phi$ begins from $\arg 1 = 0$ it must become

$$\phi = \frac{\theta}{\epsilon}\epsilon = \theta$$

after $\theta/\epsilon$ increments of $\epsilon$ each, while the magnitude must remain

$$\rho = 1.$$

Reversing the sequence of the last two equations and recalling that $\rho \equiv |\exp i\theta|$ and that $\phi \equiv \arg(\exp i\theta)$,

$$\begin{aligned} |\exp i\theta| &= 1, \\ \arg(\exp i\theta) &= \theta. \end{aligned}$$

Moreover, had we known that $\theta$ were just $\phi \equiv \arg(\exp i\theta)$, naturally we should have represented it by the symbol $\phi$ from the start. Changing $\phi \leftarrow \theta$ now, we have for real $\phi$ that

$$\begin{aligned} |\exp i\phi| &= 1, \\ \arg(\exp i\phi) &= \phi, \end{aligned}$$

which equations together say neither more nor less than that

$$\exp i\phi = \cos\phi + i\sin\phi = \operatorname{cis}\phi, \tag{5.12}$$

where the notation $\operatorname{cis}(\cdot)$ is as defined in § 3.11.

Along with the Pythagorean theorem (2.47), the fundamental theorem of calculus (7.2), Cauchy's integral formula (8.29) and Fourier's equation (18.1), eqn. (5.12) is one of the most famous results in all of mathematics. It is called *Euler's formula,*[5,6] and it opens the exponential domain fully to complex numbers, not just for the natural base $e$ but for any base. How? Consider in light of Fig. 5.3 and (5.12) that one can express any complex number in the form

$$z = x + iy = \rho \exp i\phi.$$

If a complex base $w$ is similarly expressed in the form

$$w = u + iv = \sigma \exp i\psi,$$

then it follows that

$$\begin{aligned} w^z &= \exp[\ln w^z] \\ &= \exp[z \ln w] \\ &= \exp[(x + iy)(i\psi + \ln \sigma)] \\ &= \exp[(x \ln \sigma - \psi y) + i(y \ln \sigma + \psi x)]. \end{aligned}$$

Since $\exp(\alpha + \beta) = e^{\alpha+\beta} = \exp \alpha \exp \beta$, the last equation is

$$w^z = \exp(x \ln \sigma - \psi y) \exp i(y \ln \sigma + \psi x), \tag{5.13}$$

where

$$\begin{aligned} x &= \rho \cos \phi, \\ y &= \rho \sin \phi, \\ \sigma &= \sqrt{u^2 + v^2}, \\ \tan \psi &= \frac{v}{u}. \end{aligned}$$

Equation (5.13) serves to raise any complex number to a complex power.

[5] For native English speakers who do not speak German, Leonhard Euler's name is pronounced as "oiler."

[6] An alternate derivation of Euler's formula (5.12)—less intuitive and requiring slightly more advanced mathematics, but briefer—constructs from Table 8.1 the Taylor series for $\exp i\phi$, $\cos \phi$ and $i \sin \phi$, then adds the latter two to show them equal to the first of the three. Such an alternate derivation lends little insight, perhaps, but at least it builds confidence that we actually knew what we were doing when we came up with the incredible (5.12).

Curious consequences of Euler's formula (5.12) include that

$$\begin{aligned} e^{\pm i2\pi/4} &= \pm i, \\ e^{\pm i2\pi/2} &= -1, \\ e^{in2\pi} &= 1. \end{aligned} \tag{5.14}$$

For the natural logarithm of a complex number in light of Euler's formula, we have that

$$\ln w = \ln\left(\sigma e^{i\psi}\right) = \ln\sigma + i\psi. \tag{5.15}$$

## 5.5 Complex exponentials and de Moivre's theorem

Euler's formula (5.12) implies that complex numbers $z_1$ and $z_2$ can be written

$$\begin{aligned} z_1 &= \rho_1 e^{i\phi_1}, \\ z_2 &= \rho_2 e^{i\phi_2}. \end{aligned} \tag{5.16}$$

By the basic power properties of Table 2.2, then,

$$\begin{aligned} z_1 z_2 &= \rho_1\rho_2 e^{i(\phi_1+\phi_2)} &&= \rho_1\rho_2 \exp[i(\phi_1+\phi_2)], \\ \frac{z_1}{z_2} &= \frac{\rho_1}{\rho_2} e^{i(\phi_1-\phi_2)} &&= \frac{\rho_1}{\rho_2}\exp[i(\phi_1-\phi_2)], \\ z^a &= \rho^a e^{ia\phi} &&= \rho^a \exp[ia\phi]. \end{aligned} \tag{5.17}$$

This is de Moivre's theorem, introduced in § 3.11.

## 5.6 Complex trigonometrics

Applying Euler's formula (5.12) to $+\phi$ then to $-\phi$, we have that

$$\begin{aligned} \exp(+i\phi) &= \cos\phi + i\sin\phi, \\ \exp(-i\phi) &= \cos\phi - i\sin\phi. \end{aligned}$$

Adding the two equations and solving for $\cos\phi$ yields

$$\cos\phi = \frac{\exp(+i\phi) + \exp(-i\phi)}{2}. \tag{5.18}$$

Subtracting the second equation from the first and solving for $\sin\phi$ yields

$$\sin\phi = \frac{\exp(+i\phi) - \exp(-i\phi)}{i2}. \tag{5.19}$$

Thus are the trigonometrics expressed in terms of complex exponentials.

### 5.6.1 The hyperbolic functions

The forms (5.18) and (5.19) suggest the definition of new functions

$$\cosh\phi \equiv \frac{\exp(+\phi)+\exp(-\phi)}{2}, \tag{5.20}$$

$$\sinh\phi \equiv \frac{\exp(+\phi)-\exp(-\phi)}{2}, \tag{5.21}$$

$$\tanh\phi \equiv \frac{\sinh\phi}{\cosh\phi}. \tag{5.22}$$

These are called the *hyperbolic functions.* Their inverses arccosh, etc., are defined in the obvious way. The Pythagorean theorem for trigonometrics (3.2) is that $\cos^2\phi+\sin^2\phi=1$; and from (5.20) and (5.21) one can derive the hyperbolic analog:

$$\begin{aligned}\cos^2\phi+\sin^2\phi&=1,\\ \cosh^2\phi-\sinh^2\phi&=1.\end{aligned} \tag{5.23}$$

Both lines of (5.23) hold for complex $\phi$ as well as for real.[7]

The notation $\exp i(\cdot)$ or $e^{i(\cdot)}$ is sometimes felt to be too bulky. Although less commonly seen than the other two, the notation

$$\text{cis}(\cdot)\equiv\exp i(\cdot)=\cos(\cdot)+i\sin(\cdot)$$

is also conventionally recognized, as earlier seen in § 3.11. Also conventionally recognized are $\sin^{-1}(\cdot)$ and occasionally $\text{asin}(\cdot)$ for $\arcsin(\cdot)$, and likewise for the several other trigs.

Replacing $z\leftarrow\phi$ in this section's several equations implies a coherent definition for trigonometric functions of a complex variable. Then, compar-

[7] Chapter 15 teaches that the "dot product" of a unit vector and its own conjugate is unity—$\hat{\mathbf{v}}^*\cdot\hat{\mathbf{v}}=1$, in the notation of that chapter—which tempts one incorrectly to suppose by analogy that $(\cos\phi)^*\cos\phi+(\sin\phi)^*\sin\phi=1$ and that $(\cosh\phi)^*\cosh\phi-(\sinh\phi)^*\sinh\phi=1$ when the angle $\phi$ is complex. However, (5.18) through (5.21) can generally be true only if (5.23) holds exactly as written for complex $\phi$ as well as for real. Hence in fact $(\cos\phi)^*\cos\phi+(\sin\phi)^*\sin\phi\neq1$ and $(\cosh\phi)\cosh\phi-(\sinh\phi)^*\sinh\phi\neq1$.

Such confusion probably tempts few readers unfamiliar with the material of Ch. 15, so you can ignore this footnote for now. However, if later you return after reading Ch. 15 and if the confusion then arises, then consider that the angle $\phi$ of Fig. 3.1 is a real angle, whereas we originally derived (5.23)'s first line from that figure. The figure is quite handy for real $\phi$, but what if anything the figure means when $\phi$ is complex is not obvious. If the confusion descends directly or indirectly from the figure, then such thoughts may serve to clarify the matter.

ing (5.18) and (5.19) respectively to (5.20) and (5.21), we have that

$$\begin{aligned} \cosh z &= \cos iz, \\ i \sinh z &= \sin iz, \\ i \tanh z &= \tan iz, \end{aligned} \tag{5.24}$$

by which one can immediately adapt the many trigonometric properties of Tables 3.1 and 3.3 to hyperbolic use.

At this point in the development one begins to notice that the sin, cos, exp, cis, cosh and sinh functions are each really just different facets of the same mathematical phenomenon. Likewise their respective inverses: arcsin, arccos, ln, $-i \ln$, arccosh and arcsinh. Conventional names for these two mutually inverse families of functions are unknown to the author, but one might call them the *natural exponential* and *natural logarithmic families.* Or, if the various tangent functions were included, then one might call them the *trigonometric* and *inverse trigonometric families.*

### 5.6.2 Inverse complex trigonometrics

Since one can express the several trigonometric functions in terms of complex exponentials one would like to know, complementarily, whether one cannot express the several inverse trigonometric functions in terms of complex logarithms. As it happens, one can.[8]

Let us consider the arccosine function, for instance. If per (5.18)

$$z = \cos w = \frac{e^{iw} + e^{-iw}}{2},$$

then by successive steps

$$\begin{aligned} e^{iw} &= 2z - e^{-iw}, \\ \left(e^{iw}\right)^2 &= 2z\left(e^{iw}\right) - 1, \\ e^{iw} &= z \pm \sqrt{z^2 - 1}, \end{aligned}$$

the last step of which has used the quadratic formula (2.2). Taking the logarithm, we have that

$$w = \frac{1}{i} \ln\left(z \pm i\sqrt{1 - z^2}\right);$$

[8] [57, Ch. 2]

or, since by definition $z = \cos w$, that

$$\arccos z = \frac{1}{i} \ln \left( z \pm i\sqrt{1 - z^2} \right). \tag{5.25}$$

Similarly,

$$\arcsin z = \frac{1}{i} \ln \left( iz \pm \sqrt{1 - z^2} \right). \tag{5.26}$$

The arctangent goes only a little differently:

$$\begin{aligned} z = \tan w &= -i\frac{e^{iw} - e^{-iw}}{e^{iw} + e^{-iw}}, \\ ze^{iw} + ze^{-iw} &= -ie^{iw} + ie^{-iw}, \\ (i + z)e^{iw} &= (i - z)e^{-iw}, \\ e^{i2w} &= \frac{i - z}{i + z}, \end{aligned}$$

implying that

$$\arctan z = \frac{1}{i2} \ln \frac{i - z}{i + z}. \tag{5.27}$$

By the same means, one can work out the inverse hyperbolics to be

$$\begin{aligned} \operatorname{arccosh} z &= \ln \left( z \pm \sqrt{z^2 - 1} \right), \\ \operatorname{arcsinh} z &= \ln \left( z \pm \sqrt{z^2 + 1} \right), \\ \operatorname{arctanh} z &= \frac{1}{2} \ln \frac{1 + z}{1 - z}. \end{aligned} \tag{5.28}$$

## 5.7 Summary of properties

Table 5.1 gathers properties of the complex exponential from this chapter and from §§ 2.12, 3.11 and 4.4.

## 5.8 Derivatives of complex exponentials

This section computes the derivatives of the various trigonometric and inverse trigonometric functions.

Table 5.1: Complex exponential properties.

$$
\begin{aligned}
i^2 &= -1 = (-i)^2 \\
\frac{1}{i} &= -i \\
e^{i\phi} &= \cos\phi + i\sin\phi \\
e^{iz} &= \cos z + i\sin z \\
z_1 z_2 &= \rho_1\rho_2 e^{i(\phi_1+\phi_2)} = (x_1x_2 - y_1y_2) + i(y_1x_2 + x_1y_2) \\
\frac{z_1}{z_2} &= \frac{\rho_1}{\rho_2}e^{i(\phi_1-\phi_2)} = \frac{(x_1x_2 + y_1y_2) + i(y_1x_2 - x_1y_2)}{x_2^2 + y_2^2} \\
z^a &= \rho^a e^{ia\phi} \\
w^z &= e^{x\ln\sigma - \psi y} e^{i(y\ln\sigma + \psi x)} \\
\ln w &= \ln\sigma + i\psi
\end{aligned}
$$

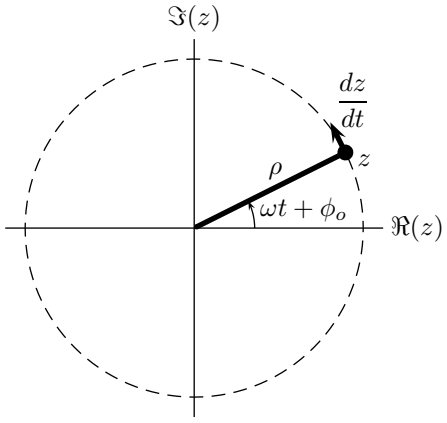
$$
\begin{aligned}
\sin z &= \frac{e^{iz} - e^{-iz}}{i2} & \sin iz &= i\sinh z & \sinh z &= \frac{e^z - e^{-z}}{2} \\
\cos z &= \frac{e^{iz} + e^{-iz}}{2} & \cos iz &= \cosh z & \cosh z &= \frac{e^z + e^{-z}}{2} \\
\tan z &= \frac{\sin z}{\cos z} & \tan iz &= i\tanh z & \tanh z &= \frac{\sinh z}{\cosh z}
\end{aligned}
$$

$$
\begin{aligned}
\arcsin z &= \frac{1}{i}\ln\left(iz \pm \sqrt{1-z^2}\right) & \operatorname{arcsinh} z &= \ln\left(z \pm \sqrt{z^2+1}\right) \\
\arccos z &= \frac{1}{i}\ln\left(z \pm i\sqrt{1-z^2}\right) & \operatorname{arccosh} z &= \ln\left(z \pm \sqrt{z^2-1}\right) \\
\arctan z &= \frac{1}{i2}\ln\frac{i-z}{i+z} & \operatorname{arctanh} z &= \frac{1}{2}\ln\frac{1+z}{1-z}
\end{aligned}
$$

$$\cos^2 z + \sin^2 z = 1 = \cosh^2 z - \sinh^2 z$$

$$
\begin{aligned}
z &\equiv x + iy = \rho e^{i\phi} & \frac{d}{dz}\exp z &= \exp z \\
w &\equiv u + iv = \sigma e^{i\psi} & \frac{d}{dw}\ln w &= \frac{1}{w} \\
\exp z &\equiv e^z & \frac{df/dz}{f(z)} &= \frac{d}{dz}\ln f(z) \\
\operatorname{cis} z &\equiv \cos z + i\sin z = e^{iz} & \log_b w &= \frac{\ln w}{\ln b}
\end{aligned}
$$

Figure 5.4: The derivatives of the sine and cosine functions.




### 5.8.1 Derivatives of sine and cosine

One can compute derivatives of the sine and cosine functions from (5.18) and (5.19), but to do it in that way doesn't seem sporting. Better applied style is to find the derivatives by observing directly the circle from which the sine and cosine functions come.

Refer to Fig. 5.4. Suppose that the point $z$ in the figure is not fixed but travels steadily about the circle such that

$$z(t) = (\rho)\left[\cos(\omega t + \phi_o) + i\sin(\omega t + \phi_o)\right]. \tag{5.29}$$

How fast then is the rate $dz/dt$, and in what Argand direction? Answer:

$$\frac{dz}{dt} = (\rho)\left[\frac{d}{dt}\cos(\omega t + \phi_o) + i\frac{d}{dt}\sin(\omega t + \phi_o)\right]. \tag{5.30}$$

Evidently however, considering the figure,

- the speed $|dz/dt|$ is also $(\rho)(d\phi/dt) = \rho\omega$;
- the direction is at right angles to the arm of $\rho$, which is to say that $\arg(dz/dt) = \phi + 2\pi/4$.

With these observations we can write that

$$\begin{aligned}\frac{dz}{dt} &= (\rho\omega)\left[\cos\left(\omega t + \phi_o + \frac{2\pi}{4}\right) + i\sin\left(\omega t + \phi_o + \frac{2\pi}{4}\right)\right] \\ &= (\rho\omega)\left[-\sin(\omega t + \phi_o) + i\cos(\omega t + \phi_o)\right].\end{aligned} \tag{5.31}$$

Matching the real and imaginary parts of (5.30) against those of (5.31), we have that

$$\begin{aligned}\frac{d}{dt}\cos(\omega t + \phi_o) &= -\omega\sin(\omega t + \phi_o),\\ \frac{d}{dt}\sin(\omega t + \phi_o) &= +\omega\cos(\omega t + \phi_o).\end{aligned} \tag{5.32}$$

If $\omega = 1$ and $\phi_o = 0$, these are

$$\begin{aligned}\frac{d}{dt}\cos t &= -\sin t,\\ \frac{d}{dt}\sin t &= +\cos t.\end{aligned} \tag{5.33}$$

### 5.8.2 Derivatives of the trigonometrics

Equations (5.4) and (5.33) give the derivatives of $\exp(\cdot)$, $\sin(\cdot)$ and $\cos(\cdot)$. From these, with the help of (5.23) and the derivative chain and product rules (§ 4.5), we can calculate the several derivatives of Table 5.2.[9]

### 5.8.3 Derivatives of the inverse trigonometrics

Observe the pair

$$\begin{aligned}\frac{d}{dz}\exp z &= \exp z,\\ \frac{d}{dw}\ln w &= \frac{1}{w}.\end{aligned}$$

The natural exponential $\exp z$ belongs to the trigonometric family of functions, as does its derivative. The natural logarithm $\ln w$, by contrast, belongs to the inverse trigonometric family of functions; but its derivative is simpler, not a trigonometric or inverse trigonometric function at all. In Table 5.2, one notices that all the trigonometrics have trigonometric derivatives. By analogy with the natural logarithm, do all the inverse trigonometrics have simpler derivatives?

It turns out that they do. Refer to the account of the natural logarithm's derivative in § 5.2. Following a similar procedure, we have by successive steps

[9] [55, back endpaper]

Table 5.2: Derivatives of the trigonometrics.

$$\begin{aligned}
\frac{d}{dz}\exp z &= +\exp z & \frac{d}{dz}\frac{1}{\exp z} &= -\frac{1}{\exp z} \\
\frac{d}{dz}\sin z &= +\cos z & \frac{d}{dz}\frac{1}{\sin z} &= -\frac{1}{\tan z \sin z} \\
\frac{d}{dz}\cos z &= -\sin z & \frac{d}{dz}\frac{1}{\cos z} &= +\frac{\tan z}{\cos z}
\end{aligned}$$

$$\begin{aligned}
\frac{d}{dz}\tan z &= +\left(1+\tan^2 z\right) &&= +\frac{1}{\cos^2 z} \\
\frac{d}{dz}\frac{1}{\tan z} &= -\left(1+\frac{1}{\tan^2 z}\right) &&= -\frac{1}{\sin^2 z}
\end{aligned}$$

$$\begin{aligned}
\frac{d}{dz}\sinh z &= +\cosh z & \frac{d}{dz}\frac{1}{\sinh z} &= -\frac{1}{\tanh z \sinh z} \\
\frac{d}{dz}\cosh z &= +\sinh z & \frac{d}{dz}\frac{1}{\cosh z} &= -\frac{\tanh z}{\cosh z}
\end{aligned}$$

$$\begin{aligned}
\frac{d}{dz}\tanh z &= 1-\tanh^2 z &&= +\frac{1}{\cosh^2 z} \\
\frac{d}{dz}\frac{1}{\tanh z} &= 1-\frac{1}{\tanh^2 z} &&= -\frac{1}{\sinh^2 z}
\end{aligned}$$

that

$$\begin{aligned}
\arcsin w &= z, \\
w &= \sin z, \\
\frac{dw}{dz} &= \cos z, \\
\frac{dw}{dz} &= \pm\sqrt{1-\sin^2 z}, \\
\frac{dw}{dz} &= \pm\sqrt{1-w^2}, \\
\frac{dz}{dw} &= \frac{\pm 1}{\sqrt{1-w^2}}, \\
\frac{d}{dw}\arcsin w &= \frac{\pm 1}{\sqrt{1-w^2}}.
\end{aligned} \tag{5.34}$$

Similarly,

$$\begin{aligned}
\arctan w &= z, \\
w &= \tan z, \\
\frac{dw}{dz} &= 1+\tan^2 z, \\
\frac{dw}{dz} &= 1+w^2, \\
\frac{dz}{dw} &= \frac{1}{1+w^2}, \\
\frac{d}{dw}\arctan w &= \frac{1}{1+w^2}.
\end{aligned} \tag{5.35}$$

Derivatives of the other inverse trigonometrics are found in the same way. Table 5.3 summarizes.

## 5.9 The actuality of complex quantities

Doing all this neat complex math, the applied mathematician can lose sight of some questions he probably ought to keep in mind: Is there really such a thing as a complex quantity in nature? If not, then hadn't we better avoid these complex quantities, leaving them to the professional mathematical theorists?

As developed by Oliver Heaviside in 1887,[10] the answer depends on your point of view. If I have 300 g of grapes and 100 g of grapes, then I have 400 g

[10] [46]

Table 5.3: Derivatives of the inverse trigonometrics.

$$\begin{aligned}
\frac{d}{dw}\ln w &= \frac{1}{w} \\
\frac{d}{dw}\arcsin w &= \frac{\pm 1}{\sqrt{1-w^2}} \\
\frac{d}{dw}\arccos w &= \frac{\mp 1}{\sqrt{1-w^2}} \\
\frac{d}{dw}\arctan w &= \frac{1}{1+w^2} \\
\frac{d}{dw}\operatorname{arcsinh} w &= \frac{\pm 1}{\sqrt{w^2+1}} \\
\frac{d}{dw}\operatorname{arccosh} w &= \frac{\pm 1}{\sqrt{w^2-1}} \\
\frac{d}{dw}\operatorname{arctanh} w &= \frac{1}{1-w^2}
\end{aligned}$$

altogether. Alternately, if I have 500 g of grapes and $-100$ g of grapes, again I have 400 g altogether. (What does it mean to have $-100$ g of grapes? Maybe that I ate some!) But what if I have $200 + i100$ g of grapes and $200 - i100$ g of grapes? Answer: again, 400 g.

Probably you would not choose to think of $200 + i100$ g of grapes and $200 - i100$ g of grapes, but because of (5.18) and (5.19), one often describes wave phenomena as linear superpositions (sums) of countervailing complex exponentials. Consider for instance the propagating wave

$$A\cos[\omega t - kz] = \frac{A}{2}\exp[+i(\omega t - kz)] + \frac{A}{2}\exp[-i(\omega t - kz)].$$

The benefit of splitting the real cosine into two complex parts is that while the magnitude of the cosine changes with time $t$, the magnitude of either exponential alone remains steady (see the circle in Fig. 5.3). It turns out to be much easier to analyze two complex wave quantities of constant magnitude than to analyze one real wave quantity of varying magnitude. Better yet, since each complex wave quantity is the complex conjugate of the other, the analyses thereof are mutually conjugate, too; so you normally needn't actually analyze the second. The one analysis suffices for both.[11] (It's like

[11] If the point is not immediately clear, an example: Suppose that by the Newton-

reflecting your sister's handwriting. To read her handwriting backward, you needn't ask her to try writing reverse with the wrong hand; you can just hold her regular script up to a mirror. Of course, this ignores the question of why one would want to reflect someone's handwriting in the first place; but anyway, reflecting—which is to say, conjugating—complex quantities often is useful.)

Some authors have gently denigrated the use of imaginary parts in physical applications as a mere mathematical trick, as though the parts were not actually there. Well, that is one way to treat the matter, but it is not the way this book recommends. Nothing in the mathematics *requires* you to regard the imaginary parts as physically nonexistent. You need not abuse Ockham's razor! (Ockham's razor, "Do not multiply objects without necessity,"[12] is not a bad philosophical indicator as far as it goes, but is overused in some circles—particularly in circles in which Aristotle[13] is mistakenly believed to be vaguely outdated. More often than one likes to believe, the necessity to multiply objects remains hidden until one has ventured the multiplication, nor reveals itself to the one who wields the razor, whose hand humility should stay.) It is true by Euler's formula (5.12) that a complex exponential $\exp i\phi$ can be decomposed into a sum of trigonometrics. However, it is equally true by the complex trigonometric formulas (5.18) and (5.19) that *a trigonometric can be decomposed into a sum of complex exponentials.* So, if each can be decomposed into the other, then which of the two is the real decomposition? Answer: it depends on your point of view. Experience seems to recommend viewing the complex exponential as the basic element—as the element of which the trigonometrics are composed—rather than the other way around. From this point of view, it is (5.18) and (5.19) which are the real decomposition. Euler's formula itself is secondary.

The complex exponential method of offsetting imaginary parts offers an elegant yet practical mathematical means to model physical wave phenomena. So go ahead: regard the imaginary parts as actual. Aristotle would regard them so (or so the author suspects). To regard the imaginary parts as actual hurts nothing, and it helps with the math.

---

Raphson iteration (§ 4.8) you have found a root of the polynomial $x^3 + 2x^2 + 3x + 4$ at $x \approx -0\text{x}0.2\text{D} + i0\text{x}1.8\text{C}$. Where is there another root? Answer: at the complex conjugate, $x \approx -0\text{x}0.2\text{D} - i0\text{x}1.8\text{C}$. One need not actually run the Newton-Raphson again to find the conjugate root.

[12] [61, Ch. 12]

[13] [**?**]

# Chapter 6

# Primes, roots and averages

This chapter gathers a few significant topics, each of whose treatment seems too brief for a chapter of its own.

## 6.1 Prime numbers

A *prime number*—or simply, a *prime*—is an integer greater than one, divisible only by one and itself. A *composite number* is an integer greater than one and not prime. A composite number can be composed as a product of two or more prime numbers. All positive integers greater than one are either composite or prime.

The mathematical study of prime numbers and their incidents constitutes *number theory,* and it is a deep area of mathematics. The deeper results of number theory seldom arise in applications,[1] however, so we will confine our study of number theory in this book to one or two of its simplest, most broadly interesting results.

### 6.1.1 The infinite supply of primes

The first primes are evidently $2, 3, 5, 7, 0\text{xB}, \ldots$. Is there a last prime? To show that there is not, suppose that there were. More precisely, suppose that there existed exactly $N$ primes, with $N$ finite, letting $p_1, p_2, \ldots, p_N$ represent these primes from least to greatest. Now consider the product of

[1]The deeper results of number theory do arise in cryptography, or so the author has been led to understand. Although cryptography is literally an application of mathematics, its spirit is that of pure mathematics rather than of applied. If you seek cryptographic derivations, this book is probably not the one you want.

all the primes,

$$C = \prod_{j=1}^{N} p_j.$$

What of $C + 1$? Since $p_1 = 2$ divides $C$, it cannot divide $C + 1$. Similarly, since $p_2 = 3$ divides $C$, it also cannot divide $C + 1$. The same goes for $p_3 = 5$, $p_4 = 7$, $p_5 = 0\text{xB}$, etc. Apparently none of the primes in the $p_j$ series divides $C + 1$, which implies either that $C + 1$ itself is prime, or that $C + 1$ is composed of primes not in the series. But the latter is assumed impossible on the ground that the $p_j$ series includes all primes; and the former is assumed impossible on the ground that $C + 1 > C > p_N$, with $p_N$ the greatest prime. The contradiction proves false the assumption which gave rise to it. The false assumption: that there were a last prime.

Thus there is no last prime. No matter how great a prime number one finds, a greater can always be found. The supply of primes is infinite.[2]

Attributed to the ancient geometer Euclid, the foregoing proof is a classic example of mathematical *reductio ad absurdum,* or as usually styled in English, *proof by contradiction.*[3]

### 6.1.2 Compositional uniqueness

Occasionally in mathematics, plausible assumptions can hide subtle logical flaws. One such plausible assumption is the assumption that every positive integer has a unique *prime factorization.* It is readily seen that the first several positive integers—$1 = ()$, $2 = (2^1)$, $3 = (3^1)$, $4 = (2^2)$, $5 = (5^1)$, $6 = (2^1)(3^1)$, $7 = (7^1)$, $8 = (2^3)$, ...—each have unique prime factorizations, but is this necessarily true of all positive integers?

To show that it is true, suppose that it were not.[4] More precisely, suppose that there did exist positive integers factorable each in two or more distinct ways, with the symbol $C$ representing the least such integer. Noting that $C$ must be composite (prime numbers by definition are each factorable

[2] [58]

[3] [52, Appendix 1][66, "Reductio ad absurdum," 02:36, 28 April 2006]

[4] Unfortunately the author knows no more elegant proof than this, yet cannot even cite this one properly. The author encountered the proof in some book over a decade ago. The identity of that book is now long forgotten.

only one way, like $5 = [5^1]$), let

$$\begin{aligned} C_p &\equiv \prod_{j=1}^{N_p} p_j, \\ C_q &\equiv \prod_{k=1}^{N_q} q_k, \\ C_p = C_q &= C, \\ p_j &\le p_{j+1}, \\ q_k &\le q_{k+1}, \\ p_1 &\le q_1, \\ N_p &> 1, \\ N_q &> 1, \end{aligned}$$

where $C_p$ and $C_q$ represent two distinct prime factorizations of the same number $C$ and where the $p_j$ and $q_k$ are the respective primes ordered from least to greatest. We see that

$$p_j \neq q_k$$

for any $j$ and $k$—that is, that the same prime cannot appear in both factorizations—because if the same prime $r$ did appear in both then $C/r$ either would be prime (in which case both factorizations would be $[r][C/r]$, defying our assumption that the two differed) or would constitute an ambiguously factorable composite integer less than $C$ when we had already defined $C$ to represent the least such. Among other effects, the fact that $p_j \neq q_k$ strengthens the definition $p_1 \le q_1$ to read

$$p_1 < q_1.$$

Let us now rewrite the two factorizations in the form

$$\begin{aligned} C_p &= p_1 A_p, \\ C_q &= q_1 A_q, \\ C_p = C_q &= C, \\ A_p &\equiv \prod_{j=2}^{N_p} p_j, \\ A_q &\equiv \prod_{k=2}^{N_q} q_k, \end{aligned}$$

where $p_1$ and $q_1$ are the least primes in their respective factorizations. Since $C$ is composite and since $p_1 < q_1$, we have that

$$1 < p_1 < q_1 \leq \sqrt{C} \leq A_q < A_p < C,$$

which implies that

$$p_1 q_1 < C.$$

The last inequality lets us compose the new positive integer

$$B = C - p_1 q_1,$$

which might be prime or composite (or unity), but which either way enjoys a unique prime factorization because $B < C$, with $C$ the least positive integer factorable two ways. Observing that some integer $s$ which divides $C$ necessarily also divides $C \pm ns$, we note that each of $p_1$ and $q_1$ necessarily divides $B$. This means that $B$'s unique factorization includes both $p_1$ and $q_1$, which further means that the product $p_1 q_1$ divides $B$. But if $p_1 q_1$ divides $B$, then it divides $B + p_1 q_1 = C$, also.

Let $E$ represent the positive integer which results from dividing $C$ by $p_1 q_1$:

$$E \equiv \frac{C}{p_1 q_1}.$$

Then,

$$Eq_1 = \frac{C}{p_1} = A_p,$$
$$Ep_1 = \frac{C}{q_1} = A_q.$$

That $Eq_1 = A_p$ says that $q_1$ divides $A_p$. But $A_p < C$, so $A_p$'s prime factorization is unique—and we see above that $A_p$'s factorization *does not include any* $q_k$, not even $q_1$. The contradiction proves false the assumption which gave rise to it. The false assumption: that there existed a least composite number $C$ prime-factorable in two distinct ways.

Thus no positive integer is ambiguously factorable. Prime factorizations are always unique.

We have observed at the start of this subsection that plausible assumptions can hide subtle logical flaws. Indeed this is so. Interestingly however, the plausible assumption of the present subsection has turned out absolutely correct; we have just had to do some extra work to prove it. Such effects

are typical on the shadowed frontier where applied shades into pure mathematics: with sufficient experience and with a firm grasp of the model at hand, if you think that it's true, then it probably is. Judging when to delve into the mathematics anyway, seeking a more rigorous demonstration of a proposition one feels pretty sure is correct, is a matter of applied mathematical style. It depends on how sure one feels, and more importantly on whether the unsureness felt is true uncertainty or is just an unaccountable desire for more precise mathematical definition (if the latter, then unlike the author you may have the right temperament to become a professional mathematician). The author does judge the present subsection's proof to be worth the applied effort; but nevertheless, when one lets logical minutiae distract him to too great a degree, one admittedly begins to drift out of the applied mathematical realm that is the subject of this book.

### 6.1.3 Rational and irrational numbers

A *rational number* is a finite real number expressible as a ratio of integers[5]

$$x = \frac{p}{q}, \quad (p, q) \in \mathbb{Z},\ q \neq 0.$$

The ratio is *fully reduced* if $p$ and $q$ have no prime factors in common. For instance, $4/6$ is not fully reduced, whereas $2/3$ is.

An *irrational number* is a finite real number which is not rational. For example, $\sqrt{2}$ is irrational. In fact any $x = \sqrt{n}$ is irrational unless integral; there is no such thing as a $\sqrt{n}$ which is not an integer but is rational.

To prove[6] the last point, suppose that there did exist a fully reduced

$$x = \frac{p}{q} = \sqrt{n}, \quad (n, p, q) \in \mathbb{Z},\ n > 0,\ p > 0,\ q > 1.$$

Squaring the equation, we have that

$$\frac{p^2}{q^2} = n,$$

which form is evidently also fully reduced. But if $q > 1$, then the fully reduced $n = p^2/q^2$ is not an integer as we had assumed that it was. The contradiction proves false the assumption which gave rise to it. Hence there exists no rational, nonintegral $\sqrt{n}$, as was to be demonstrated. The proof is readily extended to show that any $x = n^{j/k}$ is irrational if nonintegral, the

[5] Section 2.3 explains the $\in \mathbb{Z}$ notation.

[6] A proof somewhat like the one presented here is found in [52, Appendix 1].

extension by writing $p^k/q^k = n^j$ then following similar steps as those this paragraph outlines.

That's all the number theory the book treats; but in applied math, so little will take you pretty far. Now onward we go to other topics.

## 6.2 The existence and number of polynomial roots

This section shows that an $N$th-order polynomial must have exactly $N$ roots.

### 6.2.1 Polynomial roots

Consider the quotient $B(z)/A(z)$, where

$$\begin{aligned} A(z) &= z - \alpha, \\ B(z) &= \sum_{k=0}^{N} b_k z^k, \quad N > 0,\ b_N \neq 0, \\ B(\alpha) &= 0. \end{aligned}$$

In the long-division symbology of Table 2.3,

$$B(z) = A(z)Q_0(z) + R_0(z),$$

where $Q_0(z)$ is the quotient and $R_0(z)$, a remainder. In this case the divisor $A(z) = z - \alpha$ has first order, and as § 2.6.2 has observed, first-order divisors leave zeroth-order, constant remainders $R_0(z) = \rho$. Thus substituting yields

$$B(z) = (z - \alpha)Q_0(z) + \rho.$$

When $z = \alpha$, this reduces to

$$B(\alpha) = \rho.$$

But $B(\alpha) = 0$ by assumption, so

$$\rho = 0.$$

Evidently the division leaves no remainder $\rho$, which is to say that $z - \alpha$ *exactly divides every polynomial* $B(z)$ *of which* $z = \alpha$ *is a root.*

Note that if the polynomial $B(z)$ has order $N$, then the quotient $Q(z) = B(z)/(z - \alpha)$ has exactly order $N - 1$. That is, the leading, $z^{N-1}$ term of the quotient is never null. The reason is that if the leading term were null, if $Q(z)$ had order less than $N - 1$, then $B(z) = (z - \alpha)Q(z)$ could not possibly have order $N$ as we have assumed.

### 6.2.2 The fundamental theorem of algebra

The *fundamental theorem of algebra* holds that any polynomial $B(z)$ of order $N$ can be factored

$$B(z) = \sum_{k=0}^{N} b_k z^k = b_N \prod_{j=1}^{N} (z - \alpha_j), \quad b_N \neq 0, \tag{6.1}$$

where the $\alpha_k$ are the $N$ roots of the polynomial.[7]

To prove the theorem, it suffices to show that all polynomials of order $N > 0$ have at least one root; for if a polynomial of order $N$ has a root $\alpha_N$, then according to § 6.2.1 one can divide the polynomial by $z - \alpha_N$ to obtain a new polynomial of order $N - 1$. To the new polynomial the same logic applies: if it has at least one root $\alpha_{N-1}$, then one can divide *it* by $z - \alpha_{N-1}$ to obtain yet another polynomial of order $N - 2$; and so on, one root extracted at each step, factoring the polynomial step by step into the desired form $b_N \prod_{j=1}^{N} (z - \alpha_j)$.

It remains however to show that there exists no polynomial $B(z)$ of order $N > 0$ lacking roots altogether. To show that there is no such polynomial, consider the locus[8] of all $B(\rho e^{i\phi})$ in the Argand range plane (Fig. 2.5), where $z = \rho e^{i\phi}$, $\rho$ is held constant, and $\phi$ is variable. Because $e^{i(\phi + n2\pi)} = e^{i\phi}$ and no fractional powers of $z$ appear in (6.1), this locus forms a closed loop. At very large $\rho$, the $b_N z^N$ term dominates $B(z)$, so the locus there evidently has the general character of $b_N \rho^N e^{iN\phi}$. As such, the locus is nearly but not quite a circle at radius $b_N \rho^N$ from the Argand origin $B(z) = 0$, revolving $N$ times at that great distance before exactly repeating. On the other hand, when $\rho = 0$ the entire locus collapses on the single point $B(0) = b_0$.

Now consider the locus at very large $\rho$ again, but this time let $\rho$ slowly shrink. Watch the locus as $\rho$ shrinks. The locus is like a great string or rubber band, joined at the ends and looped in $N$ great loops. As $\rho$ shrinks smoothly, the string's shape changes smoothly. Eventually $\rho$ disappears and the entire string collapses on the point $B(0) = b_0$. Since the string originally has looped $N$ times at great distance about the Argand origin, but at the end has collapsed on a single point, then at some time between it must have swept through the origin and every other point within the original loops.

[7] Professional mathematicians typically state the theorem in a slightly different form. They also prove it in rather a different way. [31, Ch. 10, Prob. 74]

[8] A *locus* is the geometric collection of points which satisfy a given criterion. For example, the locus of all points in a plane at distance $\rho$ from a point $O$ is a circle; the locus of all points in three-dimensional space equidistant from two points $P$ and $Q$ is a plane; etc.

After all, $B(z)$ is everywhere differentiable, so the string can only *sweep* as $\rho$ decreases; it can never skip. The Argand origin lies inside the loops at the start but outside at the end. If so, then the values of $\rho$ and $\phi$ precisely where the string has swept through the origin by definition constitute a root $B(\rho e^{i\phi}) = 0$. Thus as we were required to show, $B(z)$ does have at least one root, which observation completes the applied demonstration of the fundamental theorem of algebra.

The fact that the roots exist is one thing. Actually finding the roots numerically is another matter. For a quadratic (second order) polynomial, (2.2) gives the roots. For cubic (third order) and quartic (fourth order) polynomials, formulas for the roots are known (see Ch. 10) though seemingly not so for quintic (fifth order) and higher-order polynomials;[9] but the Newton-Raphson iteration (§ 4.8) can be used to locate a root numerically in any case. The Newton-Raphson is used to extract one root (*any* root) at each step as described above, reducing the polynomial step by step until all the roots are found.

The reverse problem, finding the polynomial given the roots, is much easier: one just multiplies out $\prod_j (z - \alpha_j)$, as in (6.1).

## 6.3 Addition and averages

This section discusses the two basic ways to add numbers and the three basic ways to calculate averages of them.

### 6.3.1 Serial and parallel addition

Consider the following problem. There are three masons. The strongest and most experienced of the three, Adam, lays 120 bricks per hour.[10] Next is Brian who lays 90. Charles is new; he lays only 60. Given eight hours, how many bricks can the three men lay? Answer:

$$(8 \text{ hours})(120 + 90 + 60 \text{ bricks per hour}) = 2160 \text{ bricks}.$$

Now suppose that we are told that Adam can lay a brick every 30 seconds; Brian, every 40 seconds; Charles, every 60 seconds. How much time do the

[9] In a celebrated theorem of pure mathematics [64, "Abel's impossibility theorem"], it is said to be shown that no such formula even exists, given that the formula be constructed according to certain rules. Undoubtedly the theorem is interesting to the professional mathematician, but to the applied mathematician it probably suffices to observe merely that no such formula is known.

[10] The figures in the example are in decimal notation.

three men need to lay 2160 bricks? Answer:

$$\begin{aligned}\frac{\text{2160 bricks}}{\frac{1}{30}+\frac{1}{40}+\frac{1}{60}\text{ bricks per second}} &= \text{28,800 seconds}\left(\frac{\text{1 hour}}{\text{3600 seconds}}\right)\\ &= \text{8 hours.}\end{aligned}$$

The two problems are precisely equivalent. Neither is stated in simpler terms than the other. The notation used to solve the second is less elegant, but fortunately there exists a better notation:

$$(\text{2160 bricks})(30 \parallel 40 \parallel 60\text{ seconds per brick}) = \text{8 hours},$$

where

$$\frac{1}{30 \parallel 40 \parallel 60} = \frac{1}{30}+\frac{1}{40}+\frac{1}{60}.$$

The operator $\parallel$ is called the *parallel addition* operator. It works according to the law

$$\frac{1}{a \parallel b} = \frac{1}{a}+\frac{1}{b}, \tag{6.2}$$

where the familiar operator $+$ is verbally distinguished from the $\parallel$ when necessary by calling the $+$ the *serial addition* or *series addition* operator. With (6.2) and a bit of arithmetic, the several parallel-addition identities of Table 6.1 are soon derived.

The writer knows of no conventional notation for parallel sums of series, but suggests that the notation which appears in the table,

$$\sum_{k=a}^{b} \parallel f(k) \equiv f(a) \parallel f(a+1) \parallel f(a+2) \parallel \cdots \parallel f(b),$$

might serve if needed.

Assuming that none of the values involved is negative, one can readily show that[11]

$$a \parallel x \le b \parallel x \quad \text{iff} \quad a \le b. \tag{6.3}$$

This is intuitive. Counterintuitive, perhaps, is that

$$a \parallel x \le a. \tag{6.4}$$

Because we have all learned as children to count in the sensible manner $1, 2, 3, 4, 5, \ldots$—rather than as $1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \ldots$—serial addition ($+$) seems

[11]The word *iff* means, "if and only if."

Table 6.1: Parallel and serial addition identities.

$$\frac{1}{a \parallel b} = \frac{1}{a} + \frac{1}{b} \qquad \frac{1}{a+b} = \frac{1}{a} \parallel \frac{1}{b}$$

$$a \parallel b = \frac{ab}{a+b} \qquad a+b = \frac{ab}{a \parallel b}$$

$$a \parallel \frac{1}{b} = \frac{a}{1+ab} \qquad a + \frac{1}{b} = \frac{a}{1 \parallel ab}$$

$$a \parallel b = b \parallel a \qquad a+b = b+a$$

$$a \parallel (b \parallel c) = (a \parallel b) \parallel c \qquad a + (b+c) = (a+b) + c$$

$$a \parallel \infty = \infty \parallel a = a \qquad a + 0 = 0 + a = a$$

$$a \parallel (-a) = \infty \qquad a + (-a) = 0$$

$$(a)(b \parallel c) = ab \parallel ac \qquad (a)(b+c) = ab + ac$$

$$\frac{1}{\sum_k \parallel a_k} = \sum_k \frac{1}{a_k} \qquad \frac{1}{\sum_k a_k} = \sum_k \parallel \frac{1}{a_k}$$

more natural than parallel addition ($\|$) does. The psychological barrier is hard to breach, yet for many purposes parallel addition is in fact no less fundamental. Its rules are inherently neither more nor less complicated, as Table 6.1 illustrates; yet outside the electrical engineering literature the parallel addition notation is seldom seen.[12] Now that you have seen it, you can use it. There is profit in learning to think both ways. (Exercise: counting from zero serially goes $0, 1, 2, 3, 4, 5, \ldots$; how does the parallel analog go?)[13]

Convention brings no special symbol for parallel subtraction, incidentally. One merely writes

$$a \parallel (-b),$$

which means exactly what it appears to mean.

### 6.3.2 Averages

Let us return to the problem of the preceding section. Among the three masons, what is their average productivity? The answer depends on how you look at it. On the one hand,

$$\frac{120 + 90 + 60 \text{ bricks per hour}}{3} = 90 \text{ bricks per hour}.$$

On the other hand,

$$\frac{30 + 40 + 60 \text{ seconds per brick}}{3} = 43\tfrac{1}{3} \text{ seconds per brick}.$$

These two figures are not the same. That is, $1/(43\frac{1}{3} \text{ seconds per brick}) \neq 90$ bricks per hour. Yet both figures are valid. Which figure you choose depends on what you want to calculate. A common mathematical error among businessmen seems to be to fail to realize that both averages are possible and that they yield different numbers (if the businessman quotes in bricks per hour, the productivities average one way; if in seconds per brick, the other way; yet some businessmen will never clearly consider the difference). Realizing this, the clever businessman might negotiate a contract so that the average used worked to his own advantage.[14]

[12] In electric circuits, loads are connected in parallel as often as, in fact probably more often than, they are connected in series. Parallel addition gives the electrical engineer a neat way of adding the impedances of parallel-connected loads.

[13] [54, eqn. 1.27]

[14] "And what does the author know about business?" comes the rejoinder.

The rejoinder is fair enough. If the author wanted to demonstrate his business acumen (or lack thereof) he'd do so elsewhere not here! There are a lot of good business books

When it is unclear which of the two averages is more appropriate, a third average is available, the *geometric mean*

$$[(120)(90)(60)]^{1/3} \text{ bricks per hour.}$$

The geometric mean does not have the problem either of the two averages discussed above has. The inverse geometric mean

$$[(30)(40)(60)]^{1/3} \text{ seconds per brick}$$

implies the same average productivity. The mathematically savvy sometimes prefer the geometric mean over either of the others for this reason.[15]

Generally, the *arithmetic, geometric* and *harmonic means* are defined

$$\mu \equiv \frac{\sum_k w_k x_k}{\sum_k w_k} = \left(\sum_k \| \frac{1}{w_k}\right)\left(\sum_k w_k x_k\right), \tag{6.5}$$

$$\mu_\Pi \equiv \left[\prod_j x_j^{w_j}\right]^{1/\sum_k w_k} = \left[\prod_j x_j^{w_j}\right]^{\sum_k \| 1/w_k}, \tag{6.6}$$

$$\mu_\| \equiv \frac{\sum_k \| x_k/w_k}{\sum_k \| 1/w_k} = \left(\sum_k w_k\right)\left(\sum_k \| \frac{x_k}{w_k}\right), \tag{6.7}$$

---

out there and this is not one of them.

The fact remains nevertheless that businessmen sometimes use mathematics in peculiar ways, making relatively easy problems harder and more mysterious than the problems need to be. If you have ever encountered the little monstrosity of an approximation banks (at least in the author's country) actually use in place of (9.12) to accrue interest and amortize loans, then you have met the difficulty.

Trying to convince businessmen that their math is wrong, incidentally, is in the author's experience usually a waste of time. Some businessmen are mathematically rather sharp—as you presumably are if you are in business and are reading these words—but as for most: when real mathematical ability is needed, that's what they hire engineers, architects and the like for. The author is not sure, but somehow he doubts that many boards of directors would be willing to bet the company on a financial formula containing some mysterious-looking $e^x$. Business demands other talents.

[15]The writer, an American, was recently, pleasantly surprised to learn that the formula his country's relevant federal statute stipulates to implement the Constitutional requirement that representation in the country's federal House of Representatives be apportioned by population actually, properly always apportions the next available seat in the House to the state whose *geometric* mean of population per representative before and after apportionment would be greatest. Now, admittedly, the Republic does not rise or fall on the niceties of averaging techniques; but, nonetheless, some American who knew his mathematics was involved in the drafting of that statute!

where the $x_k$ are the several samples and the $w_k$ are weights. For two samples weighted equally, these are

$$\begin{aligned} \mu &= \frac{a+b}{2}, & (6.8)\\ \mu_\Pi &= \sqrt{ab}, & (6.9)\\ \mu_\parallel &= 2(a \parallel b). & (6.10) \end{aligned}$$

If $a \geq 0$ and $b \geq 0$, then by successive steps,[16]

$$\begin{aligned} 0 &\leq (a-b)^2,\\ 0 &\leq a^2 - 2ab + b^2,\\ 4ab &\leq a^2 + 2ab + b^2,\\ 2\sqrt{ab} &\leq a + b,\\ \frac{2\sqrt{ab}}{a+b} &\leq 1 \leq \frac{a+b}{2\sqrt{ab}},\\ \frac{2ab}{a+b} &\leq \sqrt{ab} \leq \frac{a+b}{2},\\ 2(a \parallel b) &\leq \sqrt{ab} \leq \frac{a+b}{2}. \end{aligned}$$

That is,

$$\mu_\parallel \leq \mu_\Pi \leq \mu. \tag{6.11}$$

The arithmetic mean is greatest and the harmonic mean, least; with the geometric mean falling between.

Does (6.11) hold when there are several nonnegative samples of various nonnegative weights? To show that it does, consider the case of $N = 2^m$ nonnegative samples of equal weight. Nothing prevents one from dividing

[16]The steps are logical enough, but the motivation behind them remains inscrutable until the reader realizes that the writer originally worked the steps out backward with his pencil, from the last step to the first. Only then did he reverse the order and write the steps formally here. The writer had no idea that he was supposed to start from $0 \leq (a-b)^2$ until his pencil working backward showed him. "Begin with the end in mind," the saying goes. In this case the saying is right.

The same reading strategy often clarifies inscrutable math. When you can follow the logic but cannot understand what could possibly have inspired the writer to conceive the logic in the first place, try reading backward.

such a set of samples in half, considering each subset separately, for if (6.11) holds for each subset individually then surely it holds for the whole set (this is so because the average of the whole set is itself the *average of the two subset averages,* where the word "average" signifies the arithmetic, geometric or harmonic mean as appropriate). But each subset can further be divided in half, then each subsubset can be divided in half again, and so on until each smallest group has two members only—in which case we already know that (6.11) obtains. Starting there and recursing back, we have that (6.11) obtains for the entire set. Now consider that a sample of any weight can be approximated arbitrarily closely by several samples of weight $1/2^m$, provided that $m$ is sufficiently large. By this reasoning, (6.11) holds for any nonnegative weights of nonnegative samples, which was to be demonstrated.

# Chapter 7

# The integral

Chapter 4 has observed that the mathematics of calculus concerns a complementary pair of questions:

- Given some function $f(t)$, what is the function's instantaneous rate of change, or *derivative,* $f'(t)$?
- Interpreting some function $f'(t)$ as an instantaneous rate of change, what is the corresponding accretion, or *integral,* $f(t)$?

Chapter 4 has built toward a basic understanding of the first question. This chapter builds toward a basic understanding of the second. The understanding of the second question constitutes the concept of the integral, one of the profoundest ideas in all of mathematics.

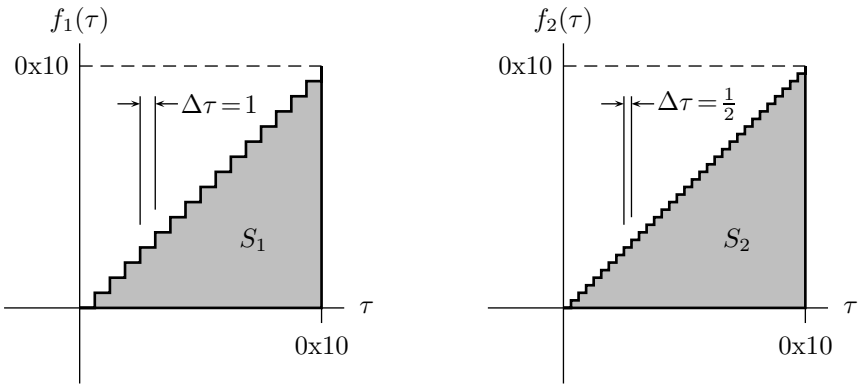
This chapter, which introduces the integral, is undeniably a hard chapter.

Experience knows no reliable way to teach the integral adequately to the uninitiated except through dozens or hundreds of pages of suitable examples and exercises, yet the book you are reading cannot be that kind of book. The sections of the present chapter concisely treat matters which elsewhere rightly command chapters or whole books of their own. Concision can be a virtue—and by design, nothing essential is omitted here—but the bold novice who wishes to learn the integral from these pages alone faces a daunting challenge. It can be done. However, for less intrepid readers who quite reasonably prefer a gentler initiation, [27] is warmly recommended.

## 7.1 The concept of the integral

An *integral* is a finite accretion or sum of an infinite number of infinitesimal elements. This section introduces the concept.

Figure 7.1: Areas representing discrete sums.




### 7.1.1 An introductory example

Consider the sums

$$\begin{aligned}
S_1 &= \sum_{k=0}^{0x10-1} k, \\
S_2 &= \frac{1}{2} \sum_{k=0}^{0x20-1} \frac{k}{2}, \\
S_4 &= \frac{1}{4} \sum_{k=0}^{0x40-1} \frac{k}{4}, \\
S_8 &= \frac{1}{8} \sum_{k=0}^{0x80-1} \frac{k}{8}, \\
&\vdots \\
S_n &= \frac{1}{n} \sum_{k=0}^{(0x10)n-1} \frac{k}{n}.
\end{aligned}$$

What do these sums represent? One way to think of them is in terms of the shaded areas of Fig. 7.1. In the figure, $S_1$ is composed of several tall, thin rectangles of width 1 and height $k$; $S_2$, of rectangles of width 1/2 and height

$k/2$.[1] As $n$ grows, the shaded region in the figure looks more and more like a triangle of base length $b = 0\text{x}10$ and height $h = 0\text{x}10$. In fact it appears that

$$\lim_{n\to\infty} S_n = \frac{bh}{2} = 0\text{x}80,$$

or more tersely

$$S_\infty = 0\text{x}80,$$

is the area the increasingly fine stairsteps approach.

Notice how we have evaluated $S_\infty$, the sum of an infinite number of infinitely narrow rectangles, without actually adding anything up. We have taken a shortcut directly to the total.

In the equation

$$S_n = \frac{1}{n} \sum_{k=0}^{(0\text{x}10)n-1} \frac{k}{n},$$

let us now change the variables

$$\begin{aligned} \tau &\leftarrow \frac{k}{n}, \\ \Delta\tau &\leftarrow \frac{1}{n}, \end{aligned}$$

to obtain the representation

$$S_n = \Delta\tau \sum_{k=0}^{(0\text{x}10)n-1} \tau;$$

or more properly,

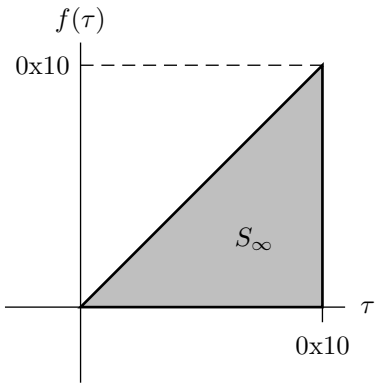
$$S_n = \sum_{k=0}^{(k|_{\tau=0\text{x}10})-1} \tau\,\Delta\tau,$$

where the notation $k|_{\tau=0\text{x}10}$ indicates the value of $k$ when $\tau = 0\text{x}10$. Then

$$S_\infty = \lim_{\Delta\tau\to 0^+} \sum_{k=0}^{(k|_{\tau=0\text{x}10})-1} \tau\,\Delta\tau,$$

[1] If the reader does not fully understand this paragraph's illustration, if the relation of the sum to the area seems unclear, the reader is urged to pause and consider the illustration carefully until he does understand it. If it still seems unclear, then the reader should probably suspend reading here and go study a good basic calculus text like [27]. The concept is important.

Figure 7.2: An area representing an infinite sum of infinitesimals. (Observe that the infinitesimal $d\tau$ is now too narrow to show on this scale. Compare against $\Delta\tau$ in Fig. 7.1.)




in which it is conventional as $\Delta\tau$ vanishes to change the symbol $d\tau \leftarrow \Delta\tau$, where $d\tau$ is the infinitesimal of Ch. 4:

$$S_\infty = \lim_{d\tau \to 0^+} \sum_{k=0}^{(k|_{\tau=0\text{x}10})-1} \tau\, d\tau.$$

The symbol $\lim_{d\tau\to 0^+} \sum_{k=0}^{(k|_{\tau=0\text{x}10})-1}$ is cumbersome, so we replace it with the new symbol[2] $\int_0^{0\text{x}10}$ to obtain the form

$$S_\infty = \int_0^{0\text{x}10} \tau\, d\tau.$$

This means, "stepping in infinitesimal intervals of $d\tau$, the sum of all $\tau\, d\tau$ from $\tau = 0$ to $\tau = 0\text{x}10$." Graphically, it is the shaded area of Fig. 7.2.

[2]Like the Greek S, $\sum$, denoting discrete summation, the seventeenth century-styled Roman S, $\int$, stands for Latin "summa," English "sum." See [66, "Long s," 14:54, 7 April 2006].

### 7.1.2 Generalizing the introductory example

Now consider a generalization of the example of § 7.1.1:

$$S_n = \frac{1}{n} \sum_{k=an}^{bn-1} f\left(\frac{k}{n}\right).$$

(In the example of § 7.1.1, $f[\tau]$ was the simple $f[\tau] = \tau$, but in general it could be any function.) With the change of variables

$$\begin{aligned} \tau &\leftarrow \frac{k}{n}, \\ \Delta\tau &\leftarrow \frac{1}{n}, \end{aligned}$$

whereby

$$\begin{aligned} k|_{\tau=a} &= an, \\ k|_{\tau=b} &= bn, \\ (k, n) &\in \mathbb{Z}, \quad n \neq 0, \end{aligned}$$

(but $a$ and $b$ need not be integers), this is

$$S_n = \sum_{k=(k|_{\tau=a})}^{(k|_{\tau=b})-1} f(\tau)\,\Delta\tau.$$

In the limit,

$$S_\infty = \lim_{d\tau\to 0^+} \sum_{k=(k|_{\tau=a})}^{(k|_{\tau=b})-1} f(\tau)\,d\tau = \int_a^b f(\tau)\,d\tau.$$

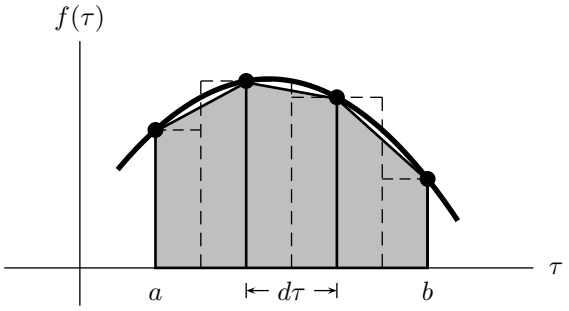
This is the *integral* of $f(\tau)$ in the interval $a < \tau < b$. It represents the area under the curve of $f(\tau)$ in that interval.

### 7.1.3 The balanced definition and the trapezoid rule

Actually, just as we have defined the derivative in the balanced form (4.15), we do well to define the integral in balanced form, too:

$$\int_a^b f(\tau)\,d\tau \equiv \lim_{d\tau\to 0^+} \left\{ \frac{f(a)\,d\tau}{2} + \sum_{k=(k|_{\tau=a})+1}^{(k|_{\tau=b})-1} f(\tau)\,d\tau + \frac{f(b)\,d\tau}{2} \right\}. \tag{7.1}$$

Figure 7.3: Integration by the trapezoid rule (7.1). Notice that the shaded and dashed areas total the same.




Here, the first and last integration samples are each balanced "on the edge," half within the integration domain and half without.

Equation (7.1) is known as the *trapezoid rule.* Figure 7.3 depicts it. The name "trapezoid" comes of the shapes of the shaded integration elements in the figure. Observe however that it makes no difference whether one regards the shaded trapezoids or the dashed rectangles as the actual integration elements; the total integration area is the same either way.[3] The important point to understand is that the integral is conceptually just a sum. It is a sum of an infinite number of infinitesimal elements as $d\tau$ tends to vanish, but a sum nevertheless; nothing more.

[3]The trapezoid rule (7.1) is perhaps the most straightforward, general, robust way to define the integral, but other schemes are possible, too. For example, taking the trapezoids in adjacent pairs—such that a pair enjoys not only a sample on each end but a third sample in the middle—one can for each pair fit a second-order curve $f(\tau) \approx (c_2)(\tau - \tau_{\text{middle}})^2 + (c_1)(\tau - \tau_{\text{middle}}) + c_0$ to the function, choosing the coefficients $c_2$, $c_1$ and $c_0$ to make the curve match the function exactly at the pair's three sample points; then substitute the area under the pair's curve (which by the end of § 7.4 we shall know how to calculate exactly) for the areas of the two trapezoids. Changing the symbol $\Delta\tau \leftarrow d\tau$ on one side of the equation to suggest coarse sampling, the result is the unexpectedly simple

$$\int_a^b f(\tau)\,\Delta\tau \approx \left[\frac{1}{3}f(a) + \frac{4}{3}f(a+\Delta\tau) + \frac{2}{3}f(a+2\,\Delta\tau) + \frac{4}{3}f(a+3\,\Delta\tau) + \frac{2}{3}f(a+4\,\Delta\tau) + \cdots + \frac{4}{3}f(b-\Delta\tau) + \frac{1}{3}f(b)\right]\Delta\tau,$$

Nothing actually requires the integration element width $d\tau$ to remain constant from element to element, incidentally. Constant widths are usually easiest to handle but variable widths find use in some cases. The only requirement is that $d\tau$ remain infinitesimal. (For further discussion of the point, refer to the treatment of the Leibnitz notation in § 4.4.2.)

## 7.2 The antiderivative and the fundamental theorem of calculus

If

$$S(x) \equiv \int_a^x g(\tau)\,d\tau,$$

then what is the derivative $dS/dx$? After some reflection, one sees that the derivative must be

$$\frac{dS}{dx} = g(x).$$

This is so because the action of the integral is to compile or accrete the area under a curve. The integral accretes area at a rate proportional to the curve's height $f(\tau)$: the higher the curve, the faster the accretion. In this way one sees that the integral and the derivative are inverse operators; the one inverts the other. The integral is the *antiderivative.*

More precisely,

$$\int_a^b \frac{df}{d\tau}\,d\tau = f(\tau)\Big|_a^b, \tag{7.2}$$

where the notation $f(\tau)|_a^b$ or $[f(\tau)]_a^b$ means $f(b) - f(a)$.

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as opposed to the trapezoidal

$$\int_a^b f(\tau)\,\Delta\tau \approx \left[\frac{1}{2}f(a) + f(a+\Delta\tau) + f(a+2\,\Delta\tau) + f(a+3\,\Delta\tau) + f(a+4\,\Delta\tau) + \cdots + f(b-\Delta\tau) + \frac{1}{2}f(b)\right]\Delta\tau$$

implied by (7.1). The curved scheme is called *Simpson's rule.* It is clever and well known.

Simpson's rule had real uses in the slide-rule era when, for practical reasons, one preferred to let $\Delta\tau$ be sloppily large, sampling a curve only a few times to estimate its integral; yet the rule is much less useful when a computer is available to do the arithmetic over an adequate number of samples. At best Simpson's rule does not help much with a computer; at worst it can yield spurious results; and because it is easy to program it tends to encourage thoughtless application. Other than in the footnote you are reading, Simpson's rule is not covered in this book.

The importance of (7.2), fittingly named the *fundamental theorem of calculus*,[4] can hardly be overstated. As the formula which ties together the complementary pair of questions asked at the chapter's start, (7.2) is of utmost importance in the practice of mathematics. The idea behind the formula is indeed simple once grasped, but to grasp the idea firmly in the first place is not entirely trivial.[5] The idea is simple but big. The reader is urged to pause now and ponder the formula thoroughly until he feels reasonably confident that indeed he does grasp it and the important idea it represents. One is unlikely to do much higher mathematics without this formula.

As an example of the formula's use, consider that because $(d/d\tau)(\tau^3/6) = \tau^2/2$, it follows that

$$\int_2^x \frac{\tau^2\,d\tau}{2} = \int_2^x \frac{d}{d\tau}\left(\frac{\tau^3}{6}\right) d\tau = \left.\frac{\tau^3}{6}\right|_2^x = \frac{x^3 - 8}{6}.$$

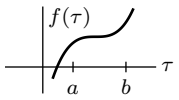
Gathering elements from (4.21) and from Tables 5.2 and 5.3, Table 7.1 lists a handful of the simplest, most useful derivatives for antiderivative use. Section 9.1 speaks further of the antiderivative.

[4] [27, § 11.6][55, § 5-4][66, "Fundamental theorem of calculus," 06:29, 23 May 2006]

[5] Having read from several calculus books and, like millions of others perhaps including the reader, having sat years ago in various renditions of the introductory calculus lectures in school, the author has never yet met a more convincing demonstration of (7.2) than the formula itself. Somehow the underlying idea is too simple, too profound to explain. It's like trying to explain how to drink water, or how to count or to add. Elaborate explanations and their attendant constructs and formalities are indeed possible to contrive, but the idea itself is so simple that somehow such contrivances seem to obscure the idea more than to reveal it.

One ponders the formula (7.2) a while, then the idea dawns on him.

If you want some help pondering, try this: Sketch some arbitrary function $f(\tau)$ on a set of axes at the bottom of a piece of paper—some squiggle of a curve like




will do nicely—then on a separate set of axes directly above the first, sketch the corresponding slope function $df/d\tau$. Mark two points $a$ and $b$ on the common horizontal axis; then on the upper, $df/d\tau$ plot, shade the integration area under the curve. Now consider (7.2) in light of your sketch.

There. Does the idea not dawn?

Another way to see the truth of the formula begins by canceling its $(1/d\tau)\,d\tau$ to obtain the form $\int_{\tau=a}^{b} df = f(\tau)|_a^b$. If this way works better for you, fine; but make sure that you understand it the other way, too.

Table 7.1: Basic derivatives for the antiderivative.

$$\int_a^b \frac{df}{d\tau}\,d\tau = f(\tau)\Big|_a^b$$

$$\begin{aligned}
\tau^{a-1} &= \frac{d}{d\tau}\left(\frac{\tau^a}{a}\right), & a &\neq 0 \\
\frac{1}{\tau} &= \frac{d}{d\tau}\ln\tau, & \ln 1 &= 0 \\
\exp\tau &= \frac{d}{d\tau}\exp\tau, & \exp 0 &= 1 \\
\cos\tau &= \frac{d}{d\tau}\sin\tau, & \sin 0 &= 0 \\
\sin\tau &= \frac{d}{d\tau}\left(-\cos\tau\right), & \cos 0 &= 1
\end{aligned}$$

## 7.3 Operators, linearity and multiple integrals

This section presents the operator concept, discusses linearity and its consequences, treats the commutivity of the summational and integrodifferential operators, and introduces the multiple integral.

### 7.3.1 Operators

An *operator* is a mathematical agent that combines several values of a function.

Such a definition, unfortunately, is extraordinarily unilluminating to those who do not already know what it means. A better way to introduce the operator is by giving examples. Operators include $+$, $-$, multiplication, division, $\sum$, $\prod$, $\int$ and $\partial$. The essential action of an operator is to take several values of a function and combine them in some way. For example, $\prod$ is an operator in

$$\prod_{j=1}^{5}(2j-1) = (1)(3)(5)(7)(9) = 0\text{x}3\text{B}1.$$

Notice that the operator has acted to remove the variable $j$ from the expression $2j-1$. The $j$ appears on the equation's left side but not on its right. The operator has used the variable up. Such a variable, used up by an operator, is a *dummy variable,* as encountered earlier in § 2.3.

### 7.3.2 A formalism

But then how are + and − operators? They don't use any dummy variables up, do they? Well, it depends on how you look at it. Consider the sum $S = 3 + 5$. One can write this as

$$S = \sum_{k=0}^{1} f(k),$$

where

$$f(k) \equiv \begin{cases} 3 & \text{if } k = 0, \\ 5 & \text{if } k = 1, \\ \text{undefined} & \text{otherwise.} \end{cases}$$

Then,

$$S = \sum_{k=0}^{1} f(k) = f(0) + f(1) = 3 + 5 = 8.$$

By such admittedly excessive formalism, the + operator can indeed be said to use a dummy variable up. The point is that + is in fact an operator just like the others.

Another example of the kind:

$$\begin{aligned} D &= g(z) - h(z) + p(z) + q(z) \\ &= g(z) - h(z) + p(z) - 0 + q(z) \\ &= \Phi(0, z) - \Phi(1, z) + \Phi(2, z) - \Phi(3, z) + \Phi(4, z) \\ &= \sum_{k=0}^{4} (-)^k \Phi(k, z), \end{aligned}$$

where

$$\Phi(k, z) \equiv \begin{cases} g(z) & \text{if } k = 0, \\ h(z) & \text{if } k = 1, \\ p(z) & \text{if } k = 2, \\ 0 & \text{if } k = 3, \\ q(z) & \text{if } k = 4, \\ \text{undefined} & \text{otherwise.} \end{cases}$$

Such unedifying formalism is essentially useless in applications, except as a vehicle for definition. Once you understand why + and − are operators just as $\sum$ and $\int$ are, you can forget the formalism. It doesn't help much.

### 7.3.3 Linearity

A function $f(z)$ is *linear* iff (if and only if) it has the properties

$$\begin{aligned} f(z_1 + z_2) &= f(z_1) + f(z_2), \\ f(\alpha z) &= \alpha f(z), \\ f(0) &= 0. \end{aligned}$$

The functions $f(z) = 3z$, $f(u, v) = 2u - v$ and $f(z) = 0$ are examples of linear functions. Nonlinear functions include[6] $f(z) = z^2$, $f(u, v) = \sqrt{uv}$, $f(t) = \cos \omega t$, $f(z) = 3z + 1$ and even $f(z) = 1$.

An operator $L$ is linear iff it has the properties

$$\begin{aligned} L(f_1 + f_2) &= Lf_1 + Lf_2, \\ L(\alpha f) &= \alpha Lf, \\ L(0) &= 0. \end{aligned}$$

The operators $\sum$, $\int$, $+$, $-$ and $\partial$ are examples of linear operators. For instance,[7]

$$\frac{d}{dz}[f_1(z) + f_2(z)] = \frac{df_1}{dz} + \frac{df_2}{dz}.$$

Nonlinear operators include multiplication, division and the various trigonometric functions, among others.

Section 16.1.2 will have more to say about operators and their notation.

### 7.3.4 Summational and integrodifferential commutivity

Consider the sum

$$S_1 = \sum_{k=a}^{b} \left[ \sum_{j=p}^{q} \frac{x^k}{j!} \right].$$

[6] If $3z + 1$ is a *linear expression,* then how is not $f(z) = 3z + 1$ a *linear function?* Answer: it is partly a matter of purposeful definition, partly of semantics. The equation $y = 3x + 1$ plots a line, so the expression $3z + 1$ is literally "linear" in this sense; but the definition has more purpose to it than merely this. When you see the linear expression $3z + 1$, think $3z + 1 = 0$, then $g(z) = 3z = -1$. The $g(z) = 3z$ is linear; the $-1$ is the constant value it targets. That's the sense of it.

[7] You don't see $d$ in the list of linear operators? But $d$ in this context is really just another way of writing $\partial$, so, yes, $d$ is linear, too. See § 4.4.2.

This is a sum of the several values of the expression $x^k/j!$, evaluated at every possible pair $(j,k)$ in the indicated domain. Now consider the sum

$$S_2 = \sum_{j=p}^{q} \left[ \sum_{k=a}^{b} \frac{x^k}{j!} \right].$$

This is evidently a sum of the same values, only added in a different order. Apparently $S_1 = S_2$. Reflection along these lines must soon lead the reader to the conclusion that, in general,

$$\sum_k \sum_j f(j,k) = \sum_j \sum_k f(j,k).$$

Now consider that an integral is just a sum of many elements, and that a derivative is just a difference of two elements. Integrals and derivatives must then have the same commutative property discrete sums have. For example,

$$\begin{aligned} \int_{v=-\infty}^{\infty} \int_{u=a}^{b} f(u,v)\,du\,dv &= \int_{u=a}^{b} \int_{v=-\infty}^{\infty} f(u,v)\,dv\,du; \\ \int \sum_k f_k(v)\,dv &= \sum_k \int f_k(v)\,dv; \\ \frac{\partial}{\partial v} \int f\,du &= \int \frac{\partial f}{\partial v}\,du. \end{aligned}$$

In general,

$$L_v L_u f(u,v) = L_u L_v f(u,v), \tag{7.3}$$

where $L$ is any of the linear operators $\sum$, $\int$ or $\partial$.

Some convergent summations, like

$$\sum_{k=0}^{\infty} \sum_{j=0}^{1} \frac{(-)^j}{2k+j+1},$$

diverge once reordered, as

$$\sum_{j=0}^{1} \sum_{k=0}^{\infty} \frac{(-)^j}{2k+j+1}.$$

One cannot blithely swap operators here. This is not because swapping is wrong, but rather because the inner sum after the swap diverges, hence the

outer sum after the swap has no concrete summand on which to work. (*Why* does the inner sum after the swap diverge? Answer: $1 + 1/3 + 1/5 + \cdots = [1] + [1/3 + 1/5] + [1/7 + 1/9 + 1/0\text{xB} + 1/0\text{xD}] + \cdots > 1[1/4] + 2[1/8] + 4[1/0\text{x}10] + \cdots = 1/4 + 1/4 + 1/4 + \cdots$. See also § 8.10.5.) For a more twisted example of the same phenomenon, consider[8]

$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots = \left(1 - \frac{1}{2} - \frac{1}{4}\right) + \left(\frac{1}{3} - \frac{1}{6} - \frac{1}{8}\right) + \cdots,$$

which associates two negative terms with each positive, but still seems to omit no term. Paradoxically, then,

$$\begin{aligned} 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots &= \left(\frac{1}{2} - \frac{1}{4}\right) + \left(\frac{1}{6} - \frac{1}{8}\right) + \cdots \\ &= \frac{1}{2} - \frac{1}{4} + \frac{1}{6} - \frac{1}{8} + \cdots \\ &= \frac{1}{2}\left(1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots\right), \end{aligned}$$

or so it would seem, but cannot be, for it claims falsely that the sum is half itself. A better way to have handled the example might have been to write the series as

$$\lim_{n \to \infty} \left\{1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots + \frac{1}{2n-1} - \frac{1}{2n}\right\}$$

in the first place, thus explicitly specifying equal numbers of positive and negative terms.[9] So specifying would have prevented the error. In the earlier example,

$$\lim_{n \to \infty} \sum_{k=0}^{n} \sum_{j=0}^{1} \frac{(-)^j}{2k + j + 1}$$

[8] [2, § 1.2.3]

[9] Some students of professional mathematics would assert that the false conclusion had been reached through lack of rigor. Well, maybe. This writer however does not feel sure that *rigor* is quite the right word for what was lacking here. Professional mathematics does bring an elegant notation and a set of formalisms which serve ably to spotlight certain limited kinds of blunders, but these are blunders no less by the applied approach. The stalwart Leonhard Euler—arguably the greatest series-smith in mathematical history—wielded his heavy analytical hammer in thunderous strokes before professional mathematics had conceived the notation or the formalisms. If the great Euler did without, then you and I might not always be forbidden to follow his robust example. See also footnote 11.

On the other hand, the professional approach is worth study if you have the time. Recommended introductions include [39], preceded if necessary by [27] and/or [2, Ch. 1].

likewise would have prevented the error, or at least have made the error explicit.

The *conditional convergence*[10] of the last paragraph, which can occur in integrals as well as in sums, seldom poses much of a dilemma in practice. One can normally swap summational and integrodifferential operators with little worry. The reader however should at least be aware that conditional convergence troubles can arise where a summand or integrand varies in sign or phase.

### 7.3.5 Multiple integrals

Consider the function

$$f(u, w) = \frac{u^2}{w}.$$

Such a function would not be plotted as a curved line in a plane, but rather as a curved *surface* in a three-dimensional space. Integrating the function seeks not the area under the curve but rather the volume under the surface:

$$V = \int_{u_1}^{u_2} \int_{w_1}^{w_2} \frac{u^2}{w} \, dw \, du.$$

This is a *double integral.* Inasmuch as it can be written in the form

$$\begin{aligned} V &= \int_{u_1}^{u_2} g(u) \, du, \\ g(u) &\equiv \int_{w_1}^{w_2} \frac{u^2}{w} \, dw, \end{aligned}$$

its effect is to cut the area under the surface into flat, upright slices, then the slices crosswise into tall, thin towers. The towers are integrated over $w$ to constitute the slice, then the slices over $u$ to constitute the volume.

In light of § 7.3.4, evidently nothing prevents us from swapping the integrations: $u$ first, then $w$. Hence

$$V = \int_{w_1}^{w_2} \int_{u_1}^{u_2} \frac{u^2}{w} \, du \, dw.$$

And indeed this makes sense, doesn't it? What difference should it make whether we add the towers by rows first then by columns, or by columns first then by rows? The total volume is the same in any case—albeit the integral

[10] [39, § 16]

over $w$ is potentially ill-behaved[11] near $w = 0$; so that, if for instance $w_1$ were negative, $w_2$ were positive, and both were real, one might rather write the double integral as[12]

$$V = \lim_{\epsilon \to 0^+} \left( \int_{w_1}^{-\epsilon} + \int_{+\epsilon}^{w_2} \right) \int_{u_1}^{u_2} \frac{u^2}{w} \, du \, dw.$$

Double integrations arise very frequently in applications. Triple integrations arise about as often. For instance, if $\mu(\mathbf{r}) = \mu(x, y, z)$ represents the variable mass density of some soil,[13] then the total soil mass in some rectangular volume is

$$M = \int_{x_1}^{x_2} \int_{y_1}^{y_2} \int_{z_1}^{z_2} \mu(x, y, z) \, dz \, dy \, dx.$$

As a concise notational convenience, the last is likely to be written

$$M = \int_V \mu(\mathbf{r}) \, d\mathbf{r},$$

where the $V$ stands for "volume" and is understood to imply a triple integration. Similarly for the double integral,

$$V = \int_S f(\boldsymbol{\rho}) \, d\boldsymbol{\rho},$$

where the $S$ stands for "surface" and is understood to imply a double integration.

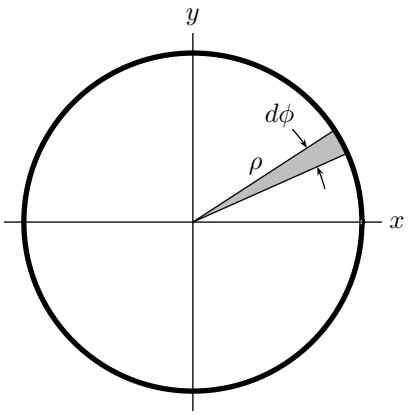
Even more than three nested integrations are possible. If we integrated over time as well as space, the integration would be fourfold. A spatial Fourier transform (§ 18.9) implies a triple integration; and its inverse, another triple: a sixfold integration altogether. Manifold nesting of integrals is thus not just a theoretical mathematical topic; it arises in sophisticated real-world engineering models. The topic concerns us here for this reason.

[11] A great deal of ink is spilled in the applied mathematical literature when summations and/or integrations are interchanged. The author tends to recommend saving the ink, for pure and applied mathematics want different styles. What usually matters in applications is not whether a particular summation or integration satisfies some formal test but rather whether one clearly understands the summand to be summed or the integrand to be integrated. See also footnote 9.

[12] It is interesting to consider the effect of withdrawing the integral's limit at $-\epsilon$ to $-2\epsilon$, as $\lim_{\epsilon \to 0^+} \left( \int_{w_1}^{-2\epsilon} + \int_{+\epsilon}^{w_2} \right) \int_{u_1}^{u_2} \frac{u^2}{w} \, du \, dw$; for, surprisingly—despite that the parameter $\epsilon$ is vanishing anyway—the withdrawal does alter the integral unless the limit at $+\epsilon$ also is withdrawn. The reason is that $\lim_{\epsilon \to 0^+} \int_{\epsilon}^{2\epsilon} (1/w) \, dw = \ln 2 \neq 0$.

[13] Conventionally the Greek letter $\rho$ not $\mu$ is used for density, but it happens that we need the letter $\rho$ for a different purpose later in the paragraph.

Figure 7.4: The area of a circle.




## 7.4 Areas and volumes

By composing and solving appropriate integrals, one can calculate the perimeters, areas and volumes of interesting common shapes and solids.

### 7.4.1 The area of a circle

Figure 7.4 depicts an element of a circle's area. The element has wedge shape, but inasmuch as the wedge is infinitesimally narrow, the wedge is indistinguishable from a triangle of base length $\rho\,d\phi$ and height $\rho$. The area of such a triangle is $A_{\text{triangle}} = \rho^2\,d\phi/2$. Integrating the many triangles, we find the circle's area to be

$$A_{\text{circle}} = \int_{\phi=-\pi}^{\pi} A_{\text{triangle}} = \int_{-\pi}^{\pi} \frac{\rho^2\,d\phi}{2} = \frac{2\pi\rho^2}{2}. \tag{7.4}$$

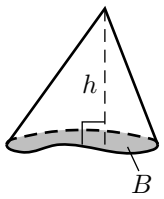
(The numerical value of $2\pi$—the circumference or perimeter of the unit circle—we have not calculated yet. We will calculate it in § 8.11.)

### 7.4.2 The volume of a cone

One can calculate the volume of any cone (or pyramid) if one knows its base area $B$ and its altitude $h$ measured normal[14] to the base. Refer to Fig. 7.5.

[14] *Normal* here means "at right angles."

Figure 7.5: The volume of a cone.




A cross-section of a cone, cut parallel to the cone's base, has the same shape the base has but a different scale. If coordinates are chosen such that the altitude $h$ runs in the $\hat{\mathbf{z}}$ direction with $z = 0$ at the cone's vertex, then the cross-sectional area is evidently[15] $(B)(z/h)^2$. For this reason, the cone's volume is

$$V_{\text{cone}} = \int_0^h (B)\left(\frac{z}{h}\right)^2 dz = \frac{B}{h^2}\int_0^h z^2\,dz = \frac{B}{h^2}\left(\frac{h^3}{3}\right) = \frac{Bh}{3}. \tag{7.5}$$

### 7.4.3 The surface area and volume of a sphere

Of a sphere, Fig. 7.6, one wants to calculate both the surface area and the volume. For the surface area, the sphere's surface is sliced vertically down the $z$ axis into narrow constant-$\phi$ tapered strips (each strip broadest at the sphere's equator, tapering to points at the sphere's $\pm z$ poles) and horizontally across the $z$ axis into narrow constant-$\theta$ rings, as in Fig. 7.7. A surface element so produced (seen as shaded in the latter figure) evidently

[15]The fact may admittedly not be evident to the reader at first glance. If it is not yet evident to you, then ponder Fig. 7.5 a moment. Consider what it means to cut parallel to a cone's base a cross-section of the cone, and how cross-sections cut nearer a cone's vertex are smaller though the same shape. What if the base were square? Would the cross-sectional area not be $(B)(z/h)^2$ in that case? What if the base were a right triangle with equal legs—in other words, half a square? What if the base were some other strange shape like the base depicted in Fig. 7.5? Could such a strange shape not also be regarded as a definite, well-characterized part of a square? (With a pair of scissors one can cut any shape from a square piece of paper, after all.) Thinking along such lines must soon lead one to the insight that the parallel-cut cross-sectional area of a cone can be nothing other than $(B)(z/h)^2$, regardless of the base's shape.

Figure 7.6: A sphere.

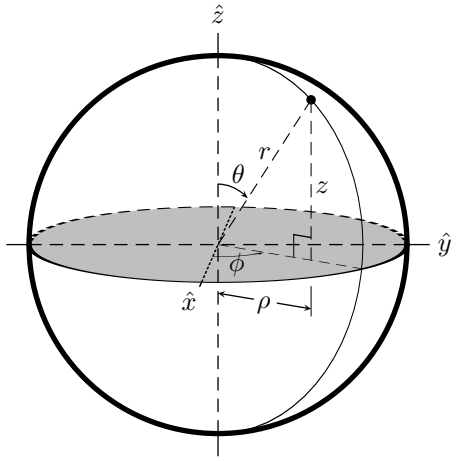
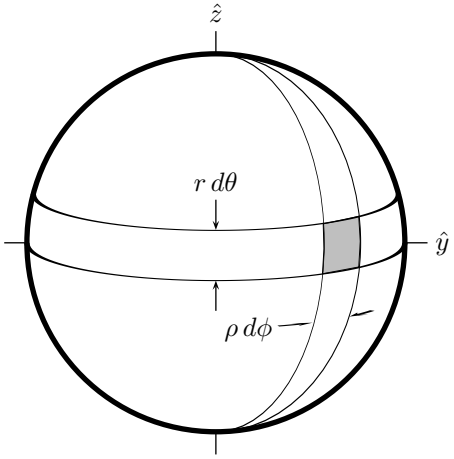



Figure 7.7: An element of the sphere's surface (see Fig. 7.6).



has the area[16]

$$dS = (r\,d\theta)(\rho\,d\phi) = r^2 \sin\theta\,d\theta\,d\phi.$$

The sphere's total surface area then is the sum of all such elements over the sphere's entire surface:

$$\begin{aligned} S_{\text{sphere}} &= \int_{\phi=-\pi}^{\pi}\int_{\theta=0}^{\pi} dS \\ &= \int_{\phi=-\pi}^{\pi}\int_{\theta=0}^{\pi} r^2 \sin\theta\,d\theta\,d\phi \\ &= r^2\int_{\phi=-\pi}^{\pi} [-\cos\theta]_0^{\pi}\,d\phi \\ &= r^2\int_{\phi=-\pi}^{\pi} [2]\,d\phi \\ &= 4\pi r^2, \end{aligned} \tag{7.6}$$

where we have used the fact from Table 7.1 that $\sin\tau = (d/d\tau)(-\cos\tau)$.

Having computed the sphere's surface area, one can find its volume just as § 7.4.1 has found a circle's area—except that instead of dividing the circle into many narrow triangles, one divides the sphere into many narrow *cones,* each cone with base area $dS$ and altitude $r$, with the vertices of all the cones meeting at the sphere's center. Per (7.5), the volume of one such cone is $V_{\text{cone}} = r\,dS/3$. Hence,

$$V_{\text{sphere}} = \oint_S V_{\text{cone}} = \oint_S \frac{r\,dS}{3} = \frac{r}{3}\oint_S dS = \frac{r}{3}S_{\text{sphere}},$$

where the useful symbol

$$\oint_S$$

indicates *integration over a closed surface.* In light of (7.6), the total volume is

$$V_{\text{sphere}} = \frac{4\pi r^3}{3}. \tag{7.7}$$

[16] It can be shown, incidentally—the details are left as an exercise—that $dS = -r\,dz\,d\phi$ also. The subsequent integration arguably goes a little easier if $dS$ is accepted in this mildly clever form. The form is interesting in any event if one visualizes the specific, annular area the expression $\int_{\phi=-\pi}^{\pi} dS = -2\pi r\,dz$ represents: evidently, unexpectedly, a precisely equal portion of the sphere's surface corresponds to each equal step along the $z$ axis, pole to pole; so, should you slice an unpeeled apple into parallel slices of equal thickness, though some slices will be bigger across and thus heavier than others, each slice curiously must take an equal share of the apple's skin. (This is true, anyway, if you judge Fig. 7.7 to represent an apple. The author's children judge it to represent "the Death Star with a square gun," so maybe it depends on your point of view.)

(One can compute the same spherical volume more prosaically, without reference to cones, by writing $dV = r^2 \sin\theta \, dr \, d\theta \, d\phi$ then integrating $\int_V dV$. The derivation given above, however, is preferred because it lends the additional insight that a sphere can sometimes be viewed as a great cone rolled up about its own vertex. The circular area derivation of § 7.4.1 lends an analogous insight: that a circle can sometimes be viewed as a great triangle rolled up about *its* own vertex.)

## 7.5 Checking an integration

Dividing 0x46B/0xD = 0x57 with a pencil, how does one check the result?[17] Answer: by multiplying (0x57)(0xD) = 0x46B. Multiplication inverts division. Easier than division, multiplication provides a quick, reliable check.

Likewise, integrating

$$\int_a^b \frac{\tau^2}{2} \, d\tau = \frac{b^3 - a^3}{6}$$

with a pencil, how does one check the result? Answer: by differentiating

$$\left[ \frac{\partial}{\partial b} \left( \frac{b^3 - a^3}{6} \right) \right]_{b=\tau} = \frac{\tau^2}{2}.$$

Differentiation inverts integration. Easier than integration, differentiation like multiplication provides a quick, reliable check.

More formally, according to (7.2),

$$S \equiv \int_a^b \frac{df}{d\tau} \, d\tau = f(b) - f(a). \tag{7.8}$$

Differentiating (7.8) with respect to $b$ and $a$,

$$\begin{aligned} \left. \frac{\partial S}{\partial b} \right|_{b=\tau} &= \frac{df}{d\tau}, \\ \left. \frac{\partial S}{\partial a} \right|_{a=\tau} &= -\frac{df}{d\tau}. \end{aligned} \tag{7.9}$$

[17] Admittedly, few readers will ever have done much such multidigit *hexadecimal* arithmetic with a pencil, but, hey, go with it. In decimal, it's 1131/13 = 87.

Actually, hexadecimal is just proxy for binary (see Appendix A), and long division in straight binary is kind of fun. If you have never tried it, you might. It is simpler than decimal or hexadecimal division, and it's how computers divide. The insight gained is worth the trial.

Either line of (7.9) can be used to check an integration. Evaluating (7.8) at $b = a$ yields

$$S|_{b=a} = 0, \tag{7.10}$$

which can be used to check further.[18]

As useful as (7.9) and (7.10) are, they nevertheless serve only integrals with variable limits. They are of little use to check *definite integrals* like (9.14) below, which lack variable limits to differentiate. However, many or most integrals one meets in practice have or can be given variable limits. Equations (7.9) and (7.10) do serve such *indefinite integrals.*

It is a rare irony of mathematics that, although numerically differentiation is indeed harder than integration, analytically precisely the opposite is true. Analytically, differentiation is the easier. So far the book has introduced only easy integrals, but Ch. 9 will bring much harder ones. Even experienced mathematicians are apt to err in analyzing these. Reversing an integration by taking an easy derivative is thus an excellent way to check a hard-earned integration result.

## 7.6 Contour integration

To this point we have considered only integrations in which the variable of integration advances in a straight line from one point to another: for instance, $\int_a^b f(\tau)\,d\tau$, in which the function $f(\tau)$ is evaluated at $\tau = a, a + d\tau, a + 2d\tau, \ldots, b$. The integration variable is a real-valued scalar which can do nothing but make a straight line from $a$ to $b$.

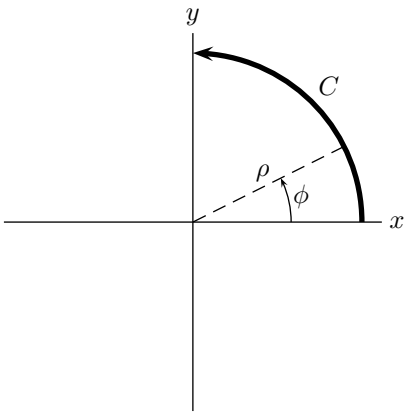
Such is not the case when the integration variable is a vector. Consider the integral

$$S = \int_{\mathbf{r}=\hat{\mathbf{x}}\rho}^{\hat{\mathbf{y}}\rho} (x^2 + y^2)\,d\ell,$$

where $d\ell$ is the infinitesimal length of a step along the path of integration. What does this integral mean? Does it mean to integrate from $\mathbf{r} = \hat{\mathbf{x}}\rho$ to $\mathbf{r} = 0$, then from there to $\mathbf{r} = \hat{\mathbf{y}}\rho$? Or does it mean to integrate along the arc of Fig. 7.8? The two paths of integration begin and end at the same points, but they differ in between, and the integral certainly does not come out the same both ways. Yet many other paths of integration from $\hat{\mathbf{x}}\rho$ to $\hat{\mathbf{y}}\rho$ are possible, not just these two.

[18] Using (7.10) to check the example, $(b^3 - a^3)/6|_{b=a} = 0$.

Figure 7.8: A contour of integration.




Because multiple paths are possible, we must be more specific:

$$S = \int_C (x^2 + y^2)\, d\ell,$$

where $C$ stands for "contour" and means in this example the specific contour of Fig. 7.8. In the example, $x^2 + y^2 = \rho^2$ (by the Pythagorean theorem) and $d\ell = \rho\, d\phi$, so

$$S = \int_C \rho^2\, d\ell = \int_0^{2\pi/4} \rho^3\, d\phi = \frac{2\pi}{4}\rho^3.$$

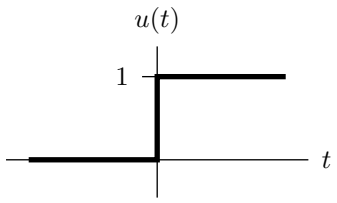
In the example the contour is open, but closed contours which begin and end at the same point are also possible, indeed common. The useful symbol

$$\oint$$

indicates *integration over a closed contour.* It means that the contour ends where it began: the loop is closed. The contour of Fig. 7.8 would be closed, for instance, if it continued to $\mathbf{r} = 0$ and then back to $\mathbf{r} = \hat{\mathbf{x}}\rho$.

Besides applying where the variable of integration is a vector, contour integration applies equally where the variable of integration is a complex scalar. In the latter case some interesting mathematics emerge, as we shall see in §§ 8.8 and 9.5.

Figure 7.9: The Heaviside unit step $u(t)$.




## 7.7 Discontinuities

The polynomials and trigonometrics studied to this point in the book offer flexible means to model many physical phenomena of interest, but one thing they do not model gracefully is the simple discontinuity. Consider a mechanical valve opened at time $t = t_o$. The flow $x(t)$ past the valve is

$$x(t) = \begin{cases} 0, & t < t_o; \\ x_o, & t > t_o. \end{cases}$$

One can write this more concisely in the form

$$x(t) = u(t - t_o)x_o,$$

where $u(t)$ is the *Heaviside unit step,*

$$u(t) \equiv \begin{cases} 0, & t < 0; \\ 1, & t > 0; \end{cases} \tag{7.11}$$

plotted in Fig. 7.9.

The derivative of the Heaviside unit step is the curious *Dirac delta*

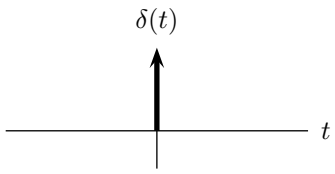
$$\delta(t) \equiv \frac{d}{dt}u(t), \tag{7.12}$$

also called[19] the *impulse function,* plotted in Fig. 7.10. This function is zero everywhere except at $t = 0$, where it is infinite, with the property that

$$\int_{-\infty}^{\infty} \delta(t)\,dt = 1, \tag{7.13}$$

[19] [35, § 19.5]

Figure 7.10: The Dirac delta $\delta(t)$.




and the interesting consequence that

$$\int_{-\infty}^{\infty} \delta(t - t_o) f(t)\, dt = f(t_o) \tag{7.14}$$

for any function $f(t)$. (Equation 7.14 is the *sifting property* of the Dirac delta.)[20]

[20]It seems inadvisable for the narrative to digress at this point to explore $u(z)$ and $\delta(z)$, the unit step and delta of a complex argument, although by means of Fourier analysis (Ch. 18) or by conceiving the Dirac delta as an infinitely narrow Gaussian pulse (§ 18.5) it could perhaps do so. The book has more pressing topics to treat. For the book's present purpose the interesting action of the two functions is with respect to the real argument $t$.

In the author's country at least, a sort of debate seems to have run for decades between professional and applied mathematicians over the Dirac delta $\delta(t)$. Some professional mathematicians seem to have objected that $\delta(t)$ is not a function, inasmuch as it lacks certain properties common to functions as they define them [48, § 2.4][17]. From the applied point of view the objection is admittedly a little hard to understand, until one realizes that it is more a dispute over methods and definitions than over facts. What the professionals seem to be saying is that $\delta(t)$ does not fit as neatly as they would like into the abstract mathematical framework they had established for functions in general before Paul Dirac came along in 1930 [66, "Paul Dirac," 05:48, 25 May 2006] and slapped his disruptive $\delta(t)$ down on the table. The objection is not so much that $\delta(t)$ is not allowed as it is that professional mathematics for years after 1930 lacked a fully coherent theory for it.

It's a little like the six-fingered man in Goldman's *The Princess Bride* [26]. If I had established a definition of "nobleman" which subsumed "human," whose relevant traits in my definition included five fingers on each hand, when the six-fingered Count Rugen appeared on the scene, then you would expect me to adapt my definition, wouldn't you? By my preëxisting definition, strictly speaking, the six-fingered count is "not a nobleman"; but such exclusion really tells one more about flaws in the definition than it does about the count.

Whether the professional mathematician's definition of the *function* is flawed, of course, is not for this writer to judge. Even if not, however, the fact of the Dirac delta dispute,

The Dirac delta is defined for vectors, too, such that

$$\int_V \delta(\mathbf{r})\,d\mathbf{r} = 1. \tag{7.15}$$

## 7.8 Remarks (and exercises)

The concept of the integral is relatively simple once grasped, but its implications are broad, deep and hard. This chapter is short. One reason introductory calculus texts run so long is that they include many, many pages of integration examples and exercises. The reader who desires a gentler introduction to the integral might consult among others the textbook the chapter's introduction has recommended.

Even if this book is not an instructional textbook, it seems not meet that it should include no exercises at all here. Here are a few. Some of them do need material from later chapters, so you should not expect to be able to complete them all now. The harder ones are marked with *asterisks. Work the exercises if you like.

1. Evaluate (a) $\int_0^x \tau\,d\tau$; (b) $\int_0^x \tau^2\,d\tau$. (Answer: $x^2/2$; $x^3/3$.)
2. Evaluate (a) $\int_1^x (1/\tau^2)\,d\tau$; (b) $\int_a^x 3\tau^{-2}\,d\tau$; (c) $\int_a^x C\tau^n\,d\tau$; (d) $\int_0^x (a_2\tau^2 + a_1\tau)\,d\tau$; *(e) $\int_1^x (1/\tau)\,d\tau$.
3. *Evaluate (a) $\int_0^x \sum_{k=0}^{\infty} \tau^k\,d\tau$; (b) $\sum_{k=0}^{\infty} \int_0^x \tau^k\,d\tau$; (c) $\int_0^x \sum_{k=0}^{\infty} (\tau^k/k!)\,d\tau$.
4. Evaluate $\int_0^x \exp \alpha\tau\,d\tau$.
5. Evaluate (a) $\int_{-2}^{5} (3\tau^2 - 2\tau^3)\,d\tau$; (b) $\int_5^{-2} (3\tau^2 - 2\tau^3)\,d\tau$. Work the exercise by hand in hexadecimal and give the answer in hexadecimal.
6. Evaluate $\int_1^{\infty} (3/\tau^2)\,d\tau$.

coupled with the difficulty we applied mathematicians experience in trying to understand the reason the dispute even exists, has unfortunately surrounded the Dirac delta with a kind of mysterious aura, an elusive sense that $\delta(t)$ hides subtle mysteries—when what it really hides is an internal discussion of words and means among the professionals. The professionals who had established the theoretical framework before 1930 justifiably felt reluctant to throw the whole framework away because some scientists and engineers like us came along one day with a useful new function which didn't quite fit, but that was the professionals' problem not ours. To us the Dirac delta $\delta(t)$ is just a function. The internal discussion of words and means, we leave to the professionals, who know whereof they speak.

7. *Evaluate the integral of the example of § 7.6 along the alternate contour suggested there, from $\hat{\mathbf{x}}\rho$ to 0 to $\hat{\mathbf{y}}\rho$.

8. Evaluate (a) $\int_0^x \cos\omega\tau\,d\tau$; (b) $\int_0^x \sin\omega\tau\,d\tau$; *(c)[21] $\int_0^x \tau\sin\omega\tau\,d\tau$.

9. *Evaluate[22] (a) $\int_1^x \sqrt{1+2\tau}\,d\tau$; (b) $\int_x^a [(\cos\sqrt{\tau})/\sqrt{\tau}]\,d\tau$.

10. *Evaluate[23] (a) $\int_0^x [1/(1+\tau^2)]\,d\tau$ (answer: $\arctan x$); (b) $\int_0^x [(4+i3)/\sqrt{2-3\tau^2}]\,d\tau$ (hint: the answer involves another inverse trigonometric).

11. **Evaluate (a) $\int_{-\infty}^x \exp[-\tau^2/2]\,d\tau$; (b) $\int_{-\infty}^{\infty} \exp[-\tau^2/2]\,d\tau$.

The last exercise in particular requires some experience to answer. Moreover, it requires a developed sense of applied mathematical style to put the answer in a pleasing form (the right form for part b is very different from that for part a). Some of the easier exercises, of course, you should be able to work right now.

The point of the exercises is to illustrate how hard integrals can be to solve, and in fact how easy it is to come up with an integral which no one really knows how to solve very well. Some solutions to the same integral are better than others (easier to manipulate, faster to numerically calculate, etc.) yet not even the masters can solve them all in practical ways. On the other hand, integrals which arise in practice often can be solved very well with sufficient cleverness—and the more cleverness you develop, the more such integrals you can solve. The ways to solve them are myriad. The mathematical art of solving diverse integrals is well worth cultivating.

Chapter 9 introduces some of the basic, most broadly useful integral-solving techniques. Before addressing techniques of integration, however, as promised earlier we turn our attention in Chapter 8 back to the derivative, applied in the form of the Taylor series.

[21] [55, § 8-2]
[22] [55, § 5-6]
[23] [55, back endpaper]

# Chapter 8

# The Taylor series

The Taylor series is a power series which fits a function in a limited domain neighborhood. Fitting a function in such a way brings two advantages:

- it lets us take derivatives and integrals in the same straightforward way (4.20) we take them with any power series; and
- it implies a simple procedure to calculate the function numerically.

This chapter introduces the Taylor series and some of its incidents. It also derives Cauchy's integral formula. The chapter's early sections prepare the ground for the treatment of the Taylor series proper in § 8.3.[1]

[1]Because even at the applied level the proper derivation of the Taylor series involves mathematical induction, analytic continuation and the matter of convergence domains, no balance of rigor the chapter might strike seems wholly satisfactory. The chapter errs maybe toward too much rigor; for, with a little less, most of §§ 8.1, 8.2, 8.4 and 8.6 would cease to be necessary. For the impatient, to read only the following sections might not be an unreasonable way to shorten the chapter: §§ 8.3, 8.5, 8.8, 8.9 and 8.11, plus the introduction of § 8.1.

From another point of view, the chapter errs maybe toward too little rigor. Some pretty constructs of pure mathematics serve the Taylor series and Cauchy's integral formula. However, such constructs drive the applied mathematician on too long a detour. The chapter as written represents the most nearly satisfactory compromise the writer has been able to attain.

## 8.1 The power-series expansion of $1/(1-z)^{n+1}$

Before approaching the Taylor series proper in § 8.3, we shall find it both interesting and useful to demonstrate that

$$\frac{1}{(1-z)^{n+1}} = \sum_{k=0}^{\infty} \binom{n+k}{n} z^k, \quad n \geq 0. \tag{8.1}$$

The demonstration comes in three stages. Of the three, it is the second stage (§ 8.1.2) which actually proves (8.1). The first stage (§ 8.1.1) comes up with the formula for the second stage to prove. The third stage (§ 8.1.3) establishes the sum's convergence. In all the section,

$$i, j, k, m, n, K \in \mathbb{Z}.$$

### 8.1.1 The formula

In § 2.6.4 we found that

$$\frac{1}{1-z} = \sum_{k=0}^{\infty} z^k = 1 + z + z^2 + z^3 + \cdots$$

for $|z| < 1$. What about $1/(1-z)^2$, $1/(1-z)^3$, $1/(1-z)^4$, and so on? By the long-division procedure of Table 2.4, one can calculate the first few terms of $1/(1-z)^2$ to be

$$\frac{1}{(1-z)^2} = \frac{1}{1-2z+z^2} = 1 + 2z + 3z^2 + 4z^3 + \cdots$$

whose coefficients $1, 2, 3, 4, \ldots$ happen to be the numbers down the first diagonal of Pascal's triangle (Fig. 4.2 on page 79; see also Fig. 4.1). Dividing $1/(1-z)^3$ seems to produce the coefficients $1, 3, 6, 0\text{xA}, \ldots$ down the second diagonal; dividing $1/(1-z)^4$, the coefficients down the third. A curious pattern seems to emerge, worth investigating more closely. The pattern recommends the conjecture (8.1).

To motivate the conjecture a bit more formally (though without actually proving it yet), suppose that $1/(1-z)^{n+1}$, $n \geq 0$, is expandable in the power series

$$\frac{1}{(1-z)^{n+1}} = \sum_{k=0}^{\infty} a_{nk} z^k, \tag{8.2}$$

where the $a_{nk}$ are coefficients to be determined. Multiplying by $1-z$, we have that

$$\frac{1}{(1-z)^n} = \sum_{k=0}^{\infty} [a_{nk} - a_{n(k-1)}] z^k.$$

This is to say that

$$a_{(n-1)k} = a_{nk} - a_{n(k-1)},$$

or in other words that

$$a_{n(k-1)} + a_{(n-1)k} = a_{nk}. \tag{8.3}$$

Thinking of Pascal's triangle, (8.3) reminds one of (4.5), transcribed here in the symbols

$$\left( \begin{array}{c} m-1 \\ j-1 \end{array} \right) + \left( \begin{array}{c} m-1 \\ j \end{array} \right) = \left( \begin{array}{c} m \\ j \end{array} \right), \tag{8.4}$$

except that (8.3) is not $a_{(m-1)(j-1)} + a_{(m-1)j} = a_{mj}$.

Various changes of variable are possible to make (8.4) better match (8.3). We might try at first a few false ones, but eventually the change

$$\begin{aligned} n + k &\leftarrow m, \\ k &\leftarrow j, \end{aligned}$$

recommends itself. Thus changing in (8.4) gives

$$\left( \begin{array}{c} n+k-1 \\ k-1 \end{array} \right) + \left( \begin{array}{c} n+k-1 \\ k \end{array} \right) = \left( \begin{array}{c} n+k \\ k \end{array} \right).$$

Transforming according to the rule (4.3), this is

$$\left( \begin{array}{c} n+[k-1] \\ n \end{array} \right) + \left( \begin{array}{c} [n-1]+k \\ n-1 \end{array} \right) = \left( \begin{array}{c} n+k \\ n \end{array} \right), \tag{8.5}$$

which fits (8.3) perfectly. Hence we conjecture that

$$a_{nk} = \left( \begin{array}{c} n+k \\ n \end{array} \right), \tag{8.6}$$

which coefficients, applied to (8.2), yield (8.1).

Equation (8.1) is thus suggestive. It works at least for the important case of $n = 0$; this much is easy to test. In light of (8.3), it seems to imply a relationship between the $1/(1-z)^{n+1}$ series and the $1/(1-z)^n$ series for any $n$. But *to seem* is not *to be.* At this point, all we can say is that (8.1) seems right. We will establish that it is right in the next subsection.

### 8.1.2 The proof by induction

Equation (8.1) is proved by induction as follows. Consider the sum

$$S_n \equiv \sum_{k=0}^{\infty} \binom{n+k}{n} z^k. \tag{8.7}$$

Multiplying by $1 - z$ yields

$$(1-z)S_n = \sum_{k=0}^{\infty} \left[ \binom{n+k}{n} - \binom{n+[k-1]}{n} \right] z^k.$$

Per (8.5), this is

$$(1-z)S_n = \sum_{k=0}^{\infty} \binom{[n-1]+k}{n-1} z^k. \tag{8.8}$$

Now suppose that (8.1) is true for $n = i - 1$ (where $i$ denotes an integer rather than the imaginary unit):

$$\frac{1}{(1-z)^i} = \sum_{k=0}^{\infty} \binom{[i-1]+k}{i-1} z^k. \tag{8.9}$$

In light of (8.8), this means that

$$\frac{1}{(1-z)^i} = (1-z)S_i.$$

Dividing by $1 - z$,

$$\frac{1}{(1-z)^{i+1}} = S_i.$$

Applying (8.7),

$$\frac{1}{(1-z)^{i+1}} = \sum_{k=0}^{\infty} \binom{i+k}{i} z^k. \tag{8.10}$$

Evidently (8.9) implies (8.10). In other words, if (8.1) is true for $n = i - 1$, then it is also true for $n = i$. Thus *by induction,* if it is true for any one $n$, then it is also true for all greater $n$.

The "if" in the last sentence is important. Like all inductions, this one needs at least one *start case* to be valid (many inductions actually need a consecutive pair of start cases). The $n = 0$ supplies the start case

$$\frac{1}{(1-z)^{0+1}} = \sum_{k=0}^{\infty} \binom{k}{0} z^k = \sum_{k=0}^{\infty} z^k,$$

which per (2.34) we know to be true.

### 8.1.3 Convergence

The question remains as to the domain over which the sum (8.1) converges.[2] To answer the question, consider that per (4.9),

$$\binom{m}{j} = \frac{m}{m-j}\binom{m-1}{j}, \quad m > 0.$$

With the substitution $n + k \leftarrow m$, $n \leftarrow j$, this means that

$$\binom{n+k}{n} = \frac{n+k}{k}\binom{n+[k-1]}{n},$$

or more tersely,

$$a_{nk} = \frac{n+k}{k} a_{n(k-1)},$$

where

$$a_{nk} \equiv \binom{n+k}{n}$$

are the coefficients of the power series (8.1). Rearranging factors,

$$\frac{a_{nk}}{a_{n(k-1)}} = \frac{n+k}{k} = 1 + \frac{n}{k}. \tag{8.11}$$

[2]The meaning of the verb *to converge* may seem clear enough from the context and from earlier references, but if explanation here helps: a series converges if and only if it approaches a specific, finite value after many terms. A more rigorous way of saying the same thing is as follows: the series

$$S = \sum_{k=0}^{\infty} \tau_k$$

converges iff (if and only if), for all possible positive constants $\epsilon$, there exists a finite $K \geq -1$ such that

$$\left| \sum_{k=K+1}^{n} \tau_k \right| < \epsilon,$$

for all $n \geq K$ (of course it is also required that the $\tau_k$ be finite, but you knew that already).

The professional mathematical literature calls such convergence "uniform convergence," distinguishing it through a test devised by Weierstrass from the weaker "pointwise convergence" [2, § 1.5]. The applied mathematician can profit substantially by learning the professional view in the matter, but the effect of trying to teach the professional view in a book like this would not be pleasing. Here, we avoid error by keeping a clear view of the physical phenomena the mathematics is meant to model.

It is interesting nevertheless to consider an example of an integral for which convergence is not so simple, such as Frullani's integral of § 9.7.

Multiplying (8.11) by $z^k/z^{k-1}$ gives the ratio

$$\frac{a_{nk}z^k}{a_{n(k-1)}z^{k-1}} = \left(1 + \frac{n}{k}\right) z,$$

which is to say that the $k$th term of (8.1) is $(1 + n/k)z$ times the $(k-1)$th term. So long as the criterion[3]

$$\left|\left(1 + \frac{n}{k}\right) z\right| \leq 1 - \delta$$

is satisfied for all sufficiently large $k > K$—where $0 < \delta \ll 1$ is a small positive constant—then the series evidently converges (see § 2.6.4 and eqn. 3.22). But we can bind $1+n/k$ as close to unity as desired by making $K$ sufficiently large, so to meet the criterion it suffices that

$$|z| < 1. \tag{8.12}$$

The bound (8.12) thus establishes a sure convergence domain for (8.1).

### 8.1.4 General remarks on mathematical induction

We have proven (8.1) by means of a mathematical induction. The virtue of induction as practiced in § 8.1.2 is that it makes a logically clean, airtight case for a formula. Its vice is that it conceals the subjective process which has led the mathematician to consider the formula in the first place. Once you obtain a formula somehow, maybe you can prove it by induction; but the induction probably does not help you to obtain the formula! A good inductive proof usually begins by motivating the formula proven, as in § 8.1.1.

Richard W. Hamming once said of mathematical induction,

> The theoretical difficulty the student has with mathematical induction arises from the reluctance to ask seriously, "How could I prove a formula for an infinite number of cases when I know that testing a finite number of cases is not enough?" Once you

[3]Although one need not ask the question to understand the proof, the reader may nevertheless wonder why the simpler $|(1 + n/k)z| < 1$ is not given as a criterion. The surprising answer is that not all series $\sum \tau_k$ with $|\tau_k/\tau_{k-1}| < 1$ converge! For example, the extremely simple $\sum 1/k$ does not converge. As we see however, all series $\sum \tau_k$ with $|\tau_k/\tau_{k-1}| < 1 - \delta$ do converge. The distinction is subtle but rather important.

The really curious reader may now ask why $\sum 1/k$ does not converge. Answer: it *majorizes* $\int_1^x (1/\tau)\, d\tau = \ln x$. See (5.8) and § 8.10.

> really face this question, you will understand the ideas behind mathematical induction. It is only when you grasp the problem clearly that the method becomes clear. [27, § 2.3]

Hamming also wrote,

> The function of rigor is mainly critical and is seldom constructive. Rigor is the hygiene of mathematics, which is needed to protect us against careless thinking. [27, § 1.6]

The applied mathematician may tend to avoid rigor for which he finds no immediate use, but he does not disdain mathematical rigor on principle. The style lies in exercising rigor at the right level for the problem at hand. Hamming, a professional mathematician who sympathized with the applied mathematician's needs, wrote further,

> Ideally, when teaching a topic the degree of rigor should follow the student's perceived need for it. . . . It is necessary to require a gradually rising level of rigor so that when faced with a real need for it you are not left helpless. As a result, [one cannot teach] a uniform level of rigor, but rather a gradually rising level. Logically, this is indefensible, but psychologically there is little else that can be done. [27, § 1.6]

Applied mathematics holds that the practice *is* defensible, on the ground that the math serves the model; but Hamming nevertheless makes a pertinent point.

Mathematical induction is a broadly applicable technique for constructing mathematical proofs. We will not always write inductions out as explicitly in this book as we have done in the present section—often we will leave the induction as an implicit exercise for the interested reader—but this section's example at least lays out the general pattern of the technique.

## 8.2 Shifting a power series' expansion point

One more question we should treat before approaching the Taylor series proper in § 8.3 concerns the shifting of a power series' expansion point. How can the expansion point of the power series

$$\begin{aligned} f(z) &= \sum_{k=K}^{\infty} (a_k)(z - z_o)^k, \\ (k, K) &\in \mathbb{Z}, \quad K \leq 0, \end{aligned} \tag{8.13}$$

which may have terms of negative order, be shifted from $z = z_o$ to $z = z_1$?

The first step in answering the question is straightforward: one rewrites (8.13) in the form

$$f(z) = \sum_{k=K}^{\infty} (a_k)([z - z_1] - [z_o - z_1])^k,$$

then changes the variables

$$\begin{aligned} w &\leftarrow \frac{z - z_1}{z_o - z_1}, \\ c_k &\leftarrow [-(z_o - z_1)]^k a_k, \end{aligned} \tag{8.14}$$

to obtain

$$f(z) = \sum_{k=K}^{\infty} (c_k)(1 - w)^k. \tag{8.15}$$

Splitting the $k < 0$ terms from the $k \geq 0$ terms in (8.15), we have that

$$\begin{aligned} f(z) &= f_-(z) + f_+(z), \\ f_-(z) &\equiv \sum_{k=0}^{-(K+1)} \frac{c_{[-(k+1)]}}{(1 - w)^{k+1}}, \\ f_+(z) &\equiv \sum_{k=0}^{\infty} (c_k)(1 - w)^k. \end{aligned} \tag{8.16}$$

Of the two subseries, the $f_-(z)$ is expanded term by term using (8.1), after which combining like powers of $w$ yields the form

$$\begin{aligned} f_-(z) &= \sum_{k=0}^{\infty} q_k w^k, \\ q_k &\equiv \sum_{n=0}^{-(K+1)} (c_{[-(n+1)]}) \binom{n+k}{n}. \end{aligned} \tag{8.17}$$

The $f_+(z)$ is even simpler to expand: one need only multiply the series out term by term per (4.12), combining like powers of $w$ to reach the form

$$\begin{aligned} f_+(z) &= \sum_{k=0}^{\infty} p_k w^k, \\ p_k &\equiv \sum_{n=k}^{\infty} (c_n) \binom{n}{k}. \end{aligned} \tag{8.18}$$

Equations (8.13) through (8.18) serve to shift a power series' expansion point, calculating the coefficients of a power series for $f(z)$ about $z = z_1$, given those of a power series about $z = z_o$. Notice that—unlike the original, $z = z_o$ power series—the new, $z = z_1$ power series has terms $(z - z_1)^k$ only for $k \geq 0$; it has no terms of negative order. At the price per (8.12) of restricting the convergence domain to $|w| < 1$, shifting the expansion point away from the pole at $z = z_o$ has resolved the $k < 0$ terms.

The method fails if $z = z_1$ happens to be a pole or other nonanalytic point of $f(z)$. The convergence domain vanishes as $z_1$ approaches such a forbidden point. (Examples of such forbidden points include $z = 0$ in $h[z] = 1/z$ and in $g[z] = \sqrt{z}$. See §§ 8.4 through 8.8.) Furthermore, even if $z_1$ does represent a fully analytic point of $f(z)$, it also must lie within the convergence domain of the original, $z = z_o$ series for the shift to be trustworthy as derived.

The attentive reader might observe that we have formally established the convergence neither of $f_-(z)$ in (8.17) nor of $f_+(z)$ in (8.18). Regarding the former convergence, that of $f_-(z)$, we have strategically framed the problem so that one needn't worry about it, running the sum in (8.13) from the finite $k = K \leq 0$ rather than from the infinite $k = -\infty$; and since according to (8.12) each term of the original $f_-(z)$ of (8.16) converges for $|w| < 1$, the reconstituted $f_-(z)$ of (8.17) safely converges in the same domain. The latter convergence, that of $f_+(z)$, is harder to establish in the abstract because that subseries has an infinite number of terms. As we will see by pursuing a different line of argument in § 8.3, however, the $f_+(z)$ of (8.18) can be nothing other than the Taylor series about $z = z_1$ of the function $f_+(z)$ in any event, enjoying the same convergence domain any such Taylor series enjoys.[4]

## 8.3 Expanding functions in Taylor series

Having prepared the ground, we now stand in position to treat the Taylor series proper. The treatment begins with a question: if you had to express

[4] A rigorous argument can be constructed without appeal to § 8.3 if desired, from the ratio $n/(n-k)$ of (4.9) and its brethren, which ratio approaches unity with increasing $n$. A more elegant rigorous argument can be made indirectly by way of a complex contour integral. In applied mathematics, however, one does not normally try to shift the expansion point of an *unspecified* function $f(z)$, anyway. Rather, one shifts the expansion point of some concrete function like $\sin z$ or $\ln(1 - z)$. The imagined difficulty (if any) vanishes in the concrete case. Appealing to § 8.3, the important point is the one made in the narrative: $f_+(z)$ can be nothing other than the Taylor series in any event.

some function $f(z)$ by a power series

$$f(z) = \sum_{k=0}^{\infty} (a_k)(z - z_o)^k,$$

with terms of nonnegative order $k \geq 0$ only, how would you do it? The procedure of § 8.1 worked well enough in the case of $f(z) = 1/(1-z)^{n+1}$, but it is not immediately obvious that the same procedure works more generally. What if $f(z) = \sin z$, for example?[5]

Fortunately a different way to attack the power-series expansion problem is known. It works by asking the question: what power series, having terms of nonnegative order only, most resembles $f(z)$ in the immediate neighborhood of $z = z_o$? To resemble $f(z)$, the desired power series should have $a_0 = f(z_o)$; otherwise it would not have the right value at $z = z_o$. Then it should have $a_1 = f'(z_o)$ for the right slope. Then, $a_2 = f''(z_o)/2$ for the right second derivative, and so on. With this procedure,

$$f(z) = \sum_{k=0}^{\infty} \left( \left. \frac{d^k f}{dz^k} \right|_{z=z_o} \right) \frac{(z - z_o)^k}{k!}. \tag{8.19}$$

Equation (8.19) is the *Taylor series.* Where it converges, it has all the same derivatives $f(z)$ has, so if $f(z)$ is infinitely differentiable then the Taylor series is an exact representation of the function.[6]

[5] The actual Taylor series for $\sin z$ is given in § 8.9.

[6] The professional mathematician reading such words is likely to blanch. To him, such words want rigor. To him, such words want pages and whole chapters [4][20][57][31] of rigor.

Before submitting unreservedly to the professional's scruples in the matter, however, let us not forget (§ 1.2.1) that the professional's approach is founded in postulation, whereas that ours is founded in physical metaphor. Our means differ from his for this reason alone.

Still, an alternate way to think about the Taylor series' sufficiency might interest some readers. It begins with an infinitely differentiable function $F(z)$ and its Taylor series $f(z)$ about $z_o$, letting $\Delta F(z) \equiv F(z) - f(z)$ be the part of $F(z)$ not representable as a Taylor series about $z_o$.

If $\Delta F(z)$ is the part of $F(z)$ not representable as a Taylor series, then $\Delta F(z_o)$ and all its derivatives at $z_o$ must be identically zero (otherwise by the Taylor series formula of eqn. 8.19, one could construct a nonzero Taylor series for $\Delta F[z_o]$ from the nonzero derivatives). However, if $F(z)$ is infinitely differentiable and if *all* the derivatives of $\Delta F(z)$ are zero at $z = z_o$ then, by the unbalanced definition of the derivative from § 4.4, all the derivatives must also be zero at $z = z_o \pm \epsilon$, hence also at $z = z_o \pm 2\epsilon$, and so on. This means that $\Delta F(z) = 0$. In other words, there is no part of $F(z)$ not representable as a Taylor series.

A more formal way to make the same argument would be to suppose that

The Taylor series is not guaranteed to converge outside some neighborhood near $z = z_o$, but where it does converge it is precise.

When $z_o = 0$, the series is also called the *Maclaurin series.* By either name, the series is a construct of great importance and tremendous practical value, as we shall soon see.

## 8.4 Analytic continuation

As earlier mentioned in § 2.12.3, an *analytic function* is a function which is infinitely differentiable in the domain neighborhood of interest—or, maybe more appropriately for our applied purpose, a function expressible as a Taylor series in that neighborhood. As we have seen, only one Taylor series

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$d^n \Delta F/dz^n|_{z=z_o+\epsilon} = h$ for some integer $n \geq 0$; whereas that this would mean that $d^{n+1} \Delta F/dz^{n+1}\big|_{z=z_o} = h/\epsilon$; but that, inasmuch as the latter is one of the derivatives of $\Delta F(z)$ at $z = z_o$, it follows that $h = 0$.

The interested reader can fill the details in, but basically that is how the alternate argument begins. After all, if the first, second, third derivatives and so forth, evaluated at some expansion point, indicate anything at all, then must they not indicate how the function and its several derivatives will evolve from that point? And if they do indicate that, then what could null derivatives indicate but null evolution? Yet such arguments even if accepted are satisfactory only from a certain point of view, and yet slightly less so once one considers the asymptotic series of [chapter not yet written] later in the book.

A more elegant rigorous argument, preferred by the professional mathematicians [40][4][20] but needing significant theoretical preparation, involves integrating over a complex contour about the expansion point. Appendix C sketches that proof.

The discussion of rigor is confined here to a footnote not to deprecate rigor as such, but to deprecate insistence on rigor which serves little known purpose in applications. Applied mathematicians normally regard mathematical functions to be imprecise analogs of, or metaphors for, physical quantities of interest. Since the functions are imprecise analogs in any case, the applied mathematician is logically free implicitly *to define* the functions he uses as Taylor series in the first place; that is, to restrict the set of infinitely differentiable functions used in the model to the subset of such functions representable as Taylor series. With such an implicit definition, whether there actually exist any infinitely differentiable functions not representable as Taylor series is more or less beside the point—at least until a concrete need for such a hypothetical function should present itself.

In applied mathematics, the definitions serve the model, not the other way around. Other than to divert the interested reader to Appendix C, this footnote will leave the matter in that form.

(It is entertaining incidentally to consider [65, "Extremum"] the Taylor series of the function $\sin[1/x]$—although in practice this particular function is readily expanded after the obvious change of variable $u \leftarrow 1/x$.)

about $z_o$ is possible for a given function $f(z)$:

$$f(z) = \sum_{k=0}^{\infty} (a_k)(z - z_o)^k.$$

However, nothing prevents one from transposing the series to a different expansion point $z = z_1$ by the method of § 8.2, except that the transposed series may there enjoy a different convergence domain. As it happens, this section's purpose finds it convenient to swap symbols $z_o \leftrightarrow z_1$, transposing rather from expansion about $z = z_1$ to expansion about $z = z_o$. In the swapped notation, so long as the expansion point $z = z_o$ lies fully within (neither outside nor right on the edge of) the $z = z_1$ series' convergence domain, the two series evidently describe the selfsame underlying analytic function.

Since an analytic function $f(z)$ is infinitely differentiable and enjoys a unique Taylor expansion $f_o(z-z_o) = f(z)$ about each point $z_o$ in its domain, it follows that if two Taylor series $f_1(z-z_1)$ and $f_2(z-z_2)$ find even a small neighborhood $|z - z_o| < \epsilon$ which lies in the domain of both, then the two can both be transposed to the common $z = z_o$ expansion point. If the two are found to have the same Taylor series there, then $f_1$ and $f_2$ both represent the same function. Moreover, if a series $f_3$ is found whose domain overlaps that of $f_2$, then a series $f_4$ whose domain overlaps that of $f_3$, and so on, and if each pair in the chain matches at least in a small neighborhood in its region of overlap, then the whole chain of overlapping series necessarily represents the same underlying analytic function $f(z)$. The series $f_1$ and the series $f_n$ represent the same analytic function even if their domains do not directly overlap at all.

This is a manifestation of the principle of *analytic continuation.* The principle holds that if two analytic functions are the same within some domain neighborhood $|z - z_o| < \epsilon$, then they are the same everywhere.[7] Observe however that the principle fails at poles and other nonanalytic points, because the function is not differentiable there.

The result of § 8.2, which shows general power series to be expressible as Taylor series except at their poles and other nonanalytic points, extends the

[7]The writer hesitates to mention that he is given to understand [57] that the domain neighborhood can technically be reduced to a domain contour of nonzero length but zero width. Having never met a significant application of this extension of the principle, the writer has neither researched the extension's proof nor asserted its truth. He does not especially recommend that the reader worry over the point. The domain neighborhood $|z - z_o| < \epsilon$ suffices.

analytic continuation principle to cover power series in general, including power series with terms of negative order.

Now, observe: though all convergent power series are indeed analytic, one need not actually expand every analytic function in a power series. Sums, products and ratios of analytic functions are no less differentiable than the functions themselves—as also, by the derivative chain rule, is an analytic function of analytic functions. For example, where $g(z)$ and $h(z)$ are analytic, there also is $f(z) \equiv g(z)/h(z)$ analytic (except perhaps at isolated points where $h[z] = 0$). Besides, given Taylor series for $g(z)$ and $h(z)$ one can make a power series for $f(z)$ by long division if desired, so that is all right. Section 8.15 speaks further on the point.

The subject of analyticity is rightly a matter of deep concern to the professional mathematician. It is also a long wedge which drives pure and applied mathematics apart. When the professional mathematician speaks generally of a "function," he means *any function at all.* One can construct some pretty unreasonable functions if one wants to, such as

$$\begin{aligned} f([2k+1]2^m) &\equiv (-)^m, \quad (k,m) \in \mathbb{Z}; \\ f(z) &\equiv 0 \text{ otherwise.} \end{aligned}$$

However, neither functions like this $f(z)$ nor more subtly unreasonable functions normally arise in the modeling of physical phenomena. When such functions do arise, one transforms, approximates, reduces, replaces and/or avoids them. The full theory which classifies and encompasses—or explicitly excludes—such functions is thus of limited interest to the applied mathematician, and this book does not cover it.[8]

This does not mean that the scientist or engineer never encounters nonanalytic functions. On the contrary, he encounters several, but they are not subtle: $|z|$; $\arg z$; $z^*$; $\Re(z)$; $\Im(z)$; $u(t)$; $\delta(t)$. Refer to §§ 2.12 and 7.7. Such functions are nonanalytic either because they lack proper derivatives in the Argand plane according to (4.19) or because one has defined them only over a real domain.

[8] Many books do cover it in varying degrees, including [20][57][31] and numerous others. The foundations of the pure theory of a complex variable, though abstract, are beautiful, and though they do not comfortably fit a book like this even an applied mathematician can profit substantially by studying them. The few pages of Appendix C trace only the pure theory's main thread. However that may be, the pure theory is probably best appreciated after one already understands its chief conclusions. Though not for the explicit purpose of serving the pure theory, the present chapter does develop just such an understanding.

## 8.5 Branch points

The function $g(z) = \sqrt{z}$ is an interesting, troublesome function. Its derivative is $dg/dz = 1/2\sqrt{z}$, so even though the function is finite at $z = 0$, its derivative is not finite there. Evidently $g(z)$ has a nonanalytic point at $z = 0$, yet the point is not a pole. What is it?

We call it a *branch point.* The defining characteristic of the branch point is that, given a function $f(z)$ with such a point at $z = z_o$, if one encircles[9] the point once alone (that is, without also encircling some other branch point) by a closed contour in the Argand domain plane, while simultaneously tracking $f(z)$ in the Argand range plane—and if one demands that $z$ and $f(z)$ move smoothly, that neither suddenly skip from one spot to another—then one finds that $f(z)$ ends in a different place than it began, even though $z$ itself has returned precisely to its own starting point. The range contour remains open even though the domain contour is closed.

> In complex analysis, a branch point may be thought of informally as a point $z_o$ at which a "multiple-valued function" changes values when one winds once around $z_o$. [66, "Branch point," 18:10, 16 May 2006]

An analytic function like $g(z) = \sqrt{z}$ having a branch point evidently is not single-valued. It is multiple-valued. For a single $z$ more than one distinct $g(z)$ is possible.

An analytic function like $h(z) = 1/z$, by contrast, is single-valued even though it has a pole. This function does not suffer the syndrome described. When a domain contour encircles a pole, the corresponding range contour is properly closed. Poles do not cause their functions to be multiple-valued and thus are not branch points.

Evidently $f(z) \equiv (z - z_o)^a$ has a branch point at $z = z_o$ if and only if $a$ is not an integer. If $f(z)$ does have a branch point—if $a$ is not an integer—then the mathematician must draw a distinction between $z_1 = z_o + \rho e^{i\phi}$ and $z_2 = z_o + \rho e^{i(\phi+2\pi)}$, *even though the two are exactly the same number.* Indeed $z_1 = z_2$, but paradoxically $f(z_1) \neq f(z_2)$.

This is difficult. It is confusing, too, until one realizes that the fact of a branch point says nothing whatsoever about the argument $z$. As far as $z$ is concerned, there really is no distinction between $z_1 = z_o + \rho e^{i\phi}$ and

[9] For readers whose native language is not English, "to encircle" means "to surround" or "to enclose." The verb does not require the boundary to have the shape of an actual, geometrical circle; any closed shape suffices. However, the circle is a typical shape, probably the most fitting shape to imagine when thinking of the concept abstractly.

$z_2 = z_o + \rho e^{i(\phi+2\pi)}$—none at all. What draws the distinction is the multiple-valued function $f(z)$ which uses the argument.

It is as though I had a mad colleague who called me Thaddeus Black, until one day I happened to walk past behind his desk (rather than in front as I usually did), whereupon for some reason he began calling me Gorbag Pfufnik. I had not changed at all, but now the colleague calls me by a different name. The change isn't really in me, is it? It's in my colleague, who seems to suffer a branch point. If it is important to me to be sure that my colleague really is addressing me when he cries, "Pfufnik!" then I had better keep a running count of how many times I have turned about his desk, hadn't I, even though the number of turns is personally of no import to me.

The usual analysis strategy when one encounters a branch point is simply to avoid the point. Where an integral follows a closed contour as in § 8.8, the strategy is to compose the contour to exclude the branch point, to shut it out. Such a strategy of avoidance usually prospers.[10]

## 8.6 Entire and meromorphic functions

Though an applied mathematician is unwise to let abstract definitions enthrall his thinking, pure mathematics nevertheless brings some technical definitions the applied mathematician can use. Two such are the definitions of *entire* and *meromorphic* functions.[11]

A function $f(z)$ which is analytic for all finite $z$ is an *entire function.* Examples include $f(z) = z^2$ and $f(z) = \exp z$, but not $f(z) = 1/z$ which has a pole at $z = 0$.

A function $f(z)$ which is analytic for all finite $z$ except at isolated poles (which can be $n$-fold poles if $n$ is a finite, positive integer), which has no branch points, of which no circle of finite radius in the Argand domain plane encompasses an infinity of poles, is a *meromorphic function.* Examples include $f(z) = 1/z$, $f(z) = 1/(z+2) + 1/(z-1)^3 + 2z^2$ and $f(z) = \tan z$—the last of which has an infinite number of poles, but of which the poles nowhere cluster in infinite numbers. The function $f(z) = \tan(1/z)$ is not meromorphic since it has an infinite number of poles within the Argand unit circle. Even the function $f(z) = \exp(1/z)$ is not meromorphic: it has only

[10] Traditionally associated with branch points in complex variable theory are the notions of *branch cuts* and *Riemann sheets.* These ideas are interesting, but are not central to the analysis as developed in this book and are not covered here. The interested reader might consult a book on complex variables or advanced calculus like [31], among many others.

[11] [65]

the one, isolated nonanalytic point at $z = 0$, and that point is no branch point; but the point is an *essential singularity,* having the character of an infinitifold ($\infty$-fold) pole.[12]

If it seems unclear that the singularities of $\tan z$ are actual poles, incidentally, then consider that

$$\tan z = \frac{\sin z}{\cos z} = -\frac{\cos w}{\sin w},$$

wherein we have changed the variable

$$w \leftarrow z - (2n+1)\frac{2\pi}{4}, \quad n \in \mathbb{Z}.$$

Section 8.9 and its Table 8.1, below, give Taylor series for $\cos z$ and $\sin z$, with which

$$\tan z = \frac{-1 + w^2/2 - w^4/0\text{x}18 - \cdots}{w - w^3/6 + w^5/0\text{x}78 - \cdots}.$$

By long division,

$$\tan z = -\frac{1}{w} + \frac{w/3 - w^3/0\text{x}1\text{E} + \cdots}{1 - w^2/6 + w^4/0\text{x}78 - \cdots}.$$

(On the other hand, if it is unclear that $z = [2n+1][2\pi/4]$ are the only singularities $\tan z$ has—that it has no singularities of which $\Im[z] \neq 0$—then consider that the singularities of $\tan z$ occur where $\cos z = 0$, which by Euler's formula, eqn. 5.18, occurs where $\exp[+iz] = \exp[-iz]$. This in turn is possible only if $|\exp[+iz]| = |\exp[-iz]|$, which happens only for real $z$.)

Sections 8.14, 8.15 and 9.6 speak further of the matter.

## 8.7 Extrema over a complex domain

If a function $f(z)$ is expanded by (8.19) or by other means about an analytic expansion point $z = z_o$ such that

$$f(z) = f(z_o) + \sum_{k=1}^{\infty} (a_k)(z - z_o)^k;$$

and if

$$\begin{aligned} a_k &= 0 \quad \text{for } k < K, \text{ but} \\ a_K &\neq 0, \\ (k, K) &\in \mathbb{Z}, \quad 0 < K < \infty, \end{aligned}$$

[12] [40]

such that $a_K$ is the series' first nonzero coefficient; then, in the immediate neighborhood of the expansion point,

$$f(z) \approx f(z_o) + (a_K)(z - z_o)^K, \quad |z - z_o| \ll 1.$$

Changing $\rho' e^{i\phi'} \leftarrow z - z_o$, this is

$$f(z) \approx f(z_o) + a_K \rho'^K e^{iK\phi'}, \quad 0 \le \rho' \ll 1. \tag{8.20}$$

Evidently one can shift the output of an analytic function $f(z)$ slightly in any desired Argand direction by shifting slightly the function's input $z$. Specifically according to (8.20), to shift $f(z)$ by $\Delta f \approx \epsilon e^{i\psi}$, one can shift $z$ by $\Delta z \approx (\epsilon/a_K)^{1/K} e^{i(\psi+n2\pi)/K}$, $n \in \mathbb{Z}$. Except at a nonanalytic point of $f(z)$ or in the trivial case that $f(z)$ were everywhere constant, this always works—even where $[df/dz]_{z=z_o} = 0$.

That one can shift an analytic function's output smoothly in any Argand direction whatsoever has the significant consequence that neither the real nor the imaginary part of the function—nor for that matter any linear combination $\Re[e^{-i\omega} f(z)]$ of the real and imaginary parts—can have an extremum within the interior of a domain over which the function is fully analytic. That is, *a function's extrema over a bounded analytic domain never lie within the domain's interior but always on its boundary.*[13,14]

## 8.8 Cauchy's integral formula

In § 7.6 we considered the problem of vector contour integration, in which the sum value of an integration depends not only on the integration's endpoints but also on the path, or *contour,* over which the integration is done, as in Fig. 7.8. Because real scalars are confined to a single line, no alternate choice of path is possible where the variable of integration is a real scalar, so the contour problem does not arise in that case. It does however arise where the variable of integration is a *complex* scalar, because there again different paths are possible. Refer to the Argand plane of Fig. 2.5.

Consider the integral

$$S_n = \int_{z_1}^{z_2} z^{n-1}\, dz, \quad n \in \mathbb{Z}. \tag{8.21}$$

[13] Professional mathematicians tend to define the domain and its boundary more carefully.

[14] [62][40]

If $z$ were always a real number, then by the antiderivative (§ 7.2) this integral would evaluate to $(z_2^n - z_1^n)/n$; or, in the case of $n = 0$, to $\ln(z_2/z_1)$. Inasmuch as $z$ is complex, however, the correct evaluation is less obvious. To evaluate the integral sensibly in the latter case, one must consider some specific path of integration in the Argand plane. One must also consider the meaning of the symbol $dz$.

### 8.8.1 The meaning of the symbol $dz$

The symbol $dz$ represents an infinitesimal step in some direction in the Argand plane:

$$\begin{aligned} dz &= [z + dz] - [z] \\ &= \left[(\rho + d\rho)e^{i(\phi+d\phi)}\right] - \left[\rho e^{i\phi}\right] \\ &= \left[(\rho + d\rho)e^{i\,d\phi}e^{i\phi}\right] - \left[\rho e^{i\phi}\right] \\ &= \left[(\rho + d\rho)(1 + i\,d\phi)e^{i\phi}\right] - \left[\rho e^{i\phi}\right]. \end{aligned}$$

Since the product of two infinitesimals is negligible even on infinitesimal scale, we can drop the $d\rho\,d\phi$ term.[15] After canceling finite terms, we are left with the peculiar but fine formula

$$dz = (d\rho + i\rho\,d\phi)e^{i\phi}. \tag{8.22}$$

### 8.8.2 Integrating along the contour

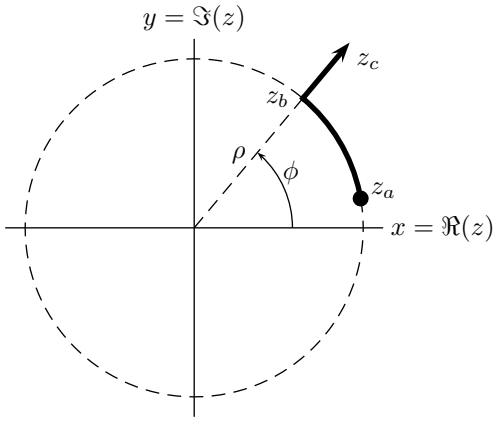
Now consider the integration (8.21) along the contour of Fig. 8.1. Integrat-

[15] The dropping of second-order infinitesimals like $d\rho\,d\phi$, added to first order infinitesimals like $d\rho$, is a standard calculus technique. One cannot *always* drop them, however. Occasionally one encounters a sum in which not only do the finite terms cancel, but also the first-order infinitesimals. In such a case, the second-order infinitesimals dominate and cannot be dropped. An example of the type is

$$\lim_{\epsilon\to 0} \frac{(1-\epsilon)^3 + 3(1+\epsilon) - 4}{\epsilon^2} = \lim_{\epsilon\to 0} \frac{(1 - 3\epsilon + 3\epsilon^2) + (3 + 3\epsilon) - 4}{\epsilon^2} = 3.$$

One typically notices that such a case has arisen when the dropping of second-order infinitesimals has left an ambiguous 0/0. To fix the problem, you simply go back to the step where you dropped the infinitesimal and you restore it, then you proceed from there. Otherwise there isn't much point in carrying second-order infinitesimals around. In the relatively uncommon event that you need them, you'll know it. The math itself will tell you.

Figure 8.1: A contour of integration in the Argand plane, in two segments: constant-$\rho$ ($z_a$ to $z_b$); and constant-$\phi$ ($z_b$ to $z_c$).




ing along the constant-$\phi$ segment,

$$\begin{aligned}
\int_{z_b}^{z_c} z^{n-1}\,dz &= \int_{\rho_b}^{\rho_c} (\rho e^{i\phi})^{n-1}(d\rho + i\rho\,d\phi)e^{i\phi} \\
&= \int_{\rho_b}^{\rho_c} (\rho e^{i\phi})^{n-1}(d\rho)e^{i\phi} \\
&= e^{in\phi}\int_{\rho_b}^{\rho_c} \rho^{n-1}d\rho \\
&= \frac{e^{in\phi}}{n}\,(\rho_c^n - \rho_b^n) \\
&= \frac{z_c^n - z_b^n}{n}.
\end{aligned}$$

Integrating along the constant-$\rho$ arc,

$$\begin{aligned}
\int_{z_a}^{z_b} z^{n-1}\,dz &= \int_{\phi_a}^{\phi_b} (\rho e^{i\phi})^{n-1}(d\rho + i\rho\,d\phi)e^{i\phi} \\
&= \int_{\phi_a}^{\phi_b} (\rho e^{i\phi})^{n-1}(i\rho\,d\phi)e^{i\phi} \\
&= i\rho^n \int_{\phi_a}^{\phi_b} e^{in\phi}\,d\phi \\
&= \frac{i\rho^n}{in}\left(e^{in\phi_b} - e^{in\phi_a}\right) \\
&= \frac{z_b^n - z_a^n}{n}.
\end{aligned}$$

Adding the two, we have that

$$\int_{z_a}^{z_c} z^{n-1}\,dz = \frac{z_c^n - z_a^n}{n},$$

surprisingly the same as for real $z$. Since any path of integration between any two complex numbers $z_1$ and $z_2$ is approximated arbitrarily closely by a succession of short constant-$\rho$ and constant-$\phi$ segments, it follows generally that

$$\int_{z_1}^{z_2} z^{n-1}\,dz = \frac{z_2^n - z_1^n}{n}, \quad n \in \mathbb{Z},\ n \neq 0. \tag{8.23}$$

The applied mathematician might reasonably ask, "Was (8.23) really worth the trouble? We knew *that* already. It's the same as for real numbers."

Well, we really didn't know it before deriving it, but the point is well taken nevertheless. However, notice the exemption of $n = 0$. Equation (8.23) does not hold in that case. Consider the $n = 0$ integral

$$S_0 = \int_{z_1}^{z_2} \frac{dz}{z}.$$

Following the same steps as before and using (5.8) and (2.39), we find that

$$\int_{\rho_1}^{\rho_2} \frac{dz}{z} = \int_{\rho_1}^{\rho_2} \frac{(d\rho + i\rho\,d\phi)e^{i\phi}}{\rho e^{i\phi}} = \int_{\rho_1}^{\rho_2} \frac{d\rho}{\rho} = \ln\frac{\rho_2}{\rho_1}. \tag{8.24}$$

This is always real-valued, but otherwise it brings no surprise. However,

$$\int_{\phi_1}^{\phi_2} \frac{dz}{z} = \int_{\phi_1}^{\phi_2} \frac{(d\rho + i\rho\,d\phi)e^{i\phi}}{\rho e^{i\phi}} = i\int_{\phi_1}^{\phi_2} d\phi = i(\phi_2 - \phi_1). \tag{8.25}$$

The odd thing about this is in what happens when the contour closes a complete loop in the Argand plane about the $z = 0$ pole. In this case, $\phi_2 = \phi_1 + 2\pi$, thus

$$S_0 = i2\pi,$$

*even though the integration ends where it begins.*

Generalizing, we have that

$$\begin{aligned} \oint (z - z_o)^{n-1}\,dz &= 0, \quad n \in \mathbb{Z},\ n \neq 0; \\ \oint \frac{dz}{z - z_o} &= i2\pi; \end{aligned} \tag{8.26}$$

where as in § 7.6 the symbol $\oint$ represents integration about a closed contour that ends where it begins, and where it is implied that the contour loops positively (counterclockwise, in the direction of increasing $\phi$) exactly once about the $z = z_o$ pole.

Notice that the formula's $i2\pi$ does not depend on the precise path of integration, but only on the fact that the path loops once positively about the pole. Notice also that nothing in the derivation of (8.23) actually requires that $n$ be an integer, so one can write

$$\int_{z_1}^{z_2} z^{a-1}\,dz = \frac{z_2^a - z_1^a}{a}, \quad a \neq 0. \tag{8.27}$$

However, (8.26) does not hold in the latter case; its integral comes to zero for nonintegral $a$ only if the contour does not enclose the branch point at $z = z_o$.

For a closed contour *which encloses no pole or other nonanalytic point,* (8.27) has that $\oint z^{a-1}\,dz = 0$, or with the change of variable $z - z_o \leftarrow z$,

$$\oint (z - z_o)^{a-1}\,dz = 0.$$

But because any analytic function can be expanded in the form $f(z) = \sum_k (c_k)(z - z_o)^{a_k - 1}$ (which is just a Taylor series if the $a_k$ happen to be positive integers), this means that

$$\oint f(z)\,dz = 0 \tag{8.28}$$

if $f(z)$ is everywhere analytic within the contour.[16]

[16]The careful reader will observe that (8.28)'s derivation does not explicitly handle

### 8.8.3 The formula

The combination of (8.26) and (8.28) is powerful. Consider the closed contour integral

$$\oint \frac{f(z)}{z - z_o}\,dz,$$

where the contour encloses no nonanalytic point of $f(z)$ itself but does enclose the pole of $f(z)/(z - z_o)$ at $z = z_o$. If the contour were a tiny circle of infinitesimal radius about the pole, then the integrand would reduce to $f(z_o)/(z - z_o)$; and then per (8.26),

$$\oint \frac{f(z)}{z - z_o}\,dz = i2\pi f(z_o). \tag{8.29}$$

But if the contour were not an infinitesimal circle but rather the larger contour of Fig. 8.2? In this case, if the dashed detour which excludes the pole is taken, then according to (8.28) the resulting integral totals zero; but the two straight integral segments evidently cancel; and similarly as we have just reasoned, the *reverse-directed* integral about the tiny detour circle is $-i2\pi f(z_o)$; so to bring the total integral to zero the integral about the main contour must be $i2\pi f(z_o)$. Thus, (8.29) holds for any positively-directed contour which once encloses a pole and no other nonanalytic point, whether the contour be small or large. Equation (8.29) is *Cauchy's integral formula.*

If the contour encloses multiple poles (§§ 2.11 and 9.6.2), then by the principle of linear superposition (§ 7.3.3),

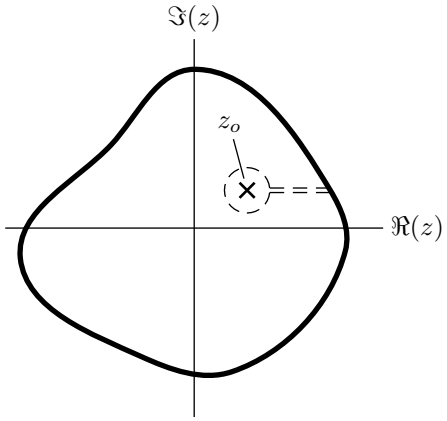
$$\oint \left[ f_o(z) + \sum_k \frac{f_k(z)}{z - z_k} \right] dz = i2\pi \sum_k f_k(z_k), \tag{8.30}$$

where the $f_o(z)$ is a *regular part*;[17] and again, where neither $f_o(z)$ nor any of the several $f_k(z)$ has a pole or other nonanalytic point within (or on) the

an $f(z)$ represented by a Taylor series with an infinite number of terms and a finite convergence domain (for example, $f[z] = \ln[1 - z]$). However, by § 8.2 one can transpose such a series from $z_o$ to an overlapping convergence domain about $z_1$. Let the contour's interior be divided into several cells, each of which is small enough to enjoy a single convergence domain. Integrate about each cell. Because the cells share boundaries within the contour's interior, each interior boundary is integrated twice, once in each direction, canceling. The original contour—each piece of which is an exterior boundary of some cell—is integrated once piecewise. This is the basis on which a more rigorous proof is constructed.

[17] [43, § 1.1]

Figure 8.2: A Cauchy contour integral.




contour. The values $f_k(z_k)$, which represent the strengths of the poles, are called *residues*. In words, (8.30) says that an integral about a closed contour in the Argand plane comes to $i2\pi$ times the sum of the residues of the poles (if any) thus enclosed. (Note however that eqn. 8.30 does not handle branch points. If there is a branch point, the contour must exclude it or the formula will not work.)

As we shall see in § 9.5, whether in the form of (8.29) or of (8.30) Cauchy's integral formula is an extremely useful result.[18]

### 8.8.4 Enclosing a multiple pole

When a complex contour of integration encloses a double, triple or other $n$-fold pole, the integration can be written

$$S = \oint \frac{f(z)}{(z - z_o)^{m+1}}\,dz, \quad m \in \mathbb{Z},\ m \geq 0,$$

where $m + 1 = n$. Expanding $f(z)$ in a Taylor series (8.19) about $z = z_o$,

$$S = \oint \sum_{k=0}^{\infty} \left( \left. \frac{d^k f}{dz^k} \right|_{z=z_o} \right) \frac{dz}{(k!)(z - z_o)^{m-k+1}}.$$

[18] [31, § 10.6][57][66, "Cauchy's integral formula," 14:13, 20 April 2006]

But according to (8.26), only the $k = m$ term contributes, so

$$\begin{aligned} S &= \oint \left( \left. \frac{d^m f}{dz^m} \right|_{z=z_o} \right) \frac{dz}{(m!)(z - z_o)} \\ &= \frac{1}{m!} \left( \left. \frac{d^m f}{dz^m} \right|_{z=z_o} \right) \oint \frac{dz}{(z - z_o)} \\ &= \frac{i2\pi}{m!} \left( \left. \frac{d^m f}{dz^m} \right|_{z=z_o} \right), \end{aligned}$$

where the integral is evaluated in the last step according to (8.29). Altogether,

$$\oint \frac{f(z)}{(z - z_o)^{m+1}} \, dz = \frac{i2\pi}{m!} \left( \left. \frac{d^m f}{dz^m} \right|_{z=z_o} \right), \quad m \in \mathbb{Z}, \ m \geq 0. \tag{8.31}$$

Equation (8.31) evaluates a contour integral about an $n$-fold pole as (8.29) does about a single pole. (When $m = 0$, the two equations are the same.)[19]

## 8.9 Taylor series for specific functions

With the general Taylor series formula (8.19), the derivatives of Tables 5.2 and 5.3, and the observation from (4.21) that

$$\frac{d(z^a)}{dz} = az^{a-1},$$

[19] [40][57]

one can calculate Taylor series for many functions. For instance, expanding about $z = 1$,

$$\begin{aligned}
\ln z\big|_{z=1} &= & \ln z\big|_{z=1} &= 0, \\
\frac{d}{dz}\ln z\Big|_{z=1} &= & \frac{1}{z}\Big|_{z=1} &= 1, \\
\frac{d^2}{dz^2}\ln z\Big|_{z=1} &= & \frac{-1}{z^2}\Big|_{z=1} &= -1, \\
\frac{d^3}{dz^3}\ln z\Big|_{z=1} &= & \frac{2}{z^3}\Big|_{z=1} &= 2, \\
& & \vdots & \\
\frac{d^k}{dz^k}\ln z\Big|_{z=1} &= \frac{-(-)^k(k-1)!}{z^k}\Big|_{z=1} & &= -(-)^k(k-1)!, \quad k > 0.
\end{aligned}$$

With these derivatives, the Taylor series about $z = 1$ is

$$\ln z = \sum_{k=1}^{\infty} \left[-(-)^k(k-1)!\right] \frac{(z-1)^k}{k!} = -\sum_{k=1}^{\infty} \frac{(1-z)^k}{k},$$

evidently convergent for $|1 - z| < 1$. (And if $z$ lies outside the convergence domain? Several strategies are then possible. One can expand the Taylor series about a different point; but cleverer and easier is to take advantage of some convenient relationship like $\ln w = -\ln[1/w]$. Section 8.10.4 elaborates.) Using such Taylor series, one can relatively efficiently calculate actual numerical values for $\ln z$ and many other functions.

Table 8.1 lists Taylor series for a few functions of interest. All the series converge for $|z| < 1$. The $\exp z$, $\sin z$ and $\cos z$ series converge for all complex $z$. Among the several series, the series for $\arctan z$ is computed indirectly[20] by way of Table 5.3 and (2.33):

$$\begin{aligned}
\arctan z &= \int_0^z \frac{1}{1+w^2}\,dw \\
&= \int_0^z \sum_{k=0}^{\infty} (-)^k w^{2k}\,dw \\
&= \sum_{k=0}^{\infty} \frac{(-)^k z^{2k+1}}{2k+1}.
\end{aligned}$$

[20] [55, § 11-7]

Table 8.1: Taylor series.

$$
\begin{aligned}
f(z) &= \sum_{k=0}^{\infty} \left( \left. \frac{d^k f}{dz^k} \right|_{z=z_o} \right) \prod_{j=1}^{k} \frac{z - z_o}{j} \\
(1+z)^{a-1} &= \sum_{k=0}^{\infty} \prod_{j=1}^{k} \left( \frac{a}{j} - 1 \right) z \\
\exp z &= \sum_{k=0}^{\infty} \prod_{j=1}^{k} \frac{z}{j} = \sum_{k=0}^{\infty} \frac{z^k}{k!} \\
\sin z &= \sum_{k=0}^{\infty} \left[ z \prod_{j=1}^{k} \frac{-z^2}{(2j)(2j+1)} \right] \\
\cos z &= \sum_{k=0}^{\infty} \prod_{j=1}^{k} \frac{-z^2}{(2j-1)(2j)} \\
\sinh z &= \sum_{k=0}^{\infty} \left[ z \prod_{j=1}^{k} \frac{z^2}{(2j)(2j+1)} \right] \\
\cosh z &= \sum_{k=0}^{\infty} \prod_{j=1}^{k} \frac{z^2}{(2j-1)(2j)} \\
-\ln(1-z) &= \sum_{k=1}^{\infty} \frac{1}{k} \prod_{j=1}^{k} z = \sum_{k=1}^{\infty} \frac{z^k}{k} \\
\arctan z &= \sum_{k=0}^{\infty} \frac{1}{2k+1} \left[ z \prod_{j=1}^{k} (-z^2) \right] = \sum_{k=0}^{\infty} \frac{(-)^k z^{2k+1}}{2k+1}
\end{aligned}
$$

It is interesting to observe from Table 8.1 the useful first-order approximations that

$$\begin{aligned}\lim_{z\to 0} \exp z &= 1 + z,\\ \lim_{z\to 0} \sin z &= z,\end{aligned} \tag{8.32}$$

among others.

## 8.10 Error bounds

One naturally cannot actually sum a Taylor series to an infinite number of terms. One must add some finite number of terms, then quit—which raises the question: how many terms are enough? How can one know that one has added adequately many terms; that the remaining terms, which constitute the tail of the series, are sufficiently insignificant? How can one set error bounds on the truncated sum?

### 8.10.1 Examples

Some series alternate sign. For these it is easy if the numbers involved happen to be real. For example, from Table 8.1,

$$\ln\frac{3}{2} = \ln\left(1 + \frac{1}{2}\right) = \frac{1}{(1)(2^1)} - \frac{1}{(2)(2^2)} + \frac{1}{(3)(2^3)} - \frac{1}{(4)(2^4)} + \cdots$$

Each term is smaller in magnitude than the last, so the true value of $\ln(3/2)$ necessarily lies between the sum of the series to $n$ terms and the sum to $n+1$ terms. The last and next partial sums bound the result. Up to but not including the fourth-order term, for instance,

$$S_4 - \frac{1}{(4)(2^4)} < \ln\frac{3}{2} < S_4,$$
$$S_4 = \frac{1}{(1)(2^1)} - \frac{1}{(2)(2^2)} + \frac{1}{(3)(2^3)}.$$

Other series however do not alternate sign. For example,

$$\begin{aligned}\ln 2 &= -\ln\frac{1}{2} = -\ln\left(1 - \frac{1}{2}\right) = S_5 + R_5,\\ S_5 &= \frac{1}{(1)(2^1)} + \frac{1}{(2)(2^2)} + \frac{1}{(3)(2^3)} + \frac{1}{(4)(2^4)},\\ R_5 &= \frac{1}{(5)(2^5)} + \frac{1}{(6)(2^6)} + \cdots\end{aligned}$$

The basic technique in such a case is to find a replacement series (or integral) $R'_n$ which one can collapse analytically, each of whose terms equals or exceeds in magnitude the corresponding term of $R_n$. For the example, one might choose

$$R'_5 = \frac{1}{5}\sum_{k=5}^{\infty}\frac{1}{2^k} = \frac{2}{(5)(2^5)},$$

wherein (2.34) had been used to collapse the summation. Then,

$$S_5 < \ln 2 < S_5 + R'_5.$$

For real $0 \leq x < 1$ generally,

$$\begin{aligned} S_n &< -\ln(1-x) < S_n + R'_n, \\ S_n &\equiv \sum_{k=1}^{n-1}\frac{x^k}{k}, \\ R'_n &\equiv \sum_{k=n}^{\infty}\frac{x^k}{n} = \frac{x^n}{(n)(1-x)}. \end{aligned}$$

Many variations and refinements are possible, some of which we will meet in the rest of the section, but that is the basic technique: to add several terms of the series to establish a lower bound, then to overestimate the remainder of the series to establish an upper bound. The overestimate $R'_n$ *majorizes* the series' true remainder $R_n$. Notice that the $R'_n$ in the example is a fairly small number, and that it would have been a lot smaller yet had we included a few more terms in $S_n$ (for instance, $n = 0\text{x}40$ would have bound $\ln 2$ tighter than the limit of a computer's typical `double`-type floating-point accuracy). The technique usually works well in practice for this reason.

### 8.10.2 Majorization

*To majorize* in mathematics is to be, or to replace by virtue of being, everywhere at least as great as. This is best explained by example. Consider the summation

$$S = \sum_{k=1}^{\infty}\frac{1}{k^2} = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \cdots$$

The exact value this summation totals to is unknown to us, but the summation does rather resemble the integral (refer to Table 7.1)

$$I = \int_1^{\infty}\frac{dx}{x^2} = -\frac{1}{x}\bigg|_1^{\infty} = 1.$$

Figure 8.3: Majorization. The area $I$ between the dashed curve and the $x$ axis majorizes the area $S - 1$ between the stairstep curve and the $x$ axis, because the height of the dashed curve is everywhere at least as great as that of the stairstep curve.

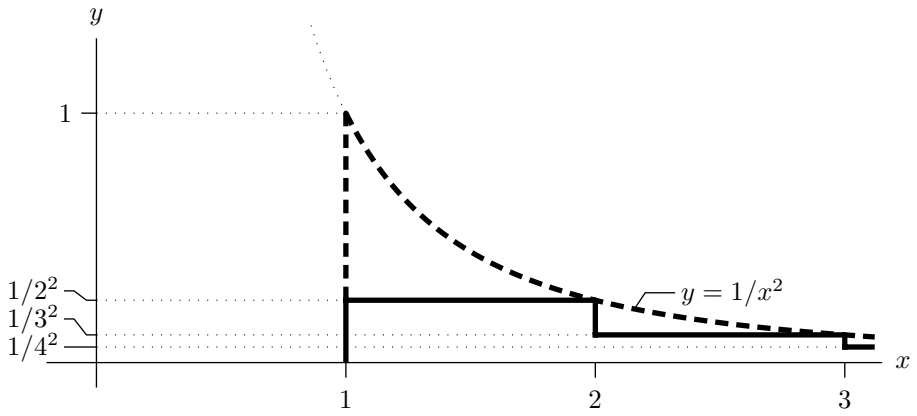



Figure 8.3 plots $S$ and $I$ together as areas—or more precisely, plots $S - 1$ and $I$ together as areas (the summation's first term is omitted). As the plot shows, the unknown area $S - 1$ cannot possibly be as great as the known area $I$. In symbols, $S - 1 < I = 1$; or,

$$S < 2.$$

The integral $I$ majorizes the summation $S - 1$, thus guaranteeing the absolute upper limit on $S$. (Of course $S < 2$ is a very loose limit, but that isn't the point of the example. In practical calculation, one would let a computer add many terms of the series first numerically, and only then majorize the remainder. Even so, cleverer ways to majorize the remainder of this particular series will occur to the reader, such as in representing the terms graphically—not as flat-topped rectangles—but as slant-topped trapezoids, shifted in the figure a half unit rightward.)

Majorization serves surely to bound an unknown quantity by a larger, known quantity. Reflecting, *minorization*[21] serves surely to bound an unknown quantity by a smaller, known quantity. The quantities in question

[21]The author does not remember ever encountering the word *minorization* heretofore in print, but as a reflection of *majorization* the word seems logical. This book at least will use the word where needed. You can use it too if you like.

are often integrals and/or series summations, the two of which are akin as Fig. 8.3 illustrates. The choice of whether to majorize a particular unknown quantity by an integral or by a series summation depends on the convenience of the problem at hand.

The series $S$ of this subsection is interesting, incidentally. It is a *harmonic series* rather than a power series, because although its terms do decrease in magnitude it has no $z^k$ factor (or seen from another point of view, it does have a $z^k$ factor, but $z = 1$), and the ratio of adjacent terms' magnitudes approaches unity as $k$ grows. Harmonic series can be hard to sum accurately, but clever majorization can help (and if majorization does not help enough, the series transformations of [chapter not yet written] can help even more).

### 8.10.3 Geometric majorization

Harmonic series can be hard to sum as § 8.10.2 has observed, but more common than harmonic series are true power series, easier to sum in that they include a $z^k$ factor in each term. There is no one, ideal bound that works equally well for all power series. However, the point of establishing a bound is not to sum a power series exactly but rather to fence the sum within some sufficiently (rather than optimally) small neighborhood. A simple, general bound which works quite adequately for most power series encountered in practice, including among many others all the Taylor series of Table 8.1, is the *geometric majorization*

$$|\epsilon_n| < \frac{|\tau_n|}{1 - |\rho_n|}. \tag{8.33}$$

Here, $\tau_n$ represents the power series' $n$th-order term (in Table 8.1's series for $\exp z$, for example, $\tau_n = z^n/[n!]$). The $|\rho_n|$ is a positive real number chosen, preferably as small as possible, such that

$$\left|\frac{\tau_{k+1}}{\tau_k}\right| \leq |\rho_n| \qquad \text{for all } k \geq n, \tag{8.34}$$

$$\left|\frac{\tau_{k+1}}{\tau_k}\right| < |\rho_n| \quad \text{for at least one } k \geq n,$$

$$0 < |\rho_n| < 1; \tag{8.35}$$

which is to say, more or less, such that each term in the series' tail is smaller than the last by at least a factor of $|\rho_n|$. Given these definitions, if[22]

$$\begin{aligned} S_n &\equiv \sum_{k=0}^{n-1} \tau_k, \\ \epsilon_n &\equiv S_\infty - S_n, \end{aligned} \tag{8.36}$$

where $S_\infty$ represents the true, exact (but uncalculatable, unknown) infinite series sum, then (2.34) and (3.22) imply the geometric majorization (8.33).

If the last paragraph seems abstract, a pair of concrete examples should serve to clarify. First, if the Taylor series

$$-\ln(1-z) = \sum_{k=1}^{\infty} \frac{z^k}{k}$$

of Table 8.1 is truncated before the $n$th-order term, then

$$\begin{aligned} -\ln(1-z) &\approx \sum_{k=1}^{n-1} \frac{z^k}{k}, \\ |\epsilon_n| &< \frac{|z^n|/n}{1-|z|}, \end{aligned}$$

where $\epsilon_n$ is the error in the truncated sum.[23] Here, $|\tau_{k+1}/\tau_k| = [k/(k+1)]\,|z| < |z|$ for all $k \geq n > 0$, so we have chosen $|\rho_n| = |z|$.

Second, if the Taylor series

$$\exp z = \sum_{k=0}^{\infty} \prod_{j=1}^{k} \frac{z}{j} = \sum_{k=0}^{\infty} \frac{z^k}{k!}$$

also of Table 8.1 is truncated before the $n$th-order term, and if we choose to stipulate that

$$n + 1 > |z|\,,$$

[22] Some scientists and engineers—as, for example, the authors of [47] and even this writer in earlier years—prefer to define $\epsilon_n \equiv S_n - S_\infty$, oppositely as we define it here. This choice is a matter of taste. Professional mathematicians—as, for example, the author of [63]—seem to tend toward the $\epsilon_n \equiv S_\infty - S_n$ of (8.36).

[23] This particular error bound fails for $n = 0$, but that is no flaw. There is no reason to use the error bound for $n = 0$ when, merely by taking one or two more terms into the truncated sum, one can quite conveniently let $n = 1$ or $n = 2$.

then

$$\begin{aligned}\exp z &\approx \sum_{k=0}^{n-1}\prod_{j=1}^{k}\frac{z}{j} = \sum_{k=0}^{n-1}\frac{z^k}{k!},\\ |\epsilon_n| &< \frac{|z^n|/n!}{1-|z|/(n+1)}.\end{aligned}$$

Here, $|\tau_{k+1}/\tau_k| = |z|/(k+1)$, whose maximum value for all $k \geq n$ occurs when $k = n$, so we have chosen $|\rho_n| = |z|/(n+1)$.

### 8.10.4 Calculation outside the fast convergence domain

Used directly, the Taylor series of Table 8.1 tend to converge slowly for some values of $z$ and not at all for others. The series for $-\ln(1-z)$ and $(1+z)^{a-1}$ for instance each converge for $|z| < 1$ (though slowly for $|z| \approx 1$); whereas each series diverges when asked to compute a quantity like $-\ln 3$ or $3^{a-1}$ directly. To shift the series' expansion points per § 8.2 is one way to seek convergence, but for nonentire functions (§ 8.6) like these a more probably profitable strategy is to find and exploit some property of the functions to transform their arguments, such as

$$\begin{aligned}-\ln\gamma &= \ln\frac{1}{\gamma},\\ \gamma^{a-1} &= \frac{1}{(1/\gamma)^{a-1}},\end{aligned}$$

which leave the respective Taylor series to compute quantities like $-\ln(1/3)$ and $(1/3)^{a-1}$ they can handle.

Let $f(1+\zeta)$ be a function whose Taylor series about $\zeta = 0$ converges for $|\zeta| < 1$ and which obeys properties of the forms[24]

$$\begin{aligned}f(\gamma) &= g\left[f\left(\frac{1}{\gamma}\right)\right],\\ f(\alpha\gamma) &= h\left[f(\alpha), f(\gamma)\right],\end{aligned} \tag{8.37}$$

where $g[\cdot]$ and $h[\cdot,\cdot]$ are functions we know how to compute like $g[\cdot] = -[\cdot]$

[24]This paragraph's notation is necessarily abstract. To make it seem more concrete, consider that the function $f(1+\zeta) = -\ln(1-z)$ has $\zeta = -z$, $f(\gamma) = g[f(1/\gamma)] = -f(1/\gamma)$ and $f(\alpha\gamma) = h[f(\alpha), f(\gamma)] = f(\alpha) + f(\gamma)$; and that the function $f(1+\zeta) = (1+z)^{a-1}$ has $\zeta = z$, $f(\gamma) = g[f(1/\gamma)] = 1/f(1/\gamma)$ and $f(\alpha\gamma) = h[f(\alpha), f(\gamma)] = f(\alpha)f(\gamma)$.

or $g[\cdot] = 1/[\cdot]$; and like $h[\cdot,\cdot] = [\cdot] + [\cdot]$ or $h[\cdot,\cdot] = [\cdot][\cdot]$. Identifying

$$\begin{aligned} \frac{1}{\gamma} &= 1 + \zeta, \\ \gamma &= \frac{1}{1+\zeta}, \\ \frac{1-\gamma}{\gamma} &= \zeta, \end{aligned} \tag{8.38}$$

we have that

$$f(\gamma) = g\left[f\left(1 + \frac{1-\gamma}{\gamma}\right)\right], \tag{8.39}$$

whose convergence domain $|\zeta| < 1$ is $|1-\gamma| \,/\, |\gamma| < 1$, which is $|\gamma - 1| < |\gamma|$ or in other words

$$\Re(\gamma) > \frac{1}{2}.$$

Although the transformation from $\zeta$ to $\gamma$ has not lifted the convergence limit altogether, we see that it has apparently opened the limit to a broader domain.

Though this writer knows no way to lift the convergence limit altogether that does not cause more problems than it solves, one can take advantage of the $h[\cdot,\cdot]$ property of (8.37) to sidestep the limit, computing $f(\omega)$ indirectly for any $\omega \neq 0$ by any of several tactics. One nonoptimal but entirely effective tactic is represented by the equations

$$\begin{aligned} \omega &\equiv i^n 2^m \gamma, \\ |\Im(\gamma)| &\leq \Re(\gamma), \\ 1 \leq \Re(\gamma) &< 2, \\ m, n &\in \mathbb{Z}, \end{aligned} \tag{8.40}$$

whereupon the formula

$$f(\omega) = h[f(i^n 2^m), f(\gamma)] \tag{8.41}$$

calculates $f(\omega)$ fast for any $\omega \neq 0$—provided only that we have other means to compute $f(i^n 2^m)$, which not infrequently we do.[25]

[25] Equation (8.41) admittedly leaves open the question of how to compute $f(i^n 2^m)$, but at least for the functions this subsection has used as examples this is not hard. For the logarithm, $-\ln(i^n 2^m) = m\ln(1/2) - in(2\pi/4)$. For the power, $(i^n 2^m)^{a-1} = \text{cis}[(n2\pi/4)(a-1)]/[(1/2)^{a-1}]^m$. The sine and cosine in the cis function are each calculated directly by Taylor series (possibly with the help of Table 3.1), as are the numbers $\ln(1/2)$ and $(1/2)^{a-1}$. The number $2\pi$, we have not calculated yet, but will in § 8.11.

Notice how (8.40) fences $\gamma$ within a comfortable zone, keeping $\gamma$ moderately small in magnitude but never too near the $\Re(\gamma) = 1/2$ frontier in the Argand plane. In theory all finite $\gamma$ rightward of the frontier let the Taylor series converge, but extreme $\gamma$ of any kind let the series converge only slowly (and due to compound floating-point rounding error inaccurately) inasmuch as they imply that $|\zeta| \approx 1$. Besides allowing all $\omega \neq 0$, the tactic (8.40) also thus significantly speeds series convergence.

The method and tactic of (8.37) through (8.41) are useful in themselves and also illustrative generally. Of course, most nonentire functions lack properties of the specific kinds that (8.37) demands, but such functions may have other properties one can analogously exploit.[26]

### 8.10.5 Nonconvergent series

Variants of this section's techniques can be used to prove that a series does not converge at all. For example,

$$\sum_{k=1}^{\infty} \frac{1}{k}$$

does not converge because

$$\frac{1}{k} > \int_k^{k+1} \frac{d\tau}{\tau};$$

hence,

$$\sum_{k=1}^{\infty} \frac{1}{k} > \sum_{k=1}^{\infty} \int_k^{k+1} \frac{d\tau}{\tau} = \int_1^{\infty} \frac{d\tau}{\tau} = \ln \infty.$$

[26] To draw another example from Table 8.1, consider that

$$\arctan \omega = \alpha + \arctan \zeta,$$
$$\zeta \equiv \frac{\omega \cos \alpha - \sin \alpha}{\omega \sin \alpha + \cos \alpha},$$

where $\arctan \omega$ is interpreted as the geometrical angle the vector $\hat{\mathbf{x}} + \hat{\mathbf{y}}\omega$ makes with $\hat{\mathbf{x}}$. Axes are rotated per (3.7) through some angle $\alpha$ to reduce the tangent from $\omega$ to $\zeta$, where $\arctan \omega$ is interpreted as the geometrical angle the vector $\hat{\mathbf{x}} + \hat{\mathbf{y}}\omega = \hat{\mathbf{x}}'(\omega \sin \alpha + \cos \alpha) + \hat{\mathbf{y}}'(\omega \cos \alpha - \sin \alpha)$ makes with $\hat{\mathbf{x}}'$, thus causing the Taylor series to converge faster or indeed to converge at all.

Any number of further examples and tactics of the kind will occur to the creative reader, shrinking a function's argument by some convenient means before feeding the argument to the function's Taylor series.

### 8.10.6 Remarks

The study of error bounds is not a matter of rules and formulas so much as of ideas, suggestions and tactics. There is normally no such thing as an optimal error bound—with sufficient cleverness, some tighter bound can usually be discovered—but often easier and more effective than cleverness is simply to add a few extra terms into the series before truncating it (that is, to increase $n$ a little). To eliminate the error entirely usually demands adding an infinite number of terms, which is impossible; but since eliminating the error entirely also requires recording the sum to infinite precision, which is impossible anyway, eliminating the error entirely is not normally a goal one seeks. To eliminate the error to the 0x34-bit (sixteen-decimal place) precision of a computer's `double`-type floating-point representation typically requires something like 0x34 terms—if the series be wisely composed and if care be taken to keep $z$ moderately small and reasonably distant from the edge of the series' convergence domain. Besides, few engineering applications really use much more than 0x10 bits (five decimal places) in any case. Perfect precision is impossible, but adequate precision is usually not hard to achieve.

Occasionally nonetheless a series arises for which even adequate precision is quite hard to achieve. An infamous example is

$$S = -\sum_{k=1}^{\infty} \frac{(-)^k}{\sqrt{k}} = 1 - \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} - \frac{1}{\sqrt{4}} + \cdots ,$$

which obviously converges, but sum it if you can! It is not easy to do.

Before closing the section, we ought to arrest one potential agent of terminological confusion. The "error" in a series summation's error bounds is unrelated to the error of probability theory (Ch. 20). The English word "error" is thus overloaded here. A series sum converges to a definite value, and to the same value every time the series is summed; no chance is involved. It is just that we do not necessarily know exactly what that value is. What we can do, by this section's techniques or perhaps by other methods, is to establish a definite neighborhood in which the unknown value is sure to lie; and we can make that neighborhood as tight as we want, merely by including a sufficient number of terms in the sum.

The topic of series error bounds is what G.S. Brown refers to as "trick-based."[27] There is no general answer to the error-bound problem, but there are several techniques which help, some of which this section has introduced. Other techniques, we shall meet later in the book as the need for them arises.

[27] [10]

## 8.11 Calculating $2\pi$

The Taylor series for $\arctan z$ in Table 8.1 implies a neat way of calculating the constant $2\pi$. We already know that $\tan(2\pi/8) = 1$, or in other words that

$$\arctan 1 = \frac{2\pi}{8}.$$

Applying the Taylor series, we have that

$$2\pi = 8\sum_{k=0}^{\infty} \frac{(-)^k}{2k+1}. \tag{8.42}$$

The series (8.42) is simple but converges extremely slowly. Much faster convergence is given by angles smaller than $2\pi/8$. For example, from Table 3.2,

$$\arctan \frac{\sqrt{3}-1}{\sqrt{3}+1} = \frac{2\pi}{0\text{x}18}.$$

Applying the Taylor series at this angle, we have that[28]

$$2\pi = 0\text{x}18 \sum_{k=0}^{\infty} \frac{(-)^k}{2k+1} \left( \frac{\sqrt{3}-1}{\sqrt{3}+1} \right)^{2k+1} \approx 0\text{x}6.487\text{F}. \tag{8.43}$$

## 8.12 Odd and even functions

An *odd function* is one for which $f(-z) = -f(z)$. Any function whose Taylor series about $z_o = 0$ includes only odd-order terms is an odd function. Examples of odd functions include $z^3$ and $\sin z$.

An *even function* is one for which $f(-z) = f(z)$. Any function whose Taylor series about $z_o = 0$ includes only even-order terms is an even function. Examples of even functions include $z^2$ and $\cos z$.

Odd and even functions are interesting because of the symmetry they bring—the plot of a real-valued odd function being symmetric about a point, the plot of a real-valued even function being symmetric about a line. Many

[28] The writer is given to understand that clever mathematicians have invented subtle, still much faster-converging iterative schemes toward $2\pi$. However, there is fast and there is fast. The relatively straightforward series this section gives converges to the best accuracy of your computer's floating-point register within a paltry 0x40 iterations or so—and, after all, you only need to compute the numerical value of $2\pi$ once.

Admittedly, the writer supposes that useful lessons lurk in the clever mathematics underlying the subtle schemes, but such schemes are not covered here.

functions are neither odd nor even, of course, but one can always split an analytic function into two components—one odd, the other even—by the simple expedient of sorting the odd-order terms from the even-order in the function's Taylor series. For example, $\exp z = \sinh z + \cosh z$. Alternately,

$$\begin{aligned} f(z) &= f_{\text{odd}}(z) + f_{\text{even}}(z), \\ f_{\text{odd}}(z) &= \frac{f(z) - f(-z)}{2}, \\ f_{\text{even}}(z) &= \frac{f(z) + f(-z)}{2}, \end{aligned} \tag{8.44}$$

the latter two lines of which are verified by substituting $-z \leftarrow z$ and observing the definitions at the section's head of odd and even, then the first line of which is verified by adding the latter two.

Section 18.2.7 will have more to say about odd and even functions.

## 8.13 Trigonometric poles

The singularities of the trigonometric functions are single poles of residue $\pm 1$ or $\pm i$. For the circular trigonometrics, all the poles lie along the real number line; for the hyperbolic trigonometrics, along the imaginary. Specifically, of the eight trigonometric functions

$$\begin{aligned} &\frac{1}{\sin z}, \frac{1}{\cos z}, \frac{1}{\tan z}, \tan z, \\ &\frac{1}{\sinh z}, \frac{1}{\cosh z}, \frac{1}{\tanh z}, \tanh z, \end{aligned}$$

the poles and their respective residues are

$$\begin{aligned}
\left.\frac{z-k\pi}{\sin z}\right|_{z=k\pi} &= (-)^k, \\
\left.\frac{z-(k-1/2)\pi}{\cos z}\right|_{z=(k-1/2)\pi} &= (-)^k, \\
\left.\frac{z-k\pi}{\tan z}\right|_{z=k\pi} &= 1, \\
[z-(k-1/2)\pi]\tan z|_{z=(k-1/2)\pi} &= -1, \\
\left.\frac{z-ik\pi}{\sinh z}\right|_{z=ik\pi} &= (-)^k, \\
\left.\frac{z-i(k-1/2)\pi}{\cosh z}\right|_{z=i(k-1/2)\pi} &= i(-)^k, \\
\left.\frac{z-ik\pi}{\tanh z}\right|_{z=ik\pi} &= 1, \\
[z-i(k-1/2)\pi]\tanh z|_{z=i(k-1/2)\pi} &= 1, \\
k &\in \mathbb{Z}.
\end{aligned} \tag{8.45}$$

To support (8.45)'s claims, we shall marshal the identities of Tables 5.1 and 5.2 plus l'Hôpital's rule (4.30). Before calculating residues and such, however, we should like to verify that the poles (8.45) lists are in fact the only poles that there are; that we have forgotten no poles. Consider for instance the function $1/\sin z = i2/(e^{iz}-e^{-iz})$. This function evidently goes infinite only when $e^{iz} = e^{-iz}$, which is possible only for real $z$; but for real $z$, the sine function's very definition establishes the poles $z = k\pi$ (refer to Fig. 3.1). With the observations from Table 5.1 that $i\sinh z = \sin iz$ and $\cosh z = \cos iz$, similar reasoning for each of the eight trigonometrics forbids poles other than those (8.45) lists. Satisfied that we have forgotten no poles, therefore, we finally apply l'Hôpital's rule to each of the ratios

$$\begin{gathered}
\frac{z-k\pi}{\sin z}, \frac{z-(k-1/2)\pi}{\cos z}, \frac{z-k\pi}{\tan z}, \frac{z-(k-1/2)\pi}{1/\tan z}, \\
\frac{z-ik\pi}{\sinh z}, \frac{z-i(k-1/2)\pi}{\cosh z}, \frac{z-ik\pi}{\tanh z}, \frac{z-i(k-1/2)\pi}{1/\tanh z}
\end{gathered}$$

to reach (8.45).

Trigonometric poles evidently are special only in that a trigonometric function has an infinite number of them. The poles are ordinary, single

poles, with residues, subject to Cauchy's integral formula and so on. The trigonometrics are meromorphic functions (§ 8.6) for this reason.[29]

The six simpler trigonometrics $\sin z$, $\cos z$, $\sinh z$, $\cosh z$, $\exp z$ and $\operatorname{cis} z$—conspicuously excluded from this section's gang of eight—have no poles for finite $z$, because $e^z$, $e^{iz}$, $e^z \pm e^{-z}$ and $e^{iz} \pm e^{-iz}$ then likewise are finite. These simpler trigonometrics are not only meromorphic but also entire. Observe however that the *inverse* trigonometrics are multiple-valued and have branch points, and thus are not meromorphic at all.

## 8.14 The Laurent series

Any analytic function can be expanded in a Taylor series, but never about a pole or branch point of the function. Sometimes one nevertheless wants to expand at least about a pole. Consider for example expanding

$$f(z) = \frac{e^{-z}}{1 - \cos z} \tag{8.46}$$

about the function's pole at $z = 0$. Expanding dividend and divisor separately,

$$\begin{aligned} f(z) &= \frac{1 - z + z^2/2 - z^3/6 + \cdots}{z^2/2 - z^4/0\text{x}18 + \cdots} \\ &= \frac{\sum_{j=0}^{\infty} \left[(-)^j z^j / j!\right]}{-\sum_{k=1}^{\infty} (-)^k z^{2k}/(2k)!} \\ &= \frac{\sum_{k=0}^{\infty} \left[-z^{2k}/(2k)! + z^{2k+1}/(2k+1)!\right]}{\sum_{k=1}^{\infty} (-)^k z^{2k}/(2k)!}. \end{aligned}$$

[29] [40]

By long division,

$$\begin{aligned}
f(z) &= \frac{2}{z^2} - \frac{2}{z} + \left\{ \left[ -\frac{2}{z^2} + \frac{2}{z} \right] \sum_{k=1}^{\infty} \frac{(-)^k z^{2k}}{(2k)!} \right. \\
&\qquad \left. + \sum_{k=0}^{\infty} \left[ -\frac{z^{2k}}{(2k)!} + \frac{z^{2k+1}}{(2k+1)!} \right] \right\} \Bigg/ \sum_{k=1}^{\infty} \frac{(-)^k z^{2k}}{(2k)!} \\
&= \frac{2}{z^2} - \frac{2}{z} + \left\{ \sum_{k=1}^{\infty} \left[ -\frac{(-)^k 2 z^{2k-2}}{(2k)!} + \frac{(-)^k 2 z^{2k-1}}{(2k)!} \right] \right. \\
&\qquad \left. + \sum_{k=0}^{\infty} \left[ -\frac{z^{2k}}{(2k)!} + \frac{z^{2k+1}}{(2k+1)!} \right] \right\} \Bigg/ \sum_{k=1}^{\infty} \frac{(-)^k z^{2k}}{(2k)!} \\
&= \frac{2}{z^2} - \frac{2}{z} + \left\{ \sum_{k=0}^{\infty} \left[ \frac{(-)^k 2 z^{2k}}{(2k+2)!} - \frac{(-)^k 2 z^{2k+1}}{(2k+2)!} \right] \right. \\
&\qquad \left. + \sum_{k=0}^{\infty} \left[ -\frac{z^{2k}}{(2k)!} + \frac{z^{2k+1}}{(2k+1)!} \right] \right\} \Bigg/ \sum_{k=1}^{\infty} \frac{(-)^k z^{2k}}{(2k)!} \\
&= \frac{2}{z^2} - \frac{2}{z} + \sum_{k=0}^{\infty} \left[ \frac{-(2k+1)(2k+2) + (-)^k 2}{(2k+2)!} z^{2k} \right. \\
&\qquad \left. + \frac{(2k+2) - (-)^k 2}{(2k+2)!} z^{2k+1} \right] \Bigg/ \sum_{k=1}^{\infty} \frac{(-)^k z^{2k}}{(2k)!}.
\end{aligned}$$

The remainder's $k = 0$ terms now disappear as intended; so, factoring $z^2/z^2$ from the division leaves

$$\begin{aligned}
f(z) &= \frac{2}{z^2} - \frac{2}{z} + \sum_{k=0}^{\infty} \left[ \frac{(2k+3)(2k+4) + (-)^k 2}{(2k+4)!} z^{2k} \right. \\
&\qquad \left. - \frac{(2k+4) + (-)^k 2}{(2k+4)!} z^{2k+1} \right] \Bigg/ \sum_{k=0}^{\infty} \frac{(-)^k z^{2k}}{(2k+2)!}.
\end{aligned} \tag{8.47}$$

One can continue dividing to extract further terms if desired, and if all the terms

$$f(z) = \frac{2}{z^2} - \frac{2}{z} + \frac{7}{6} - \frac{z}{2} + \cdots$$

are extracted the result is the *Laurent series* proper,

$$f(z) = \sum_{k=K}^{\infty} (a_k)(z - z_o)^k, \quad (k, K) \in \mathbb{Z}, \ K \leq 0. \tag{8.48}$$

However for many purposes (as in eqn. 8.47) the partial Laurent series

$$\begin{aligned} f(z) &= \sum_{k=K}^{-1} (a_k)(z - z_o)^k + \frac{\sum_{k=0}^{\infty} (b_k)(z - z_o)^k}{\sum_{k=0}^{\infty} (c_k)(z - z_o)^k}, \\ (k, K) &\in \mathbb{Z}, \quad K \leq 0, \ c_0 \neq 0, \end{aligned} \tag{8.49}$$

suffices and may even be preferable. In either form,

$$f(z) = \sum_{k=K}^{-1} (a_k)(z - z_o)^k + f_o(z - z_o), \quad (k, K) \in \mathbb{Z}, \ K \leq 0, \tag{8.50}$$

where, unlike $f(z)$, $f_o(z - z_o)$ is analytic at $z = z_o$. The $f_o(z - z_o)$ of (8.50) is $f(z)$'s *regular part* at $z = z_o$.

The ordinary Taylor series diverges at a function's pole. Handling the pole separately, the Laurent series remedies this defect.[30,31]

Sections 9.5 and 9.6 tell more about poles generally, including multiple poles like the one in the example here.

## 8.15 Taylor series in $1/z$

A little imagination helps the Taylor series a lot. The Laurent series of § 8.14 represents one way to extend the Taylor series. Several other ways are possible. The typical trouble one has with the Taylor series is that a function's poles and branch points limit the series' convergence domain. Thinking flexibly, however, one can often evade the trouble.

Consider the function

$$f(z) = \frac{\sin(1/z)}{\cos z}.$$

This function has a nonanalytic point of a most peculiar nature at $z = 0$. The point is an essential singularity, and one cannot expand the function directly about it. One could expand the function directly about some other point like $z = 1$, but calculating the Taylor coefficients would take a lot of effort and, even then, the resulting series would suffer a straitly limited

[30] The professional mathematician's treatment of the Laurent series usually begins by defining an annular convergence domain (a convergence domain bounded without by a large circle and within by a small) in the Argand plane. From an applied point of view however what interests us is the basic technique to remove the poles from an otherwise analytic function. Further rigor is left to the professionals.

[31] [31, § 10.8][20, § 2.5]

convergence domain. All that however tries too hard. Depending on the application, it may suffice to write

$$f(z) = \frac{\sin w}{\cos z}, \quad w \equiv \frac{1}{z}.$$

This is

$$f(z) = \frac{z^{-1} - z^{-3}/3! + z^{-5}/5! - \cdots}{1 - z^2/2! + z^4/4! - \cdots},$$

which is all one needs to calculate $f(z)$ numerically—and may be all one needs for analysis, too.

As an example of a different kind, consider

$$g(z) = \frac{1}{(z-2)^2}.$$

Most often, one needs no Taylor series to handle such a function (one simply does the indicated arithmetic). Suppose however that a Taylor series specifically about $z = 0$ were indeed needed for some reason. Then by (8.1) and (4.2),

$$g(z) = \frac{1/4}{(1 - z/2)^2} = \frac{1}{4} \sum_{k=0}^{\infty} \binom{1+k}{1} \left[\frac{z}{2}\right]^k = \sum_{k=0}^{\infty} \frac{k+1}{2^{k+2}} z^k,$$

That expansion is good only for $|z| < 2$, but for $|z| > 2$ we also have that

$$g(z) = \frac{1/z^2}{(1 - 2/z)^2} = \frac{1}{z^2} \sum_{k=0}^{\infty} \binom{1+k}{1} \left[\frac{2}{z}\right]^k = \sum_{k=2}^{\infty} \frac{2^{k-2}(k-1)}{z^k},$$

which expands in negative rather than positive powers of $z$. Note that we have computed the two series for $g(z)$ without ever actually taking a derivative.

Neither of the section's examples is especially interesting in itself, but their point is that it often pays to think flexibly in extending and applying the Taylor series. One is not required immediately to take the Taylor series of a function as it presents itself; one can first change variables or otherwise rewrite the function in some convenient way, then take the Taylor series either of the whole function at once or of pieces of it separately. One can expand in negative powers of $z$ equally validly as in positive powers. And, though taking derivatives per (8.19) may be the canonical way to determine Taylor coefficients, any effective means to find the coefficients suffices.

## 8.16 The multidimensional Taylor series

Equation (8.19) has given the Taylor series for functions of a single variable. The idea of the Taylor series does not differ where there are two or more independent variables, only the details are a little more complicated. For example, consider the function $f(z_1, z_2) = z_1^2 + z_1 z_2 + 2z_2$, which has terms $z_1^2$ and $2z_2$—these we understand—but also has the cross-term $z_1 z_2$ for which the relevant derivative is the cross-derivative $\partial^2 f / \partial z_1 \, \partial z_2$. Where two or more independent variables are involved, one must account for the cross-derivatives, too.

With this idea in mind, the multidimensional Taylor series is

$$f(\mathbf{z}) = \sum_{\mathbf{k}} \left( \left. \frac{\partial^{\mathbf{k}} f}{\partial \mathbf{z}^{\mathbf{k}}} \right|_{\mathbf{z}=\mathbf{z}_o} \right) \frac{(\mathbf{z} - \mathbf{z}_o)^{\mathbf{k}}}{\mathbf{k}!}. \tag{8.51}$$

Well, that's neat. What does it mean?

- The $\mathbf{z}$ is a *vector*[32] incorporating the several independent variables $z_1, z_2, \ldots, z_N$.
- The $\mathbf{k}$ is a nonnegative integer vector of $N$ counters—$k_1, k_2, \ldots, k_N$—one for each of the independent variables. Each of the $k_n$ runs independently from 0 to $\infty$, and every permutation is possible. For example, if $N = 2$ then

$$\begin{aligned} \mathbf{k} &= (k_1, k_2) \\ &= (0,0), (0,1), (0,2), (0,3), \ldots; \\ &\quad (1,0), (1,1), (1,2), (1,3), \ldots; \\ &\quad (2,0), (2,1), (2,2), (2,3), \ldots; \\ &\quad (3,0), (3,1), (3,2), (3,3), \ldots; \\ &\quad \ldots \end{aligned}$$

- The $\partial^{\mathbf{k}} f / \partial \mathbf{z}^{\mathbf{k}}$ represents the $\mathbf{k}$th cross-derivative of $f(\mathbf{z})$, meaning that

$$\frac{\partial^{\mathbf{k}} f}{\partial \mathbf{z}^{\mathbf{k}}} \equiv \left( \prod_{n=1}^{N} \frac{\partial^{k_n}}{(\partial z_n)^{k_n}} \right) f.$$

[32] In this generalized sense of the word, a *vector* is an ordered set of $N$ elements. The geometrical vector $\mathbf{v} = \hat{\mathbf{x}} x + \hat{\mathbf{y}} y + \hat{\mathbf{z}} z$ of § 3.3, then, is a vector with $N = 3$, $v_1 = x$, $v_2 = y$ and $v_3 = z$. (Generalized vectors of arbitrary $N$ will figure prominently in the book from Ch. 11 onward.)

- The $(\mathbf{z} - \mathbf{z}_o)^{\mathbf{k}}$ represents

$$(\mathbf{z} - \mathbf{z}_o)^{\mathbf{k}} \equiv \prod_{n=1}^{N} (z_n - z_{on})^{k_n}.$$

- The $\mathbf{k}!$ represents

$$\mathbf{k}! \equiv \prod_{n=1}^{N} k_n!.$$

With these definitions, the multidimensional Taylor series (8.51) yields all the right derivatives and cross-derivatives at the expansion point $\mathbf{z} = \mathbf{z}_o$. Thus within some convergence domain about $\mathbf{z} = \mathbf{z}_o$, the multidimensional Taylor series (8.51) represents a function $f(\mathbf{z})$ as accurately as the simple Taylor series (8.19) represents a function $f(z)$, and for the same reason.

# Chapter 9

# Integration techniques

Equation (4.19) implies a general technique for calculating a derivative symbolically. Its counterpart (7.1), unfortunately, implies a general technique only for calculating an integral *numerically*—and even for this purpose it is imperfect; for, when it comes to adding an infinite number of infinitesimal elements, how is one actually to do the sum?

It turns out that there is no one general answer to this question. Some functions are best integrated by one technique, some by another. It is hard to guess in advance which technique might work best.

This chapter surveys several weapons of the intrepid mathematician's arsenal against the integral.

## 9.1 Integration by antiderivative

The simplest way to solve an integral is just to look at it, recognizing its integrand to be the derivative of something already known:[1]

$$\int_a^z \frac{df}{d\tau}\,d\tau = f(\tau)\Big|_a^z. \tag{9.1}$$

For instance,

$$\int_1^x \frac{1}{\tau}\,d\tau = \ln\tau\Big|_1^x = \ln x.$$

One merely looks at the integrand $1/\tau$, recognizing it to be the derivative of $\ln\tau$, then directly writes down the solution $\ln\tau|_1^x$. Refer to § 7.2.

[1] The notation $f(\tau)|_a^z$ or $[f(\tau)]_a^z$ means $f(z) - f(a)$.

The technique by itself is pretty limited. However, the frequent object of other integration techniques is to transform an integral into a form to which this basic technique can be applied.

Besides the essential

$$\tau^{a-1} = \frac{d}{d\tau}\left(\frac{\tau^a}{a}\right), \tag{9.2}$$

Tables 7.1, 5.2, 5.3 and 9.1 provide several further good derivatives this antiderivative technique can use.

One particular, nonobvious, useful variation on the antiderivative technique seems worth calling out specially here. If $z = \rho e^{i\phi}$, then (8.24) and (8.25) have that

$$\int_{z_1}^{z_2} \frac{dz}{z} = \ln\frac{\rho_2}{\rho_1} + i(\phi_2 - \phi_1). \tag{9.3}$$

This helps, for example, when $z_1$ and $z_2$ are real but negative numbers.

## 9.2 Integration by substitution

Consider the integral

$$S = \int_{x_1}^{x_2} \frac{x\,dx}{1+x^2}.$$

This integral is not in a form one immediately recognizes. However, with the change of variable

$$u \leftarrow 1 + x^2,$$

whose differential is (by successive steps)

$$\begin{aligned} d(u) &= d(1+x^2), \\ du &= 2x\,dx, \end{aligned}$$

the integral is

$$\begin{aligned} S &= \int_{x=x_1}^{x_2} \frac{x\,dx}{u} \\ &= \int_{x=x_1}^{x_2} \frac{2x\,dx}{2u} \\ &= \int_{u=1+x_1^2}^{1+x_2^2} \frac{du}{2u} \\ &= \frac{1}{2}\ln u\Big|_{u=1+x_1^2}^{1+x_2^2} \\ &= \frac{1}{2}\ln\frac{1+x_2^2}{1+x_1^2}. \end{aligned}$$

To check the result, we can take the derivative per § 7.5 of the final expression with respect to $x_2$:

$$\begin{aligned} \frac{\partial}{\partial x_2}\frac{1}{2}\ln\frac{1+x_2^2}{1+x_1^2}\Big|_{x_2=x} &= \left[\frac{1}{2}\frac{\partial}{\partial x_2}\left\{\ln\left(1+x_2^2\right) - \ln\left(1+x_1^2\right)\right\}\right]_{x_2=x} \\ &= \frac{x}{1+x^2}, \end{aligned}$$

which indeed has the form of the integrand we started with.

The technique is *integration by substitution.* It does not solve all integrals but it does solve many, whether alone or in combination with other techniques.

## 9.3 Integration by parts

Integration by parts is a curious but very broadly applicable technique which begins with the derivative product rule (4.25),

$$d(uv) = u\,dv + v\,du,$$

where $u(\tau)$ and $v(\tau)$ are functions of an independent variable $\tau$. Reordering terms,

$$u\,dv = d(uv) - v\,du.$$

Integrating,

$$\int_{\tau=a}^{b} u\,dv = uv\Big|_{\tau=a}^{b} - \int_{\tau=a}^{b} v\,du. \tag{9.4}$$

Equation (9.4) is the rule of *integration by parts*.

For an example of the rule's operation, consider the integral

$$S(x) = \int_0^x \tau \cos \alpha\tau \, d\tau.$$

Unsure how to integrate this, we can begin by integrating *part* of it. We can begin by integrating the $\cos \alpha\tau \, d\tau$ part. Letting

$$\begin{aligned} u &\leftarrow \tau, \\ dv &\leftarrow \cos \alpha\tau \, d\tau, \end{aligned}$$

we find that[2]

$$\begin{aligned} du &= d\tau, \\ v &= \frac{\sin \alpha\tau}{\alpha}. \end{aligned}$$

According to (9.4), then,

$$S(x) = \left. \frac{\tau \sin \alpha\tau}{\alpha} \right|_0^x - \int_0^x \frac{\sin \alpha\tau}{\alpha} \, d\tau = \frac{x}{\alpha} \sin \alpha x + \cos \alpha x - 1.$$

Though integration by parts is a powerful technique, one should understand clearly what it does and does not do. The technique does not just integrate each part of an integral separately. It isn't that simple. What it does is to integrate one part of an integral separately—whichever part one has chosen to identify as $dv$—while contrarily differentiating the other part $u$, upon which it rewards the mathematician only with a whole new integral $\int v \, du$. The new integral may or may not be easier to integrate than was the original $\int u \, dv$. The virtue of the technique lies in that one often can find a part $dv$ which does yield an easier $\int v \, du$. The technique is powerful for this reason.

For another kind of example of the rule's operation, consider the definite integral[3]

$$\Gamma(z) \equiv \int_0^\infty e^{-\tau} \tau^{z-1} \, d\tau, \quad \Re(z) > 0. \tag{9.5}$$

[2]The careful reader will observe that $v = (\sin \alpha\tau)/\alpha + C$ matches the chosen $dv$ for any value of $C$, not just for $C = 0$. This is true. However, nothing in the integration by parts technique requires us to consider all possible $v$. Any convenient $v$ suffices. In this case, we choose $v = (\sin \alpha\tau)/\alpha$.

[3][43]

Letting

$$\begin{aligned} u &\leftarrow e^{-\tau}, \\ dv &\leftarrow \tau^{z-1}\,d\tau, \end{aligned}$$

we evidently have that

$$\begin{aligned} du &= -e^{-\tau}\,d\tau, \\ v &= \frac{\tau^z}{z}. \end{aligned}$$

Substituting these according to (9.4) into (9.5) yields

$$\begin{aligned} \Gamma(z) &= \left[e^{-\tau}\frac{\tau^z}{z}\right]_{\tau=0}^{\infty} - \int_0^\infty \left(\frac{\tau^z}{z}\right)\left(-e^{-\tau}\,d\tau\right) \\ &= [0-0] + \int_0^\infty \frac{\tau^z}{z}e^{-\tau}\,d\tau \\ &= \frac{\Gamma(z+1)}{z}. \end{aligned}$$

When written

$$\Gamma(z+1) = z\Gamma(z), \tag{9.6}$$

this is an interesting result. Since per (9.5)

$$\Gamma(1) = \int_0^\infty e^{-\tau}\,d\tau = \left[-e^{-\tau}\right]_0^\infty = 1,$$

it follows by induction on (9.6) that

$$(n-1)! = \Gamma(n). \tag{9.7}$$

Thus (9.5), called the *gamma function,* can be taken as an extended definition of the factorial $(z-1)!$ for all $z$, $\Re(z) > 0$. Integration by parts has made this finding possible.

## 9.4 Integration by unknown coefficients

One of the more powerful integration techniques is relatively inelegant, yet it easily cracks some integrals that give other techniques trouble. The technique is the *method of unknown coefficients,* and it is based on the antiderivative (9.1) plus intelligent guessing. It is best illustrated by example.

Consider the integral (which arises in probability theory)

$$S(x) = \int_0^x e^{-(\rho/\sigma)^2/2} \rho \, d\rho. \tag{9.8}$$

If one does not know how to solve the integral in a more elegant way, one can *guess* a likely-seeming antiderivative form, such as

$$e^{-(\rho/\sigma)^2/2} \rho = \frac{d}{d\rho} a e^{-(\rho/\sigma)^2/2},$$

where the $a$ is an *unknown coefficient.* Having guessed, one has no guarantee that the guess is right, but see: if the guess *were* right, then the antiderivative would have the form

$$\begin{aligned} e^{-(\rho/\sigma)^2/2} \rho &= \frac{d}{d\rho} a e^{-(\rho/\sigma)^2/2} \\ &= -\frac{a\rho}{\sigma^2} e^{-(\rho/\sigma)^2/2}, \end{aligned}$$

implying that

$$a = -\sigma^2$$

(evidently the guess is right, after all). Using this value for $a$, one can write the specific antiderivative

$$e^{-(\rho/\sigma)^2/2} \rho = \frac{d}{d\rho} \left[ -\sigma^2 e^{-(\rho/\sigma)^2/2} \right],$$

with which one can solve the integral, concluding that

$$S(x) = \left[ -\sigma^2 e^{-(\rho/\sigma)^2/2} \right]_0^x = \left( \sigma^2 \right) \left[ 1 - e^{-(x/\sigma)^2/2} \right]. \tag{9.9}$$

The same technique solves differential equations, too. Consider for example the differential equation

$$dx = (Ix - P)\, dt, \quad x|_{t=0} = x_o, \; x|_{t=T} = 0, \tag{9.10}$$

which conceptually represents[4] the changing balance $x$ of a bank loan account over time $t$, where $I$ is the loan's interest rate and $P$ is the borrower's payment rate. If it is desired to find the correct payment rate $P$ which pays

[4]Real banks (in the author's country, at least) by law or custom actually use a needlessly more complicated formula—and not only more complicated, but mathematically slightly incorrect, too.

the loan off in the time $T$, then (perhaps after some bad guesses) we guess the form

$$x(t) = Ae^{\alpha t} + B,$$

where $\alpha$, $A$ and $B$ are unknown coefficients. The guess' derivative is

$$dx = \alpha A e^{\alpha t}\, dt.$$

Substituting the last two equations into (9.10) and dividing by $dt$ yields

$$\alpha A e^{\alpha t} = IAe^{\alpha t} + IB - P,$$

which at least is satisfied if both of the equations

$$\begin{aligned} \alpha A e^{\alpha t} &= IAe^{\alpha t}, \\ 0 &= IB - P, \end{aligned}$$

are satisfied. Evidently good choices for $\alpha$ and $B$, then, are

$$\begin{aligned} \alpha &= I, \\ B &= \frac{P}{I}. \end{aligned}$$

Substituting these coefficients into the $x(t)$ equation above yields the general solution

$$x(t) = Ae^{It} + \frac{P}{I} \tag{9.11}$$

to (9.10). The constants $A$ and $P$, we establish by applying the given *boundary conditions* $x|_{t=0} = x_o$ and $x|_{t=T} = 0$. For the former condition, (9.11) is

$$x_o = Ae^{(I)(0)} + \frac{P}{I} = A + \frac{P}{I};$$

and for the latter condition,

$$0 = Ae^{IT} + \frac{P}{I}.$$

Solving the last two equations simultaneously, we have that

$$\begin{aligned} A &= \frac{-e^{-IT}x_o}{1 - e^{-IT}}, \\ P &= \frac{Ix_o}{1 - e^{-IT}}. \end{aligned} \tag{9.12}$$

Applying these to the general solution (9.11) yields the specific solution

$$x(t) = \frac{x_o}{1 - e^{-IT}} \left[ 1 - e^{(I)(t-T)} \right] \tag{9.13}$$

to (9.10) meeting the boundary conditions, with the payment rate $P$ required of the borrower given by (9.12).

The virtue of the method of unknown coefficients lies in that it permits one to try an entire family of candidate solutions at once, with the family members distinguished by the values of the coefficients. If a solution exists anywhere in the family, the method usually finds it.

The method of unknown coefficients is an elephant. Slightly inelegant the method may be, but it is pretty powerful, too—and it has surprise value (for some reason people seem not to expect it). Such are the kinds of problems the method can solve.

## 9.5 Integration by closed contour

We pass now from the elephant to the falcon, from the inelegant to the sublime. Consider the definite integral[5]

$$S = \int_0^\infty \frac{\tau^a}{\tau + 1} \, d\tau, \quad -1 < a < 0.$$

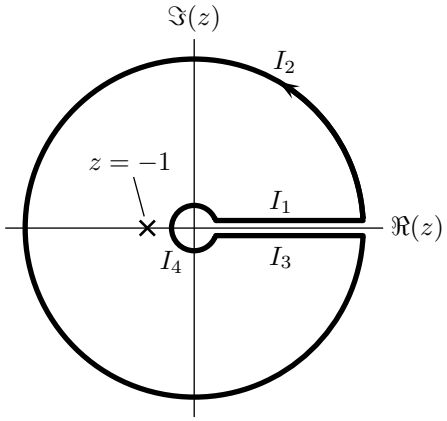
This is a hard integral. No obvious substitution, no evident factoring into parts, seems to solve the integral; but there is a way. The integrand has a pole at $\tau = -1$. Observing that $\tau$ is only a dummy integration variable, if one writes the same integral using the complex variable $z$ in place of the real variable $\tau$, then Cauchy's integral formula (8.29) has that integrating once counterclockwise about a closed complex contour, with the contour enclosing the pole at $z = -1$ but shutting out the branch point at $z = 0$, yields

$$I = \oint \frac{z^a}{z + 1} \, dz = i2\pi z^a |_{z=-1} = i2\pi \left( e^{i2\pi/2} \right)^a = i2\pi e^{i2\pi a/2}.$$

The trouble, of course, is that the integral $S$ does not go about a closed complex contour. One can however construct a closed complex contour $I$ of which $S$ is a part, as in Fig 9.1. If the outer circle in the figure is of infinite

[5] [43, § 1.2]

Figure 9.1: Integration by closed contour.




radius and the inner, of infinitesimal, then the closed contour $I$ is composed of the four parts

$$\begin{aligned} I &= I_1 + I_2 + I_3 + I_4 \\ &= (I_1 + I_3) + I_2 + I_4. \end{aligned}$$

The figure tempts one to make the mistake of writing that $I_1 = S = -I_3$, but besides being incorrect this defeats the purpose of the closed contour technique. More subtlety is needed. One must take care to interpret the four parts correctly. The integrand $z^a/(z+1)$ is multiple-valued; so, in fact, the two parts $I_1 + I_3 \neq 0$ do not cancel. The integrand has a branch point at $z = 0$, which, in passing from $I_3$ through $I_4$ to $I_1$, the contour has circled. Even though $z$ itself takes on the same values along $I_3$ as along $I_1$, the multiple-valued integrand $z^a/(z+1)$ does not. Indeed,

$$\begin{aligned} I_1 &= \int_0^\infty \frac{(\rho e^{i0})^a}{(\rho e^{i0}) + 1}\, d\rho &&= \int_0^\infty \frac{\rho^a}{\rho + 1}\, d\rho &&= S, \\ -I_3 &= \int_0^\infty \frac{(\rho e^{i2\pi})^a}{(\rho e^{i2\pi}) + 1}\, d\rho &&= e^{i2\pi a} \int_0^\infty \frac{\rho^a}{\rho + 1}\, d\rho &&= e^{i2\pi a} S. \end{aligned}$$

Therefore,

$$\begin{aligned}
I &= I_1 + I_2 + I_3 + I_4 \\
&= (I_1 + I_3) + I_2 + I_4 \\
&= (1 - e^{i2\pi a})S + \lim_{\rho\to\infty} \int_{\phi=0}^{2\pi} \frac{z^a}{z+1}\,dz - \lim_{\rho\to 0} \int_{\phi=0}^{2\pi} \frac{z^a}{z+1}\,dz \\
&= (1 - e^{i2\pi a})S + \lim_{\rho\to\infty} \int_{\phi=0}^{2\pi} z^{a-1}\,dz - \lim_{\rho\to 0} \int_{\phi=0}^{2\pi} z^a\,dz \\
&= (1 - e^{i2\pi a})S + \lim_{\rho\to\infty} \frac{z^a}{a}\bigg|_{\phi=0}^{2\pi} - \lim_{\rho\to 0} \frac{z^{a+1}}{a+1}\bigg|_{\phi=0}^{2\pi}.
\end{aligned}$$

Since $a < 0$, the first limit vanishes; and because $a > -1$, the second limit vanishes, too, leaving

$$I = (1 - e^{i2\pi a})S.$$

But by Cauchy's integral formula we have already found an expression for $I$. Substituting this expression into the last equation yields, by successive steps,

$$\begin{aligned}
i2\pi e^{i2\pi a/2} &= (1 - e^{i2\pi a})S, \\
S &= \frac{i2\pi e^{i2\pi a/2}}{1 - e^{i2\pi a}}, \\
S &= \frac{i2\pi}{e^{-i2\pi a/2} - e^{i2\pi a/2}}, \\
S &= -\frac{2\pi/2}{\sin(2\pi a/2)}.
\end{aligned}$$

That is,

$$\int_0^\infty \frac{\tau^a}{\tau+1}\,d\tau = -\frac{2\pi/2}{\sin(2\pi a/2)}, \quad -1 < a < 0, \tag{9.14}$$

an astonishing result.[6]

Another example[7] is

$$T = \int_0^{2\pi} \frac{d\theta}{1 + a\cos\theta}, \quad \Im(a) = 0,\ |\Re(a)| < 1.$$

[6]So astonishing is the result, that one is unlikely to believe it at first encounter. However, straightforward (though computationally highly inefficient) numerical integration per (7.1) confirms the result, as the interested reader and his computer can check. Such results vindicate the effort we have spent in deriving Cauchy's integral formula (8.29).

[7][40]

As in the previous example, here again the contour is not closed. The previous example closed the contour by extending it, excluding the branch point. In this example there is no branch point to exclude, nor need one extend the contour. Rather, one changes the variable

$$z \leftarrow e^{i\theta}$$

and takes advantage of the fact that $z$, unlike $\theta$, *begins and ends the integration at the same point.* One thus obtains the equivalent integral

$$\begin{aligned} T &= \oint \frac{dz/iz}{1+(a/2)(z+1/z)} = -\frac{i2}{a}\oint \frac{dz}{z^2+2z/a+1} \\ &= -\frac{i2}{a}\oint \frac{dz}{\left[z-\left(-1+\sqrt{1-a^2}\right)/a\right]\left[z-\left(-1-\sqrt{1-a^2}\right)/a\right]}, \end{aligned}$$

whose contour is the unit circle in the Argand plane. The integrand evidently has poles at

$$z = \frac{-1 \pm \sqrt{1-a^2}}{a},$$

whose magnitudes are such that

$$|z|^2 = \frac{2-a^2 \mp 2\sqrt{1-a^2}}{a^2}.$$

One of the two magnitudes is less than unity and one is greater, meaning that one of the two poles lies within the contour and one lies without, as is

seen by the successive steps[8]

$$\begin{aligned}
a^2 &< 1, \\
0 &< 1-a^2, \\
(-a^2)(0) &> (-a^2)(1-a^2), \\
0 &> -a^2+a^4, \\
1-a^2 &> 1-2a^2+a^4, \\
1-a^2 &> \left(1-a^2\right)^2, \\
\sqrt{1-a^2} &> 1-a^2, \\
-\sqrt{1-a^2} < -(1-a^2) &< \sqrt{1-a^2}, \\
1-\sqrt{1-a^2} < a^2 &< 1+\sqrt{1-a^2}, \\
2-2\sqrt{1-a^2} < 2a^2 &< 2+2\sqrt{1-a^2}, \\
2-a^2-2\sqrt{1-a^2} < a^2 &< 2-a^2+2\sqrt{1-a^2}, \\
\frac{2-a^2-2\sqrt{1-a^2}}{a^2} < 1 &< \frac{2-a^2+2\sqrt{1-a^2}}{a^2}.
\end{aligned}$$

Per Cauchy's integral formula (8.29), integrating about the pole within the contour yields

$$T = i2\pi \frac{-i2/a}{z-\left(-1-\sqrt{1-a^2}\right)/a}\bigg|_{z=\left(-1+\sqrt{1-a^2}\right)/a} = \frac{2\pi}{\sqrt{1-a^2}}.$$

Observe that by means of a complex variable of integration, each example has indirectly evaluated an integral whose integrand is purely real. If it seems unreasonable to the reader to expect so flamboyant a technique actually to work, this seems equally unreasonable to the writer—but work it does, nevertheless. It is a great technique.

The technique, *integration by closed contour,* is found in practice to solve many integrals other techniques find almost impossible to crack. The key to making the technique work lies in closing a contour one knows how to treat. The robustness of the technique lies in that any contour of any shape will work, so long as the contour encloses appropriate poles in the Argand domain plane while shutting branch points out.

[8] These steps are perhaps best read from bottom to top. See Ch. 6's footnote 16.

The extension

$$\left|\int_{z_1}^{z_2} f(z)\,dz\right| \le \int_{z_1}^{z_2} |f(z)\,dz| \tag{9.15}$$

of the complex triangle sum inequality (3.22) from the discrete to the continuous case sometimes proves useful in evaluating integrals by this section's technique, as in § 17.5.4.

## 9.6 Integration by partial-fraction expansion

This section treats integration by partial-fraction expansion. It introduces the expansion itself first.[9] Throughout the section,

$$j, j', k, \ell, m, n, p, p_{(\cdot)}, M, N \in \mathbb{Z}.$$

### 9.6.1 Partial-fraction expansion

Consider the function

$$f(z) = \frac{-4}{z-1} + \frac{5}{z-2}.$$

Combining the two fractions over a common denominator[10] yields

$$f(z) = \frac{z+3}{(z-1)(z-2)}.$$

Of the two forms, the former is probably the more amenable to analysis. For example, using (9.3),

$$\begin{aligned}\int_{-1}^{0} f(\tau)\,d\tau &= \int_{-1}^{0} \frac{-4}{\tau-1}\,d\tau + \int_{-1}^{0} \frac{5}{\tau-2}\,d\tau \\ &= \left[-4\ln(1-\tau) + 5\ln(2-\tau)\right]_{-1}^{0}.\end{aligned}$$

The trouble is that one is not always given the function in the amenable form.

Given a *rational function*

$$f(z) = \frac{\sum_{k=0}^{N-1} b_k z^k}{\prod_{j=1}^{N} (z-\alpha_j)} \tag{9.16}$$

[9] [48, Appendix F][31, §§ 2.7 and 10.12]

[10] Terminology (you probably knew this already): A *fraction* is the ratio of two numbers or expressions $B/A$. In the fraction, $B$ is the *numerator* and $A$ is the *denominator*. The *quotient* is $Q = B/A$.

in which no two of the several poles $\alpha_j$ are the same, the *partial-fraction expansion* has the form

$$f(z) = \sum_{k=1}^{N} \frac{A_k}{z - \alpha_k}, \tag{9.17}$$

where multiplying each fraction of (9.17) by

$$\frac{\left[\prod_{j=1}^{N}(z - \alpha_j)\right]/(z - \alpha_k)}{\left[\prod_{j=1}^{N}(z - \alpha_j)\right]/(z - \alpha_k)}$$

puts the several fractions over a common denominator, yielding (9.16). Dividing (9.16) by (9.17) gives the ratio

$$1 = \frac{\sum_{k=0}^{N-1} b_k z^k}{\prod_{j=1}^{N}(z - \alpha_j)} \Bigg/ \sum_{k=1}^{N} \frac{A_k}{z - \alpha_k}.$$

In the immediate neighborhood of $z = \alpha_m$, the $m$th term $A_m/(z - \alpha_m)$ dominates the summation of (9.17). Hence,

$$1 = \lim_{z \to \alpha_m} \frac{\sum_{k=0}^{N-1} b_k z^k}{\prod_{j=1}^{N}(z - \alpha_j)} \Bigg/ \frac{A_m}{z - \alpha_m}.$$

Rearranging factors, we have that

$$A_m = \left.\frac{\sum_{k=0}^{N-1} b_k z^k}{\left[\prod_{j=1}^{N}(z - \alpha_j)\right]/(z - \alpha_m)}\right|_{z=\alpha_m} = \lim_{z \to \alpha_m} \left[(z - \alpha_m) f(z)\right], \tag{9.18}$$

where $A_m$, the value of $f(z)$ with the pole canceled, is called the *residue* of $f(z)$ at the pole $z = \alpha_m$. Equations (9.17) and (9.18) together give the partial-fraction expansion of (9.16)'s rational function $f(z)$.

### 9.6.2 Repeated poles

The weakness of the partial-fraction expansion of § 9.6.1 is that it cannot directly handle repeated poles. That is, if $\alpha_n = \alpha_j$, $n \neq j$, then the residue formula (9.18) finds an uncanceled pole remaining in its denominator and thus fails for $A_n = A_j$ (it still works for the other $A_m$). The conventional way to expand a fraction with repeated poles is presented in § 9.6.5 below; but because at least to this writer that way does not lend much applied

insight, the present subsection treats the matter in a different way. Here, we *separate the poles.*

Consider the function

$$g(z) = \sum_{k=0}^{N-1} \frac{Ce^{i2\pi k/N}}{z - \epsilon e^{i2\pi k/N}}, \quad N > 1,\ 0 < \epsilon \ll 1, \tag{9.19}$$

where $C$ is a real-valued constant. This function evidently has a small circle of poles in the Argand plane at $\alpha_k = \epsilon e^{i2\pi k/N}$. Factoring,

$$g(z) = \frac{C}{z} \sum_{k=0}^{N-1} \frac{e^{i2\pi k/N}}{1 - (\epsilon e^{i2\pi k/N})/z}.$$

Using (2.34) to expand the fraction,

$$\begin{aligned} g(z) &= \frac{C}{z} \sum_{k=0}^{N-1} \left[ e^{i2\pi k/N} \sum_{j=0}^{\infty} \left( \frac{\epsilon e^{i2\pi k/N}}{z} \right)^j \right] \\ &= C \sum_{k=0}^{N-1} \sum_{j=1}^{\infty} \frac{\epsilon^{j-1} e^{i2\pi jk/N}}{z^j} \\ &= C \sum_{j=1}^{\infty} \frac{\epsilon^{j-1}}{z^j} \sum_{k=0}^{N-1} \left( e^{i2\pi j/N} \right)^k. \end{aligned}$$

But[11]

$$\sum_{k=0}^{N-1} \left( e^{i2\pi j/N} \right)^k = \begin{cases} N & \text{if } j = mN, \\ 0 & \text{otherwise,} \end{cases}$$

so

$$g(z) = NC \sum_{m=1}^{\infty} \frac{\epsilon^{mN-1}}{z^{mN}}.$$

For $|z| \gg \epsilon$—that is, except in the immediate neighborhood of the small circle of poles—the first term of the summation dominates. Hence,

$$g(z) \approx NC \frac{\epsilon^{N-1}}{z^N}, \quad |z| \gg \epsilon.$$

[11] If you don't see why, then for $N = 8$ and $j = 3$ plot the several $(e^{i2\pi j/N})^k$ in the Argand plane. Do the same for $j = 2$ then $j = 8$. Only in the $j = 8$ case do the terms add coherently; in the other cases they cancel.

This effect—reinforcing when $j = nN$, canceling otherwise—is a classic manifestation of *Parseval's principle,* which § 17.1 will formally introduce later in the book.

Having achieved this approximation, if we strategically choose

$$C = \frac{1}{N\epsilon^{N-1}},$$

then

$$g(z) \approx \frac{1}{z^N}, \quad |z| \gg \epsilon.$$

But given the chosen value of $C$, (9.19) is

$$g(z) = \frac{1}{N\epsilon^{N-1}} \sum_{k=0}^{N-1} \frac{e^{i2\pi k/N}}{z - \epsilon e^{i2\pi k/N}}, \quad N > 1,\ 0 < \epsilon \ll 1.$$

Joining the last two equations together, changing $z - z_o \leftarrow z$, and writing more formally, we have that

$$\frac{1}{(z - z_o)^N} = \lim_{\epsilon \to 0} \frac{1}{N\epsilon^{N-1}} \sum_{k=0}^{N-1} \frac{e^{i2\pi k/N}}{z - \left[z_o + \epsilon e^{i2\pi k/N}\right]}, \quad N > 1. \tag{9.20}$$

The significance of (9.20) is that it lets one replace an $N$-fold pole with a small circle of ordinary poles, which per § 9.6.1 we already know how to handle. Notice incidentally that $1/N\epsilon^{N-1}$ is a large number not a small. The poles are close together but very strong.

An example to illustrate the technique, separating a double pole:

$$\begin{aligned}
f(z) &= \frac{z^2 - z + 6}{(z-1)^2(z+2)} \\
&= \lim_{\epsilon\to 0} \frac{z^2 - z + 6}{(z - [1 + \epsilon e^{i2\pi(0)/2}])(z - [1 + \epsilon e^{i2\pi(1)/2}])(z+2)} \\
&= \lim_{\epsilon\to 0} \frac{z^2 - z + 6}{(z - [1 + \epsilon])(z - [1 - \epsilon])(z+2)} \\
&= \lim_{\epsilon\to 0} \left\{ \left(\frac{1}{z - [1+\epsilon]}\right) \left[\frac{z^2 - z + 6}{(z - [1-\epsilon])(z+2)}\right]_{z=1+\epsilon} \right. \\
&\qquad + \left(\frac{1}{z - [1-\epsilon]}\right) \left[\frac{z^2 - z + 6}{(z - [1+\epsilon])(z+2)}\right]_{z=1-\epsilon} \\
&\qquad \left. + \left(\frac{1}{z+2}\right) \left[\frac{z^2 - z + 6}{(z - [1+\epsilon])(z - [1-\epsilon])}\right]_{z=-2} \right\} \\
&= \lim_{\epsilon\to 0} \left\{ \left(\frac{1}{z - [1+\epsilon]}\right) \left[\frac{6+\epsilon}{6\epsilon + 2\epsilon^2}\right] \right. \\
&\qquad + \left(\frac{1}{z - [1-\epsilon]}\right) \left[\frac{6-\epsilon}{-6\epsilon + 2\epsilon^2}\right] \\
&\qquad \left. + \left(\frac{1}{z+2}\right) \left[\frac{\mathrm{0xC}}{9}\right] \right\} \\
&= \lim_{\epsilon\to 0} \left\{ \frac{1/\epsilon - 1/6}{z - [1+\epsilon]} + \frac{-1/\epsilon - 1/6}{z - [1-\epsilon]} + \frac{4/3}{z+2} \right\} \\
&= \lim_{\epsilon\to 0} \left\{ \frac{1/\epsilon}{z - [1+\epsilon]} + \frac{-1/\epsilon}{z - [1-\epsilon]} + \frac{-1/3}{z-1} + \frac{4/3}{z+2} \right\}.
\end{aligned}$$

Notice how the calculation has discovered an additional, single pole at $z = 1$, the pole hiding under dominant, double pole there.

### 9.6.3 Integrating a rational function

If one can find the poles of a rational function of the form (9.16), then one can use (9.17) and (9.18)—and, if needed, (9.20)—to expand the function into a sum of partial fractions, each of which one can integrate individually.

Continuing the example of § 9.6.2, for $0 \le x < 1$,

$$\begin{aligned}
\int_0^x f(\tau)\,d\tau &= \int_0^x \frac{\tau^2-\tau+6}{(\tau-1)^2(\tau+2)}\,d\tau \\
&= \lim_{\epsilon\to 0}\int_0^x \left\{\frac{1/\epsilon}{\tau-[1+\epsilon]} + \frac{-1/\epsilon}{\tau-[1-\epsilon]} + \frac{-1/3}{\tau-1} + \frac{4/3}{\tau+2}\right\} d\tau \\
&= \lim_{\epsilon\to 0}\left\{\frac{1}{\epsilon}\ln([1+\epsilon]-\tau) - \frac{1}{\epsilon}\ln([1-\epsilon]-\tau)\right. \\
&\qquad \left. -\frac{1}{3}\ln(1-\tau) + \frac{4}{3}\ln(\tau+2)\right\}_0^x \\
&= \lim_{\epsilon\to 0}\left\{\frac{1}{\epsilon}\ln\left(\frac{[1+\epsilon]-\tau}{[1-\epsilon]-\tau}\right) - \frac{1}{3}\ln(1-\tau) + \frac{4}{3}\ln(\tau+2)\right\}_0^x \\
&= \lim_{\epsilon\to 0}\left\{\frac{1}{\epsilon}\ln\left(\frac{[1-\tau]+\epsilon}{[1-\tau]-\epsilon}\right) - \frac{1}{3}\ln(1-\tau) + \frac{4}{3}\ln(\tau+2)\right\}_0^x \\
&= \lim_{\epsilon\to 0}\left\{\frac{1}{\epsilon}\ln\left(1+\frac{2\epsilon}{1-\tau}\right) - \frac{1}{3}\ln(1-\tau) + \frac{4}{3}\ln(\tau+2)\right\}_0^x \\
&= \lim_{\epsilon\to 0}\left\{\frac{1}{\epsilon}\left(\frac{2\epsilon}{1-\tau}\right) - \frac{1}{3}\ln(1-\tau) + \frac{4}{3}\ln(\tau+2)\right\}_0^x \\
&= \lim_{\epsilon\to 0}\left\{\frac{2}{1-\tau} - \frac{1}{3}\ln(1-\tau) + \frac{4}{3}\ln(\tau+2)\right\}_0^x \\
&= \frac{2}{1-x} - 2 - \frac{1}{3}\ln(1-x) + \frac{4}{3}\ln\left(\frac{x+2}{2}\right).
\end{aligned}$$

To check (§ 7.5) that the result is correct, we can take the derivative of the final expression:

$$\begin{aligned}
\left[\frac{d}{dx}\left\{\frac{2}{1-x} - 2 - \frac{1}{3}\ln(1-x) + \frac{4}{3}\ln\left(\frac{x+2}{2}\right)\right\}\right]_{x=\tau} &= \frac{2}{(\tau-1)^2} + \frac{-1/3}{\tau-1} + \frac{4/3}{\tau+2} \\
&= \frac{\tau^2-\tau+6}{(\tau-1)^2(\tau+2)},
\end{aligned}$$

which indeed has the form of the integrand we started with, confirming the result. (Notice incidentally how much easier it is symbolically to differentiate than to integrate!)

Section 18.7 exercises the technique in a more sophisticated way, applying it in the context of Ch. 18's Laplace transform to solve a linear differential equation.

### 9.6.4 The derivatives of a rational function

Not only the integral of a rational function interests us; its derivatives interest us, too. One needs no special technique to compute such derivatives, of course, but the derivatives do bring some noteworthy properties.

First of interest is the property that a function in the general rational form

$$\Phi(w) = \frac{w^p h_0(w)}{g(w)}, \quad g(0) \neq 0, \tag{9.21}$$

enjoys derivatives in the general rational form

$$\frac{d^k \Phi}{dw^k} = \frac{w^{p-k} h_k(w)}{[g(w)]^{k+1}}, \quad 0 \leq k \leq p, \tag{9.22}$$

where $g$ and $h_k$ are polynomials in nonnegative powers of $w$. The property is proved by induction. When $k = 0$, (9.22) is (9.21), so (9.22) is good at least for this case. Then, if (9.22) holds for $k = n - 1$,

$$\begin{aligned} \frac{d^n \Phi}{dw^n} &= \frac{d}{dw}\left[\frac{d^{n-1}\Phi}{dw^{n-1}}\right] = \frac{d}{dw}\left[\frac{w^{p-n+1} h_{n-1}(w)}{[g(w)]^n}\right] = \frac{w^{p-n} h_n(w)}{[g(w)]^{n+1}}, \\ h_n(w) &\equiv wg\frac{dh_{n-1}}{dw} - nwh_{n-1}\frac{dg}{dw} + (p-n+1)gh_{n-1}, \quad 0 < n \leq p, \end{aligned}$$

which makes $h_n$ (like $h_{n-1}$) a polynomial in nonnegative powers of $w$. By induction on this basis, (9.22) holds for all $0 \leq k \leq p$, as was to be demonstrated.

A related property is that

$$\left.\frac{d^k \Phi}{dw^k}\right|_{w=0} = 0 \quad \text{for } 0 \leq k < p. \tag{9.23}$$

That is, the function and its first $p-1$ derivatives are all zero at $w = 0$. The reason is that (9.22)'s denominator is $[g(w)]^{k+1} \neq 0$, whereas its numerator has a $w^{p-k} = 0$ factor, when $0 \leq k < p$ and $w = 0$.

### 9.6.5 Repeated poles (the conventional technique)

Though the technique of §§ 9.6.2 and 9.6.3 affords extra insight, it is not the conventional technique to expand in partial fractions a rational function having a repeated pole. The conventional technique is worth learning not only because it is conventional but also because it is usually quicker to apply in practice. This subsection derives it.

A rational function with repeated poles,

$$\begin{aligned} f(z) &= \frac{\sum_{k=0}^{N-1} b_k z^k}{\prod_{j=1}^{M} (z-\alpha_j)^{p_j}}, \\ N &\equiv \sum_{j=1}^{M} p_j, \\ p_j &\geq 0, \\ \alpha_{j'} &\neq \alpha_j \quad \text{if } j' \neq j, \end{aligned} \tag{9.24}$$

where $j$, $k$, $M$, $N$ and the several $p_j$ are integers, cannot be expanded solely in the first-order fractions of § 9.6.1, but can indeed be expanded if higher-order fractions are allowed:

$$f(z) = \sum_{j=1}^{M} \sum_{\ell=0}^{p_j-1} \frac{A_{j\ell}}{(z-\alpha_j)^{p_j-\ell}}. \tag{9.25}$$

What the partial-fraction expansion (9.25) lacks are the values of its several coefficients $A_{j\ell}$.

One can determine the coefficients with respect to one (possibly repeated) pole at a time. To determine them with respect to the $p_m$-fold pole at $z = \alpha_m$, $1 \leq m \leq M$, one multiplies (9.25) by $(z-\alpha_m)^{p_m}$ to obtain the form

$$(z-\alpha_m)^{p_m} f(z) = \sum_{\substack{j=1,\\ j\neq m}}^{M} \sum_{\ell=0}^{p_j-1} \frac{(A_{j\ell})(z-\alpha_m)^{p_m}}{(z-\alpha_j)^{p_j-\ell}} + \sum_{\ell=0}^{p_m-1} (A_{m\ell})(z-\alpha_m)^{\ell}.$$

But (9.23) with $w = z - \alpha_m$ reveals the double summation and its first $p_m - 1$ derivatives all to be null at $z = \alpha_m$; that is,

$$\left. \frac{d^k}{dz^k} \sum_{\substack{j=1,\\ j\neq m}}^{M} \sum_{\ell=0}^{p_j-1} \frac{(A_{j\ell})(z-\alpha_m)^{p_m}}{(z-\alpha_j)^{p_j-\ell}} \right|_{z=\alpha_m} = 0, \qquad 0 \leq k < p_m;$$

so, the $(z-\alpha_m)^{p_m} f(z)$ equation's $k$th derivative reduces at that point to

$$\begin{aligned} \left. \frac{d^k}{dz^k} \Big[ (z-\alpha_m)^{p_m} f(z) \Big] \right|_{z=\alpha_m} &= \sum_{\ell=0}^{p_m-1} \left. \frac{d^k}{dz^k} \Big[ (A_{m\ell})(z-\alpha_m)^{\ell} \Big] \right|_{z=\alpha_m} \\ &= k! A_{mk}, \quad 0 \leq k < p_m. \end{aligned}$$

Changing $j \leftarrow m$ and $\ell \leftarrow k$ and solving for $A_{j\ell}$ then produces the coefficients

$$A_{j\ell} = \left(\frac{1}{\ell!}\right) \frac{d^\ell}{dz^\ell} \Big[ (z - \alpha_j)^{p_j} f(z) \Big] \Big|_{z=\alpha_j}, \qquad 0 \le \ell < p, \tag{9.26}$$

to weight the expansion (9.25)'s partial fractions. In case of a repeated pole, these coefficients evidently depend not only on the residual function itself but also on its several derivatives, one derivative per repetition of the pole.

### 9.6.6 The existence and uniqueness of solutions

Equation (9.26) has solved (9.24) and (9.25). A professional mathematician might object however that it has done so without first proving that a unique solution actually exists.

Comes from us the reply, "Why should we prove that a solution exists, once we have actually found it?"

Ah, but the professional's point is that we have found the solution only if in fact it does exist, and uniquely; otherwise what we have *found* is a phantom. A careful review of § 9.6.5's logic discovers no guarantee that all of (9.26)'s coefficients actually come from the same expansion. Maybe there exist two distinct expansions, and some of the coefficients come from the one, some from the other. On the other hand, maybe there exists no expansion at all, in which event it is not even clear what (9.26) means.

"But these are quibbles, cavils and nitpicks!" we are inclined to grumble. "The present book is a book of applied mathematics."

Well, yes, but on this occasion let us nonetheless follow the professional's line of reasoning, if only a short way.

*Uniqueness* is proved by positing two solutions

$$f(z) = \sum_{j=1}^{M} \sum_{\ell=0}^{p_j-1} \frac{A_{j\ell}}{(z - \alpha_j)^{p_j-\ell}} = \sum_{j=1}^{M} \sum_{\ell=0}^{p_j-1} \frac{B_{j\ell}}{(z - \alpha_j)^{p_j-\ell}}$$

and computing the difference

$$\sum_{j=1}^{M} \sum_{\ell=0}^{p_j-1} \frac{B_{j\ell} - A_{j\ell}}{(z - \alpha_j)^{p_j-\ell}}$$

between them. Logically this difference must be zero for all $z$ if the two solutions are actually to represent the same function $f(z)$. This however

is seen to be possible only if $B_{j\ell} = A_{j\ell}$ for each $(j, \ell)$. Therefore, the two solutions are one and the same.

*Existence* comes of combining the several fractions of (9.25) over a common denominator and comparing the resulting numerator against the numerator of (9.24). Each coefficient $b_k$ is seen thereby to be a linear combination of the several $A_{j\ell}$, where the combination's weights depend solely on the locations $\alpha_j$ and multiplicities $p_j$ of $f(z)$'s several poles. From the $N$ coefficients $b_k$ and the $N$ coefficients $A_{j\ell}$, an $N \times N$ system of $N$ linear equations in $N$ unknowns results—which might for example (if, say, $N = 3$) look like

$$\begin{aligned} b_0 &= -2A_{00} + A_{01} + 3A_{10}, \\ b_1 &= A_{00} + A_{01} + A_{10}, \\ b_2 &= 2A_{01} - 5A_{10}. \end{aligned}$$

We will show in Chs. 11 through 14 that when such a system has no solution, there always exist an alternate set of $b_k$ for which the same system has multiple solutions. But uniqueness, which we have already established, forbids such multiple solutions in all cases. Therefore it is not possible for the system to have no solution—which is to say, the solution necessarily exists.

We will not often in this book prove existence and uniqueness explicitly, but such proofs when desired tend to fit the pattern outlined here.

## 9.7 Frullani's integral

One occasionally meets an integral of the form

$$S = \int_0^\infty \frac{f(b\tau) - f(a\tau)}{\tau}\, d\tau,$$

where $a$ and $b$ are real, positive coefficients and $f(\tau)$ is an arbitrary complex expression in $\tau$. One wants to split such an integral in two as $\int [f(b\tau)/\tau]\, d\tau - \int [f(a\tau)/\tau]\, d\tau$; but if $f(0^+) \neq f(+\infty)$, one cannot, because each half-integral alone diverges. Nonetheless, splitting the integral in two is the right idea, provided that one first relaxes the limits of integration as

$$S = \lim_{\epsilon \to 0^+} \left\{ \int_\epsilon^{1/\epsilon} \frac{f(b\tau)}{\tau}\, d\tau - \int_\epsilon^{1/\epsilon} \frac{f(a\tau)}{\tau}\, d\tau \right\}.$$

Changing $\sigma \leftarrow b\tau$ in the left integral and $\sigma \leftarrow a\tau$ in the right yields

$$\begin{aligned} S &= \lim_{\epsilon\to 0^+} \left\{ \int_{b\epsilon}^{b/\epsilon} \frac{f(\sigma)}{\sigma}\,d\sigma - \int_{a\epsilon}^{a/\epsilon} \frac{f(\sigma)}{\sigma}\,d\sigma \right\} \\ &= \lim_{\epsilon\to 0^+} \left\{ \int_{a\epsilon}^{b\epsilon} \frac{-f(\sigma)}{\sigma}\,d\sigma + \int_{b\epsilon}^{a/\epsilon} \frac{f(\sigma)-f(\sigma)}{\sigma}\,d\sigma + \int_{a/\epsilon}^{b/\epsilon} \frac{f(\sigma)}{\sigma}\,d\sigma \right\} \\ &= \lim_{\epsilon\to 0^+} \left\{ \int_{a/\epsilon}^{b/\epsilon} \frac{f(\sigma)}{\sigma}\,d\sigma - \int_{a\epsilon}^{b\epsilon} \frac{f(\sigma)}{\sigma}\,d\sigma \right\} \end{aligned}$$

(here on the face of it, we have split the integration as though $a \le b$, but in fact it does not matter which of $a$ and $b$ is the greater, as is easy to verify). So long as each of $f(\epsilon)$ and $f(1/\epsilon)$ approaches a constant value as $\epsilon$ vanishes, this is

$$\begin{aligned} S &= \lim_{\epsilon\to 0^+} \left\{ f(+\infty) \int_{a/\epsilon}^{b/\epsilon} \frac{d\sigma}{\sigma} - f(0^+) \int_{a\epsilon}^{b\epsilon} \frac{d\sigma}{\sigma} \right\} \\ &= \lim_{\epsilon\to 0^+} \left\{ f(+\infty) \ln\frac{b/\epsilon}{a/\epsilon} - f(0^+) \ln\frac{b\epsilon}{a\epsilon} \right\} \\ &= [f(\tau)]_0^\infty \ln\frac{b}{a}. \end{aligned}$$

Thus we have *Frullani's integral,*

$$\int_0^\infty \frac{f(b\tau) - f(a\tau)}{\tau}\,d\tau = [f(\tau)]_0^\infty \ln\frac{b}{a}, \tag{9.27}$$

which, if $a$ and $b$ are both real and positive, works for any $f(\tau)$ which has definite $f(0^+)$ and $f(+\infty)$.[12]

## 9.8 Integrating products of exponentials, powers and logarithms

The products $\exp(\alpha\tau)\tau^n$ (where $n \in \mathbb{Z}$) and $\tau^{a-1}\ln\tau$ tend to arise[13] among other places in integrands related to special functions ([chapter not yet written]). The two occur often enough to merit investigation here.

[12] [43, § 1.3][2, § 2.5.1][65, "Frullani's integral"]

[13] One could write the latter product more generally as $\tau^{a-1}\ln\beta\tau$. According to Table 2.5, however, $\ln\beta\tau = \ln\beta + \ln\tau$; wherein $\ln\beta$ is just a constant.

Concerning $\exp(\alpha\tau)\tau^n$, by § 9.4's method of unknown coefficients we guess its antiderivative to fit the form

$$\begin{aligned}\exp(\alpha\tau)\tau^n &= \frac{d}{d\tau}\sum_{k=0}^{n} a_k \exp(\alpha\tau)\tau^k \\ &= \sum_{k=0}^{n} \alpha a_k \exp(\alpha\tau)\tau^k + \sum_{k=1}^{n} \frac{a_k}{k}\exp(\alpha\tau)\tau^{k-1} \\ &= \alpha a_n \exp(\alpha\tau)\tau^n + \sum_{k=0}^{n-1}\left(\alpha a_k + \frac{a_{k+1}}{k+1}\right)\exp(\alpha\tau)\tau^k.\end{aligned}$$

If so, then evidently

$$\begin{aligned}a_n &= \frac{1}{\alpha}; \\ a_k &= -\frac{a_{k+1}}{(k+1)(\alpha)}, \quad 0 \le k < n.\end{aligned}$$

That is,

$$a_k = \frac{1}{\alpha\prod_{j=k+1}^{n}(-j\alpha)} = \frac{(-)^{n-k}}{(n!/k!)\alpha^{n-k+1}}, \quad 0 \le k \le n.$$

Therefore,[14]

$$\exp(\alpha\tau)\tau^n = \frac{d}{d\tau}\sum_{k=0}^{n}\frac{(-)^{n-k}}{(n!/k!)\alpha^{n-k+1}}\exp(\alpha\tau)\tau^k, \quad n \in \mathbb{Z},\ n \ge 0,\ \alpha \ne 0. \tag{9.28}$$

The right form to guess for the antiderivative of $\tau^{a-1}\ln\tau$ is less obvious. Remembering however § 5.3's observation that $\ln\tau$ is of zeroth order in $\tau$, after maybe some false tries we eventually do strike the right form

$$\begin{aligned}\tau^{a-1}\ln\tau &= \frac{d}{d\tau}\tau^a[B\ln\tau + C] \\ &= \tau^{a-1}[aB\ln\tau + (B + aC)],\end{aligned}$$

which demands that $B = 1/a$ and that $C = -1/a^2$. Therefore,[15]

$$\tau^{a-1}\ln\tau = \frac{d}{d\tau}\frac{\tau^a}{a}\left(\ln\tau - \frac{1}{a}\right), \quad a \ne 0. \tag{9.29}$$

[14] [55, Appendix 2, eqn. 73]
[15] [55, Appendix 2, eqn. 74]

Table 9.1: Antiderivatives of products of exponentials, powers and logarithms.

$$\begin{aligned}
\exp(\alpha\tau)\tau^n &= \frac{d}{d\tau}\sum_{k=0}^{n}\frac{(-)^{n-k}}{(n!/k!)\alpha^{n-k+1}}\exp(\alpha\tau)\tau^k, \quad n \in \mathbb{Z},\ n \geq 0,\ \alpha \neq 0 \\
\tau^{a-1}\ln\tau &= \frac{d}{d\tau}\frac{\tau^a}{a}\left(\ln\tau - \frac{1}{a}\right), \quad a \neq 0 \\
\frac{\ln\tau}{\tau} &= \frac{d}{d\tau}\frac{(\ln\tau)^2}{2}
\end{aligned}$$

Antiderivatives of terms like $\tau^{a-1}(\ln\tau)^2$, $\exp(\alpha\tau)\tau^n\ln\tau$ and so on can be computed in like manner as the need arises.

Equation (9.29) fails when $a = 0$, but in this case with a little imagination the antiderivative is not hard to guess:

$$\frac{\ln\tau}{\tau} = \frac{d}{d\tau}\frac{(\ln\tau)^2}{2}. \tag{9.30}$$

If (9.30) seemed hard to guess nevertheless, then l'Hôpital's rule (4.30), applied to (9.29) as $a \to 0$, with the observation from (2.41) that

$$\tau^a = \exp(a\ln\tau), \tag{9.31}$$

would yield the same (9.30).

Table 9.1 summarizes.

## 9.9 Integration by Taylor series

With sufficient cleverness the techniques of the foregoing sections solve many, many integrals. But not all. When all else fails, as sometimes it does, the Taylor series of Ch. 8 and the antiderivative of § 9.1 together offer a concise, practical way to integrate some functions, at the price of losing the functions'

known closed analytic forms. For example,

$$\begin{aligned}
\int_0^x \exp\left(-\frac{\tau^2}{2}\right) d\tau &= \int_0^x \sum_{k=0}^{\infty} \frac{(-\tau^2/2)^k}{k!}\, d\tau \\
&= \int_0^x \sum_{k=0}^{\infty} \frac{(-)^k \tau^{2k}}{2^k k!}\, d\tau \\
&= \left[\sum_{k=0}^{\infty} \frac{(-)^k \tau^{2k+1}}{(2k+1)2^k k!}\right]_0^x \\
&= \sum_{k=0}^{\infty} \frac{(-)^k x^{2k+1}}{(2k+1)2^k k!} = (x) \sum_{k=0}^{\infty} \frac{1}{2k+1} \prod_{j=1}^{k} \frac{-x^2}{2j}.
\end{aligned}$$

The result is no function one recognizes; it is just a series. This is not necessarily bad, however. After all, when a Taylor series from Table 8.1 is used to calculate $\sin z$, then $\sin z$ is just a series, too. The series above converges just as accurately and just as fast.

Sometimes it helps to give the series a name like

$$\text{myf}\, z \equiv \sum_{k=0}^{\infty} \frac{(-)^k z^{2k+1}}{(2k+1)2^k k!} = (z) \sum_{k=0}^{\infty} \frac{1}{2k+1} \prod_{j=1}^{k} \frac{-z^2}{2j}.$$

Then,

$$\int_0^x \exp\left(-\frac{\tau^2}{2}\right) d\tau = \text{myf}\, x.$$

The myf $z$ is no less a function than $\sin z$ is; it's just a function you hadn't heard of before. You can plot the function, or take its derivative

$$\frac{d}{d\tau} \text{myf}\, \tau = \exp\left(-\frac{\tau^2}{2}\right),$$

or calculate its value, or do with it whatever else one does with functions. It works just the same.

Beyond the several integration techniques this chapter has introduced, a special-purpose technique of integration by cylindrical transformation will surface in § 18.5.

# Chapter 10

# Cubics and quartics

Under the heat of noonday, between the hard work of the morning and the heavy lifting of the afternoon, one likes to lay down one's burden and rest a spell in the shade. Chapters 2 through 9 have established the applied mathematical foundations upon which coming chapters will build; and Ch. 11, hefting the weighty topic of the matrix, will indeed begin to build on those foundations. But in this short chapter which rests between, we shall refresh ourselves with an interesting but lighter mathematical topic: the topic of cubics and quartics.

The expression

$$z + a_0$$

is a *linear* polynomial, the lone root $z = -a_0$ of which is plain to see. The *quadratic* polynomial

$$z^2 + a_1 z + a_0$$

has of course two roots, which though not plain to see the quadratic formula (2.2) extracts with little effort. So much algebra has been known since antiquity. The roots of higher-order polynomials, the Newton-Raphson iteration (4.31) locates swiftly, but that is an approximate iteration rather than an exact formula like (2.2), and as we have seen in § 4.8 it can occasionally fail to converge. One would prefer an actual formula to extract the roots.

No general formula to extract the roots of the $n$th-order polynomial seems to be known.[1] However, to extract the roots of the *cubic* and *quartic* polynomials

$$z^3 + a_2 z^2 + a_1 z + a_0,$$
$$z^4 + a_3 z^3 + a_2 z^2 + a_1 z + a_0,$$

[1] Refer to Ch. 6's footnote 9.

though the ancients never discovered how, formulas do exist. The 16th-century algebraists Ferrari, Vieta, Tartaglia and Cardano have given us the clever technique. This chapter explains.[2]

## 10.1 Vieta's transform

There is a sense to numbers by which 1/2 resembles 2, 1/3 resembles 3, 1/4 resembles 4, and so forth. To capture this sense, one can transform a function $f(z)$ into a function $f(w)$ by the change of variable[3]

$$w + \frac{1}{w} \leftarrow z,$$

or, more generally,

$$w + \frac{w_o^2}{w} \leftarrow z. \tag{10.1}$$

Equation (10.1) is *Vieta's transform.*[4]

For $|w| \gg |w_o|$, we have that $z \approx w$; but as $|w|$ approaches $|w_o|$ this ceases to be true. For $|w| \ll |w_o|$, $z \approx w_o^2/w$. The constant $w_o$ is the *corner value,* in the neighborhood of which $w$ transitions from the one domain to the other. Figure 10.1 plots Vieta's transform for real $w$ in the case $w_o = 1$.

An interesting alternative to Vieta's transform is

$$w \parallel \frac{w_o^2}{w} \leftarrow z, \tag{10.2}$$

which in light of § 6.3 might be named *Vieta's parallel transform.*

Section 10.2 shows how Vieta's transform can be used.

## 10.2 Cubics

The general cubic polynomial is too hard to extract the roots of directly, so one begins by changing the variable

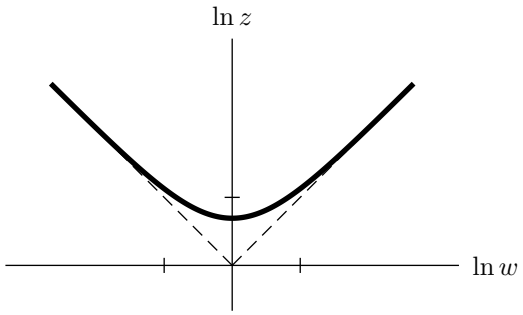
$$x + h \leftarrow z \tag{10.3}$$

[2] [65, "Cubic equation"][65, "Quartic equation"][66, "Quartic equation," 00:26, 9 Nov. 2006][66, "François Viète," 05:17, 1 Nov. 2006][66, "Gerolamo Cardano," 22:35, 31 Oct. 2006][59, § 1.5]

[3] This change of variable broadly recalls the sum-of-exponentials form (5.20) of the $\cosh(\cdot)$ function, inasmuch as $\exp[-\phi] = 1/\exp\phi$.

[4] Also called "Vieta's substitution." [65, "Vieta's substitution"]

Figure 10.1: Vieta's transform (10.1) for $w_o = 1$, plotted logarithmically.




to obtain the polynomial

$$x^3 + (a_2 + 3h)x^2 + (a_1 + 2ha_2 + 3h^2)x + (a_0 + ha_1 + h^2a_2 + h^3).$$

The choice

$$h \equiv -\frac{a_2}{3} \tag{10.4}$$

casts the polynomial into the improved form

$$x^3 + \left[a_1 - \frac{a_2^2}{3}\right] x + \left[a_0 - \frac{a_1 a_2}{3} + 2\left(\frac{a_2}{3}\right)^3\right],$$

or better yet

$$x^3 - px - q,$$

where

$$\begin{aligned} p &\equiv -a_1 + \frac{a_2^2}{3}, \\ q &\equiv -a_0 + \frac{a_1 a_2}{3} - 2\left(\frac{a_2}{3}\right)^3. \end{aligned} \tag{10.5}$$

The solutions to the equation

$$x^3 = px + q, \tag{10.6}$$

then, are the cubic polynomial's three roots.

So we have struck the $a_2 z^2$ term. That was the easy part; what to do next is not so obvious. If one could strike the $px$ term as well, then the

roots would follow immediately, but no very simple substitution like (10.3) achieves this—or rather, such a substitution does achieve it, but at the price of reintroducing an unwanted $x^2$ or $z^2$ term. That way is no good. Lacking guidance, one might try many, various substitutions, none of which seems to help much; but after weeks or months of such frustration one might eventually discover Vieta's transform (10.1), with the idea of balancing the equation between offsetting $w$ and $1/w$ terms. This works.

Vieta-transforming (10.6) by the change of variable

$$w + \frac{w_o^2}{w} \leftarrow x \tag{10.7}$$

we get the new equation

$$w^3 + (3w_o^2 - p)w + (3w_o^2 - p)\frac{w_o^2}{w} + \frac{w_o^6}{w^3} = q, \tag{10.8}$$

which invites the choice

$$w_o^2 \equiv \frac{p}{3}, \tag{10.9}$$

reducing (10.8) to read

$$w^3 + \frac{(p/3)^3}{w^3} = q.$$

Multiplying by $w^3$ and rearranging terms, we have the quadratic equation

$$(w^3)^2 = 2\left(\frac{q}{2}\right)w^3 - \left(\frac{p}{3}\right)^3, \tag{10.10}$$

which by (2.2) we know how to solve.

Vieta's transform has reduced the original cubic to a quadratic.

The careful reader will observe that (10.10) seems to imply six roots, double the three the fundamental theorem of algebra (§ 6.2.2) allows a cubic polynomial to have. We shall return to this point in § 10.3. For the moment, however, we should like to improve the notation by defining[5]

$$\begin{aligned} P &\leftarrow -\frac{p}{3}, \\ Q &\leftarrow +\frac{q}{2}, \end{aligned} \tag{10.11}$$

[5]Why did we not define $P$ and $Q$ so to begin with? Well, before unveiling (10.10), we lacked motivation to do so. To define inscrutable coefficients unnecessarily before the need for them is apparent seems poor applied mathematical style.

Table 10.1: A method to extract the three roots of the general cubic polynomial. (In the definition of $w^3$, one can choose either sign.)

$$\begin{aligned}
0 &= z^3 + a_2 z^2 + a_1 z + a_0 \\
P &\equiv \frac{a_1}{3} - \left(\frac{a_2}{3}\right)^2 \\
Q &\equiv \frac{1}{2}\left[-a_0 + 3\left(\frac{a_1}{3}\right)\left(\frac{a_2}{3}\right) - 2\left(\frac{a_2}{3}\right)^3\right] \\
w^3 &\equiv \begin{cases} 2Q & \text{if } P = 0, \\ Q \pm \sqrt{Q^2 + P^3} & \text{otherwise.} \end{cases} \\
x &\equiv \begin{cases} 0 & \text{if } P = 0 \text{ and } Q = 0, \\ w - P/w & \text{otherwise.} \end{cases} \\
z &= x - \frac{a_2}{3}
\end{aligned}$$

with which (10.6) and (10.10) are written

$$x^3 = 2Q - 3Px, \tag{10.12}$$

$$(w^3)^2 = 2Qw^3 + P^3. \tag{10.13}$$

Table 10.1 summarizes the complete cubic polynomial root extraction method in the revised notation—including a few fine points regarding superfluous roots and edge cases, treated in §§ 10.3 and 10.4 below.

## 10.3 Superfluous roots

As § 10.2 has observed, the equations of Table 10.1 seem to imply six roots, double the three the fundamental theorem of algebra (§ 6.2.2) allows a cubic polynomial to have. However, what the equations really imply is not six distinct roots but six distinct $w$. The definition $x \equiv w - P/w$ maps two $w$ to any one $x$, so in fact the equations imply only three $x$ and thus three roots $z$. The question then is: of the six $w$, which three do we really need and which three can we ignore as superfluous?

The six $w$ naturally come in two groups of three: one group of three from the one $w^3$ and a second from the other. For this reason, we will guess—and logically it is only a guess—that a single $w^3$ generates three distinct $x$ and thus (because $z$ differs from $x$ only by a constant offset) all three roots $z$. If

the guess is right, then the second $w^3$ cannot but yield the same three roots, which means that the second $w^3$ is superfluous and can safely be overlooked. But is the guess right? Does a single $w^3$ in fact generate three distinct $x$?

To prove that it does, let us suppose that it did not. Let us suppose that a single $w^3$ did generate two $w$ which led to the same $x$. Letting the symbol $w_1$ represent the third $w$, then (since all three $w$ come from the same $w^3$) the two $w$ are $e^{+i2\pi/3}w_1$ and $e^{-i2\pi/3}w_1$. Because $x \equiv w - P/w$, by successive steps,

$$\begin{aligned}
e^{+i2\pi/3}w_1 - \frac{P}{e^{+i2\pi/3}w_1} &= e^{-i2\pi/3}w_1 - \frac{P}{e^{-i2\pi/3}w_1}, \\
e^{+i2\pi/3}w_1 + \frac{P}{e^{-i2\pi/3}w_1} &= e^{-i2\pi/3}w_1 + \frac{P}{e^{+i2\pi/3}w_1}, \\
e^{+i2\pi/3}\left(w_1 + \frac{P}{w_1}\right) &= e^{-i2\pi/3}\left(w_1 + \frac{P}{w_1}\right),
\end{aligned}$$

which can only be true if

$$w_1^2 = -P.$$

Cubing[6] the last equation,

$$w_1^6 = -P^3;$$

but squaring the table's $w^3$ definition for $w = w_1$,

$$w_1^6 = 2Q^2 + P^3 \pm 2Q\sqrt{Q^2 + P^3}.$$

Combining the last two on $w_1^6$,

$$-P^3 = 2Q^2 + P^3 \pm 2Q\sqrt{Q^2 + P^3},$$

or, rearranging terms and halving,

$$Q^2 + P^3 = \mp Q\sqrt{Q^2 + P^3}.$$

Squaring,

$$Q^4 + 2Q^2P^3 + P^6 = Q^4 + Q^2P^3,$$

then canceling offsetting terms and factoring,

$$(P^3)(Q^2 + P^3) = 0.$$

[6]The verb *to cube* in this context means "to raise to the third power," as to change $y$ to $y^3$, just as the verb *to square* means "to raise to the second power."

The last equation demands rigidly that either $P = 0$ or $P^3 = -Q^2$. Some cubic polynomials do meet the demand—§ 10.4 will treat these and the reader is asked to set them aside for the moment—but most cubic polynomials do not meet it. For most cubic polynomials, then, the contradiction proves false the assumption which gave rise to it. The assumption: that the three $x$ descending from a single $w^3$ were not distinct. Therefore, provided that $P \neq 0$ and $P^3 \neq -Q^2$, the three $x$ descending from a single $w^3$ are indeed distinct, as was to be demonstrated.

The conclusion: *either, not both, of the two signs in the table's quadratic solution* $w^3 \equiv Q \pm \sqrt{Q^2 + P^3}$ *demands to be considered.* One can choose either sign; it matters not which.[7] The one sign alone yields all three roots of the general cubic polynomial.

In calculating the three $w$ from $w^3$, one can apply the Newton-Raphson iteration (4.33), the Taylor series of Table 8.1, or any other convenient root-finding technique to find a single root $w_1$ such that $w_1^3 = w^3$. Then the other two roots come easier. They are $e^{\pm i2\pi/3}w_1$; but $e^{\pm i2\pi/3} = (-1 \pm i\sqrt{3})/2$, so

$$w = w_1, \frac{-1 \pm i\sqrt{3}}{2}w_1. \tag{10.14}$$

We should observe, incidentally, that nothing prevents two actual roots of a cubic polynomial from having the same value. This certainly is possible, and it does not mean that one of the two roots is superfluous or that the polynomial has fewer than three roots. For example, the cubic polynomial $(z - 1)(z - 1)(z - 2) = z^3 - 4z^2 + 5z - 2$ has roots at 1, 1 and 2, with a single root at $z = 2$ and a double root—that is, two roots—at $z = 1$. When this happens, the method of Table 10.1 properly yields the single root once and the double root twice, just as it ought to do.

## 10.4 Edge cases

Section 10.3 excepts the edge cases $P = 0$ and $P^3 = -Q^2$. Mostly the book does not worry much about edge cases, but the effects of these cubic edge cases seem sufficiently nonobvious that the book might include here a few words about them, if for no other reason than to offer the reader a model of how to think about edge cases on his own. Table 10.1 gives the quadratic solution

$$w^3 \equiv Q \pm \sqrt{Q^2 + P^3},$$

[7] Numerically, it can matter. As a simple rule, because $w$ appears in the denominator of $x$'s definition, when the two $w^3$ differ in magnitude one might choose the larger.

in which § 10.3 generally finds it sufficient to consider either of the two signs. In the edge case $P = 0$,

$$w^3 = 2Q \text{ or } 0.$$

In the edge case $P^3 = -Q^2$,

$$w^3 = Q.$$

Both edge cases are interesting. In this section, we shall consider first the edge cases themselves, then their effect on the proof of § 10.3.

The edge case $P = 0$, like the general non-edge case, gives two distinct quadratic solutions $w^3$. One of the two however is $w^3 = Q - Q = 0$, which is awkward in light of Table 10.1's definition that $x \equiv w - P/w$. For this reason, in applying the table's method when $P = 0$, one chooses the other quadratic solution, $w^3 = Q + Q = 2Q$.

The edge case $P^3 = -Q^2$ gives only the one quadratic solution $w^3 = Q$; or more precisely, it gives two quadratic solutions which happen to have the same value. This is fine. One merely accepts that $w^3 = Q$, and does not worry about choosing one $w^3$ over the other.

The double edge case, or *corner case,* arises where the two edges meet—where $P = 0$ and $P^3 = -Q^2$, or equivalently where $P = 0$ and $Q = 0$. At the corner, the trouble is that $w^3 = 0$ and that no alternate $w^3$ is available. However, according to (10.12), $x^3 = 2Q - 3Px$, which in this case means that $x^3 = 0$ and thus that $x = 0$ absolutely, no other $x$ being possible. This implies the triple root $z = -a_2/3$.

Section 10.3 has excluded the edge cases from its proof of the sufficiency of a single $w^3$. Let us now add the edge cases to the proof. In the edge case $P^3 = -Q^2$, both $w^3$ are the same, so the one $w^3$ suffices by default because the other $w^3$ brings nothing different. The edge case $P = 0$ however does give two distinct $w^3$, one of which is $w^3 = 0$, which puts an awkward $0/0$ in the table's definition of $x$. We address this edge in the spirit of l'Hôpital's rule, by sidestepping it, changing $P$ infinitesimally from $P = 0$ to $P = \epsilon$. Then, choosing the $-$ sign in the definition of $w^3$,

$$\begin{aligned}
w^3 &= Q - \sqrt{Q^2 + \epsilon^3} = Q - (Q)\left(1 + \frac{\epsilon^3}{2Q^2}\right) = -\frac{\epsilon^3}{2Q}, \\
w &= -\frac{\epsilon}{(2Q)^{1/3}}, \\
x &= w - \frac{\epsilon}{w} = -\frac{\epsilon}{(2Q)^{1/3}} + (2Q)^{1/3} = (2Q)^{1/3}.
\end{aligned}$$

But choosing the + sign,

$$\begin{aligned} w^3 &= Q + \sqrt{Q^2 + \epsilon^3} = 2Q, \\ w &= (2Q)^{1/3}, \\ x &= w - \frac{\epsilon}{w} = (2Q)^{1/3} - \frac{\epsilon}{(2Q)^{1/3}} = (2Q)^{1/3}. \end{aligned}$$

Evidently the roots come out the same, either way. This completes the proof.

## 10.5 Quartics

Having successfully extracted the roots of the general cubic polynomial, we now turn our attention to the general quartic. The kernel of the cubic technique lay in reducing the cubic to a quadratic. The kernel of the quartic technique lies likewise in reducing the quartic to a cubic. The details differ, though; and, strangely enough, in some ways the quartic reduction is actually the simpler.[8]

As with the cubic, one begins solving the quartic by changing the variable

$$x + h \leftarrow z \tag{10.15}$$

to obtain the equation

$$x^4 = sx^2 + px + q, \tag{10.16}$$

where

$$\begin{aligned} h &\equiv -\frac{a_3}{4}, \\ s &\equiv -a_2 + 6\left(\frac{a_3}{4}\right)^2, \\ p &\equiv -a_1 + 2a_2\left(\frac{a_3}{4}\right) - 8\left(\frac{a_3}{4}\right)^3, \\ q &\equiv -a_0 + a_1\left(\frac{a_3}{4}\right) - a_2\left(\frac{a_3}{4}\right)^2 + 3\left(\frac{a_3}{4}\right)^4. \end{aligned} \tag{10.17}$$

[8] Even stranger, historically Ferrari discovered it earlier [65, "Quartic equation"]. Apparently Ferrari discovered the quartic's resolvent cubic (10.22), which he could not solve until Tartaglia applied Vieta's transform to it. What motivated Ferrari to chase the quartic solution while the cubic solution remained still unknown, this writer does not know, but one supposes that it might make an interesting story.

The reason the quartic is simpler to reduce is probably related to the fact that $(1)^{1/4} = \pm 1, \pm i$, whereas $(1)^{1/3} = 1, (-1 \pm i\sqrt{3})/2$. The $(1)^{1/4}$ brings a much neater result, the roots lying nicely along the Argand axes. This may also be why the quintic is intractable—but here we trespass the professional mathematician's territory and stray from the scope of this book. See Ch. 6's footnote 9.

To reduce (10.16) further, one must be cleverer. Ferrari[9] supplies the cleverness. The clever idea is to transfer some but not all of the $sx^2$ term to the equation's left side by

$$x^4 + 2ux^2 = (2u + s)x^2 + px + q,$$

where $u$ remains to be chosen; then to complete the square on the equation's left side as in § 2.2, but with respect to $x^2$ rather than $x$, as

$$\left(x^2 + u\right)^2 = k^2x^2 + px + j^2, \tag{10.18}$$

where

$$\begin{aligned} k^2 &\equiv 2u + s, \\ j^2 &\equiv u^2 + q. \end{aligned} \tag{10.19}$$

Now, one must regard (10.18) and (10.19) properly. In these equations, $s$, $p$ and $q$ have definite values fixed by (10.17), but not so $u$, $j$ or $k$. The variable $u$ is completely free; we have introduced it ourselves and can assign it any value we like. And though $j^2$ and $k^2$ depend on $u$, still, even after specifying $u$ we remain free at least to choose signs for $j$ and $k$. As for $u$, though no choice would truly be wrong, one supposes that a wise choice might at least render (10.18) easier to simplify.

So, what choice for $u$ would be wise? Well, look at (10.18). The left side of that equation is a perfect square. The right side would be, too, if it were that $p = \pm 2jk$; so, arbitrarily choosing the + sign, we propose the constraint that

$$p = 2jk, \tag{10.20}$$

or, better expressed,

$$j = \frac{p}{2k}. \tag{10.21}$$

Squaring (10.20) and substituting for $j^2$ and $k^2$ from (10.19), we have that

$$p^2 = 4(2u + s)(u^2 + q);$$

or, after distributing factors, rearranging terms and scaling, that

$$0 = u^3 + \frac{s}{2}u^2 + qu + \frac{4sq - p^2}{8}. \tag{10.22}$$

[9] [65, "Quartic equation"]

Equation (10.22) is the *resolvent cubic,* which we know by Table 10.1 how to solve for $u$, and which we now specify as a second constraint. If the constraints (10.21) and (10.22) are both honored, then we can safely substitute (10.20) into (10.18) to reach the form

$$\left(x^2 + u\right)^2 = k^2x^2 + 2jkx + j^2,$$

which is

$$\left(x^2 + u\right)^2 = \left(kx + j\right)^2. \tag{10.23}$$

The resolvent cubic (10.22) of course yields three $u$ not one, but the resolvent cubic is a voluntary constraint, so we can just pick one $u$ and ignore the other two. Equation (10.19) then gives $k$ (again, we can just pick one of the two signs), and (10.21) then gives $j$. With $u$, $j$ and $k$ established, (10.23) implies the quadratic

$$x^2 = \pm(kx + j) - u, \tag{10.24}$$

which (2.2) solves as

$$x = \pm\frac{k}{2} \pm_o \sqrt{\left(\frac{k}{2}\right)^2 \pm j - u}, \tag{10.25}$$

wherein the two $\pm$ signs are tied together but the third, $\pm_o$ sign is independent of the two. Equation (10.25), with the other equations and definitions of this section, reveals the four roots of the general quartic polynomial.

In view of (10.25), the change of variables

$$\begin{aligned} K &\leftarrow \frac{k}{2}, \\ J &\leftarrow j, \end{aligned} \tag{10.26}$$

improves the notation. Using the improved notation, Table 10.2 summarizes the complete quartic polynomial root extraction method.

## 10.6 Guessing the roots

It is entertaining to put pencil to paper and use Table 10.1's method to extract the roots of the cubic polynomial

$$0 = [z - 1][z - i][z + i] = z^3 - z^2 + z - 1.$$

Table 10.2: A method to extract the four roots of the general quartic polynomial. (In the table, the resolvent cubic is solved for $u$ by the method of Table 10.1, where any one of the three resulting $u$ serves. Either of the two $K$ similarly serves. Of the three $\pm$ signs in $x$'s definition, the $\pm_o$ is independent but the other two are tied together, the four resulting combinations giving the four roots of the general quartic.)

$$
\begin{aligned}
0 &= z^4 + a_3 z^3 + a_2 z^2 + a_1 z + a_0 \\
s &\equiv -a_2 + 6\left(\frac{a_3}{4}\right)^2 \\
p &\equiv -a_1 + 2a_2\left(\frac{a_3}{4}\right) - 8\left(\frac{a_3}{4}\right)^3 \\
q &\equiv -a_0 + a_1\left(\frac{a_3}{4}\right) - a_2\left(\frac{a_3}{4}\right)^2 + 3\left(\frac{a_3}{4}\right)^4 \\
0 &= u^3 + \frac{s}{2}u^2 + qu + \frac{4sq - p^2}{8} \\
K &\equiv \pm\frac{\sqrt{2u+s}}{2} \\
J &\equiv \begin{cases} \pm\sqrt{u^2+q} & \text{if } K = 0, \\ p/4K & \text{otherwise.} \end{cases} \\
x &\equiv \pm K \pm_o \sqrt{K^2 \pm J - u} \\
z &= x - \frac{a_3}{4}
\end{aligned}
$$

One finds that

$$\begin{aligned} z &= w + \frac{1}{3} - \frac{2}{3^2 w}, \\ w^3 &\equiv \frac{2\left(5 + \sqrt{3^3}\right)}{3^3}, \end{aligned}$$

which says indeed that $z = 1, \pm i$, but just you try to simplify it! A more baroque, more impenetrable way to write the number 1 is not easy to conceive. One has found the number 1 but cannot recognize it. Figuring the square and cube roots in the expression numerically, the root of the polynomial comes mysteriously to 1.0000, but why? The root's symbolic form gives little clue.

In general no better way is known;[10] we are stuck with the cubic baroquity. However, to the extent to which a cubic, a quartic, a quintic or any other polynomial has real, rational roots, a trick is known to sidestep Tables 10.1 and 10.2 and guess the roots directly. Consider for example the quintic polynomial

$$z^5 - \frac{7}{2}z^4 + 4z^3 + \frac{1}{2}z^2 - 5z + 3.$$

Doubling to make the coefficients all integers produces the polynomial

$$2z^5 - 7z^4 + 8z^3 + 1z^2 - 0\text{xA}z + 6,$$

which naturally has the same roots. If the roots are complex or irrational, they are hard to guess; but if any of the roots happens to be real and rational, it must belong to the set

$$\left\{ \pm 1, \pm 2, \pm 3, \pm 6, \pm\frac{1}{2}, \pm\frac{2}{2}, \pm\frac{3}{2}, \pm\frac{6}{2} \right\}.$$

No other real, rational root is possible. Trying the several candidates on the polynomial, one finds that 1, $-1$ and $3/2$ are indeed roots. Dividing these out leaves a quadratic which is easy to solve for the remaining roots.

The real, rational candidates are the factors of the polynomial's trailing coefficient (in the example, 6, whose factors are $\pm 1$, $\pm 2$, $\pm 3$ and $\pm 6$) divided by the factors of the polynomial's leading coefficient (in the example, 2,

[10] At least, no better way is known to this author. If any reader can straightforwardly simplify the expression without solving a cubic polynomial of some kind, the author would like to hear of it.

whose factors are $\pm 1$ and $\pm 2$). The reason no other real, rational root is possible is seen[11] by writing $z = p/q$—where $p, q \in \mathbb{Z}$ are integers and the fraction $p/q$ is fully reduced—then multiplying the $n$th-order polynomial by $q^n$ to reach the form

$$a_n p^n + a_{n-1} p^{n-1} q + \cdots + a_1 p q^{n-1} + a_0 q^n = 0,$$

where all the coefficients $a_k$ are integers. Moving the $q^n$ term to the equation's right side, we have that

$$\left(a_n p^{n-1} + a_{n-1} p^{n-2} q + \cdots + a_1 q^{n-1}\right) p = -a_0 q^n,$$

which implies that $a_0 q^n$ is a multiple of $p$. But by demanding that the fraction $p/q$ be fully reduced, we have defined $p$ and $q$ to be *relatively prime* to one another—that is, we have defined them to have no factors but $\pm 1$ in common—so, not only $a_0 q^n$ but $a_0$ itself is a multiple of $p$. By similar reasoning, $a_n$ is a multiple of $q$. But if $a_0$ is a multiple of $p$, and $a_n$, a multiple of $q$, then $p$ and $q$ are factors of $a_0$ and $a_n$ respectively. We conclude for this reason, as was to be demonstrated, that no real, rational root is possible except a factor of $a_0$ divided by a factor of $a_n$.[12]

Such root-guessing is little more than an algebraic trick, of course, but it can be a pretty useful trick if it saves us the embarrassment of inadvertently expressing simple rational numbers in ridiculous ways.

One could write much more about higher-order algebra, but now that the reader has tasted the topic he may feel inclined to agree that, though the general methods this chapter has presented to solve cubics and quartics are interesting, further effort were nevertheless probably better spent elsewhere. The next several chapters turn to the topic of the matrix, harder but much more profitable, toward which we mean to put substantial effort.

[11] The presentation here is quite informal. We do not want to spend many pages on this.

[12] [59, § 3.2]

# Part II

# Matrices and vectors

# Chapter 11

# The matrix

Chapters 2 through 9 have laid solidly the basic foundations of applied mathematics. This chapter begins to build on those foundations, demanding some heavier mathematical lifting.

Taken by themselves, most of the foundational methods of the earlier chapters have handled only one or at most a few numbers (or functions) at a time. However, in practical applications the need to handle large arrays of numbers at once arises often. Some nonobvious effects emerge then, as, for example, the eigenvalue of Ch. 14.

Regarding the eigenvalue: the eigenvalue was always there, but prior to this point in the book it was usually trivial—the eigenvalue of 5 is just 5, for instance—so we didn't bother much to talk about it. It is when numbers are laid out in orderly grids like

$$C = \begin{bmatrix} 6 & 4 & 0 \\ 3 & 0 & 1 \\ 3 & 1 & 0 \end{bmatrix}$$

that nontrivial eigenvalues arise (though you cannot tell just by looking, the eigenvalues of $C$ happen to be $-1$ and $[7 \pm \sqrt{0\text{x}49}]/2$). But, just what is an *eigenvalue?* Answer: an eigenvalue is the value by which an object like $C$ scales an eigenvector without altering the eigenvector's direction. Of course, we have not yet said what an *eigenvector* is, either, or how $C$ might scale something, but it is to answer precisely such questions that this chapter and the three which follow it are written.

So, we are getting ahead of ourselves. Let's back up.

An object like $C$ is called a *matrix.* It serves as a generalized coefficient or multiplier. Where we have used single numbers as coefficients or multipliers

heretofore, one can with sufficient care often use matrices instead. The matrix interests us for this reason among others.

The technical name for the "single number" is the *scalar.* Such a number, as for instance 5 or $-4 + i3$, is called a scalar because its action alone during multiplication is simply to scale the thing it multiplies. Besides acting alone, however, scalars can also act in concert—in orderly formations—thus constituting any of three basic kinds of arithmetical object:

- the *scalar* itself, a single number like $\alpha = 5$ or $\beta = -4 + i3$;
- the *vector,* a column of $m$ scalars like

$$\mathbf{u} = \begin{bmatrix} 5 \\ -4 + i3 \end{bmatrix},$$

  which can be written in-line with the notation $\mathbf{u} = [5 \;\; -4 + i3]^T$ (here there are two scalar elements, 5 and $-4+i3$, so in this example $m = 2$);
- the *matrix,* an $m \times n$ grid of scalars, or equivalently a row of $n$ vectors, like

$$A = \begin{bmatrix} 0 & 6 & 2 \\ 1 & 1 & -1 \end{bmatrix},$$

  which can be written in-line with the notation $A = [0\ 6\ 2; 1\ 1\ -1]$ or the notation $A = [0\ 1; 6\ 1; 2\ -1]^T$ (here there are two rows and three columns of scalar elements, so in this example $m = 2$ and $n = 3$).

Several general points are immediately to be observed about these various objects. First, despite the geometrical Argand interpretation of the complex number, a complex number is not a two-element vector but a scalar; therefore any or all of a vector's or matrix's scalar elements can be complex. Second, an $m$-element vector does not differ for most purposes from an $m \times 1$ matrix; generally the two can be regarded as the same thing. Third, the three-element (that is, three-dimensional) geometrical vector of § 3.3 is just an $m$-element vector with $m = 3$. Fourth, $m$ and $n$ can be any nonnegative integers, even one, even zero, even infinity.[1]

Where one needs visually to distinguish a symbol like $A$ representing a matrix, one can write it $[A]$, in square brackets.[2] Normally however a simple $A$ suffices.

[1] Fifth, though the progression *scalar, vector, matrix* suggests next a "matrix stack" or stack of $p$ matrices, such objects in fact are seldom used. As we shall see in § 11.1, the chief advantage of the standard matrix is that it neatly represents the linear transformation of one vector into another. "Matrix stacks" bring no such advantage. This book does not treat them.

[2] Alternate notations seen in print include $\overline{\overline{A}}$ and $\mathbf{A}$.

The matrix is a notoriously hard topic to motivate. The idea of the matrix is deceptively simple. The mechanics of matrix arithmetic are deceptively intricate. The most basic body of matrix theory, without which little or no useful matrix work can be done, is deceptively extensive. The matrix neatly encapsulates a substantial knot of arithmetical tedium and clutter, but to understand the matrix one must first understand the tedium and clutter the matrix encapsulates. As far as the author is aware, no one has ever devised a way to introduce the matrix which does not seem shallow, tiresome, irksome, even interminable at first encounter; yet the matrix is too important to ignore. Applied mathematics brings nothing else quite like it.[3]

Chapters 11 through 14 treat the matrix and its algebra. This chapter,

[3]In most of its chapters, the book seeks a balance between terseness the determined beginner cannot penetrate and prolixity the seasoned veteran will not abide. The matrix upsets this balance.

Part of the trouble with the matrix is that its arithmetic is just that, an arithmetic, no more likely to be mastered by mere theoretical study than was the classical arithmetic of childhood. To master matrix arithmetic, one must drill it; yet the book you hold is fundamentally one of theory not drill.

The reader who has previously drilled matrix arithmetic will meet here the essential applied theory of the matrix. That reader will find this chapter and the next three tedious enough. The reader who has not previously drilled matrix arithmetic, however, is likely to find these chapters positively hostile. Only the doggedly determined beginner will learn the matrix here alone; others will find it more amenable to drill matrix arithmetic first in the early chapters of an introductory linear algebra textbook, dull though such chapters be (see [42] or better yet the fine, surprisingly less dull [30] for instance, though the early chapters of almost any such book give the needed arithmetical drill.) Returning here thereafter, the beginner can expect to find *these* chapters still tedious but no longer impenetrable. The reward is worth the effort. That is the approach the author recommends.

To the mathematical rebel, the young warrior with face painted and sword agleam, still determined to learn the matrix here alone, the author salutes his honorable defiance. Would the rebel consider alternate counsel? If so, then the rebel might compose a dozen matrices of various sizes and shapes, broad, square and tall, decomposing each carefully by pencil per the Gauss-Jordan method of § 12.3, checking results (again by pencil; using a machine defeats the point of the exercise, and using a sword, well, it won't work) by multiplying factors to restore the original matrices. Several hours of such drill should build the young warrior the practical arithmetical foundation to master—with commensurate effort—the theory these chapters bring. The way of the warrior is hard, but conquest is not impossible.

To the matrix veteran, the author presents these four chapters with grim enthusiasm. Substantial, logical, necessary the chapters may be, but exciting they are not. At least, the earlier parts are not very exciting (later parts are better). As a reasonable compromise, the veteran seeking more interesting reading might skip directly to Chs. 13 and 14, referring back to Chs. 11 and 12 as need arises.

Ch. 11, introduces the rudiments of the matrix itself.[4]

## 11.1 Provenance and basic use

It is in the study of linear transformations that the concept of the matrix first arises. We begin there.

### 11.1.1 The linear transformation

Section 7.3.3 has introduced the idea of linearity. The *linear transformation*[5] is the operation of an $m \times n$ matrix $A$, as in

$$A\mathbf{x} = \mathbf{b}, \tag{11.1}$$

to transform an $n$-element vector $\mathbf{x}$ into an $m$-element vector $\mathbf{b}$, while respecting the rules of linearity

$$\begin{aligned} A(\mathbf{x}_1 + \mathbf{x}_2) &= A\mathbf{x}_1 + A\mathbf{x}_2 &&= \mathbf{b}_1 + \mathbf{b}_2, \\ A(\alpha\mathbf{x}) &= \alpha A\mathbf{x} &&= \alpha\mathbf{b}, \\ A(0) &= 0. \end{aligned} \tag{11.2}$$

For example,

$$A = \begin{bmatrix} 0 & 6 & 2 \\ 1 & 1 & -1 \end{bmatrix}$$

is the $2 \times 3$ matrix which transforms a three-element vector $\mathbf{x}$ into a two-element vector $\mathbf{b}$ such that

$$A\mathbf{x} = \begin{bmatrix} 0x_1 + 6x_2 + 2x_3 \\ 1x_1 + 1x_2 - 1x_3 \end{bmatrix} = \mathbf{b},$$

where

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}.$$

[4] [6][21][30][42]

[5] Professional mathematicians conventionally are careful to begin by drawing a clear distinction between the ideas of the linear transformation, the basis set and the simultaneous system of linear equations—proving from suitable axioms that the three amount more or less to the same thing, rather than implicitly assuming the fact. The professional approach [6, Chs. 1 and 2][42, Chs. 1, 2 and 5] has much to recommend it, but it is not the approach we will follow here.

In general, the operation of a matrix $A$ is that[6,7]

$$b_i = \sum_{j=1}^{n} a_{ij} x_j, \tag{11.3}$$

where $x_j$ is the $j$th element of $\mathbf{x}$, $b_i$ is the $i$th element of $\mathbf{b}$, and

$$a_{ij} \equiv [A]_{ij}$$

is the element at the $i$th row and $j$th column of $A$, counting from top left (in the example for instance, $a_{12} = 6$).

Besides representing linear transformations as such, matrices can also represent simultaneous systems of linear equations. For example, the system

$$\begin{aligned} 0x_1 + 6x_2 + 2x_3 &= 2, \\ 1x_1 + 1x_2 - 1x_3 &= 4, \end{aligned}$$

is compactly represented as

$$A\mathbf{x} = \mathbf{b},$$

with $A$ as given above and $\mathbf{b} = [2\ 4]^T$. Seen from this point of view, a simultaneous system of linear equations is itself neither more nor less than a linear transformation.

### 11.1.2 Matrix multiplication (and addition)

Nothing prevents one from lining several vectors $\mathbf{x}_k$ up in a row, industrial mass production-style, transforming them at once into the corresponding

[6] As observed in Appendix B, there are unfortunately not enough distinct Roman and Greek letters available to serve the needs of higher mathematics. In matrix work, the Roman letters $ijk$ conventionally serve as indices, but the same letter $i$ also serves as the imaginary unit, which is not an index and has nothing to do with indices. Fortunately, the meaning is usually clear from the context: $i$ in $\sum_i$ or $a_{ij}$ is an index; $i$ in $-4 + i3$ or $e^{i\phi}$ is the imaginary unit. Should a case arise in which the meaning is not clear, one can use $\ell jk$ or some other convenient letters for the indices.

[7] Whether to let the index $j$ run from 0 to $n-1$ or from 1 to $n$ is an awkward question of applied mathematical style. In computers, the index normally runs from 0 to $n-1$, and in many ways this really is the more sensible way to do it. In mathematical theory, however, a 0 index normally implies something special or basic about the object it identifies. The book you are reading tends to let the index run from 1 to $n$, following mathematical convention in the matter for this reason.

Conceived more generally, an $m \times n$ matrix can be considered an $\infty \times \infty$ matrix with zeros in the unused cells. Here, both indices $i$ and $j$ run from $-\infty$ to $+\infty$ anyway, so the computer's indexing convention poses no dilemma in this case. See § 11.3.

vectors $\mathbf{b}_k$ by the same matrix $A$. In this case,

$$\begin{aligned} X &\equiv [\ \mathbf{x}_1 \quad \mathbf{x}_2 \quad \cdots \quad \mathbf{x}_p\ ], \\ B &\equiv [\ \mathbf{b}_1 \quad \mathbf{b}_2 \quad \cdots \quad \mathbf{b}_p\ ], \\ AX &= B, \\ b_{ik} &= \sum_{j=1}^{n} a_{ij} x_{jk}. \end{aligned} \tag{11.4}$$

Equation (11.4) implies a definition for matrix multiplication. Such matrix multiplication is associative since

$$\begin{aligned} [(A)(XY)]_{ik} &= \sum_{j=1}^{n} a_{ij} [XY]_{jk} \\ &= \sum_{j=1}^{n} a_{ij} \left[ \sum_{\ell=1}^{p} x_{j\ell} y_{\ell k} \right] \\ &= \sum_{\ell=1}^{p} \sum_{j=1}^{n} a_{ij} x_{j\ell} y_{\ell k} \\ &= [(AX)(Y)]_{ik}\,. \end{aligned} \tag{11.5}$$

Matrix multiplication is not generally commutative, however;

$$AX \neq XA, \tag{11.6}$$

as one can show by a suitable counterexample like $A = [0\ 1; 0\ 0]$, $X = [1\ 0; 0\ 0]$. To multiply a matrix by a scalar, one multiplies each of the matrix's elements individually by the scalar:

$$[\alpha A]_{ij} = \alpha a_{ij}. \tag{11.7}$$

Evidently multiplication by a scalar is commutative: $\alpha A\mathbf{x} = A\alpha\mathbf{x}$.

Matrix addition works in the way one would expect, element by element; and as one can see from (11.4), under multiplication, matrix addition is indeed distributive:

$$\begin{aligned} [X + Y]_{ij} &= x_{ij} + y_{ij}; \\ (A)(X + Y) &= AX + AY; \\ (A + C)(X) &= AX + CX. \end{aligned} \tag{11.8}$$

### 11.1.3 Row and column operators

The matrix equation $A\mathbf{x} = \mathbf{b}$ represents the linear transformation of $\mathbf{x}$ into $\mathbf{b}$, as we have seen. Viewed from another perspective, however, the same matrix equation represents something else; it represents a weighted sum of the columns of $A$, with the elements of $\mathbf{x}$ as the weights. In this view, one writes (11.3) as

$$\mathbf{b} = \sum_{j=1}^{n} [A]_{*j} x_j, \tag{11.9}$$

where $[A]_{*j}$ is the $j$th column of $A$. Here $\mathbf{x}$ is not only a vector; it is also an operator. It operates on $A$'s columns. By virtue of multiplying $A$ from the right, the vector $\mathbf{x}$ is a *column operator* acting on $A$.

If several vectors $\mathbf{x}_k$ line up in a row to form a matrix $X$, such that $AX = B$, then the matrix $X$ is likewise a column operator:

$$[B]_{*k} = \sum_{j=1}^{n} [A]_{*j} x_{jk}. \tag{11.10}$$

The $k$th column of $X$ weights the several columns of $A$ to yield the $k$th column of $B$.

If a matrix multiplying from the right is a column operator, is a matrix multiplying from the left a *row operator?* Indeed it is. Another way to write $AX = B$, besides (11.10), is

$$[B]_{i*} = \sum_{j=1}^{n} a_{ij} [X]_{j*}. \tag{11.11}$$

The $i$th row of $A$ weights the several rows of $X$ to yield the $i$th row of $B$. The matrix $A$ is a row operator. (Observe the notation. The $*$ here means "any" or "all." Hence $[X]_{j*}$ means "$j$th row, all columns of $X$"—that is, the $j$th row of $X$. Similarly, $[A]_{*j}$ means "all rows, $j$th column of $A$"—that is, the $j$th column of $A$.)

*Column operators attack from the right; row operators, from the left.* This rule is worth memorizing; the concept is important. In $AX = B$, the matrix $X$ operates on $A$'s columns; the matrix $A$ operates on $X$'s rows.

Since matrix multiplication produces the same result whether one views it as a linear transformation (11.4), a column operation (11.10) or a row operation (11.11), one might wonder what purpose lies in defining matrix multiplication three separate ways. However, it is not so much for the sake

of the mathematics that we define it three ways as it is for the sake of the mathematician. We do it for ourselves. Mathematically, the latter two do indeed expand to yield (11.4), but as written the three represent three different perspectives on the matrix. A tedious, nonintuitive matrix theorem from one perspective can appear suddenly obvious from another (see for example eqn. 11.63). Results hard to visualize one way are easy to visualize another. It is worth developing the mental agility to view and handle matrices all three ways for this reason.

### 11.1.4 The transpose and the adjoint

One function peculiar to matrix algebra is the *transpose*

$$\begin{aligned} C &= A^T, \\ c_{ij} &= a_{ji}, \end{aligned} \tag{11.12}$$

which mirrors an $m \times n$ matrix into an $n \times m$ matrix. For example,

$$A^T = \begin{bmatrix} 0 & 1 \\ 6 & 1 \\ 2 & -1 \end{bmatrix}.$$

Similar and even more useful is the *conjugate transpose* or *adjoint*[8]

$$\begin{aligned} C &= A^*, \\ c_{ij} &= a^*_{ji}, \end{aligned} \tag{11.13}$$

which again mirrors an $m \times n$ matrix into an $n \times m$ matrix, but conjugates each element as it goes.

The transpose is convenient notationally to write vectors and matrices inline and to express certain matrix-arithmetical mechanics; but algebraically the transpose is artificial. It is the adjoint rather which mirrors a matrix properly. (If the transpose and adjoint functions applied to words as to matrices, then the transpose of "derivations" would be "snoitavired," whereas the adjoint would be "ƨnoiʇɐvirɐb." See the difference?) On real-valued matrices like the $A$ in the example, of course, the transpose and the adjoint amount to the same thing.

[8] Alternate notations sometimes seen in print for the adjoint include $A^\dagger$ (a notation which in this book means something unrelated) and $A^H$ (a notation which recalls the name of the mathematician Charles Hermite). However, the book you are reading writes the adjoint only as $A^*$, a notation which better captures the sense of the thing in the author's view.

If one needed to conjugate the elements of a matrix without transposing the matrix itself, one could contrive notation like $A^{*T}$. Such a need seldom arises, however.

Observe that

$$\begin{aligned}(A_2A_1)^T &= A_1^TA_2^T,\\ (A_2A_1)^* &= A_1^*A_2^*,\end{aligned} \tag{11.14}$$

and more generally that[9]

$$\begin{aligned}\left(\prod_k A_k\right)^T &= \coprod_k A_k^T,\\ \left(\prod_k A_k\right)^* &= \coprod_k A_k^*.\end{aligned} \tag{11.15}$$

## 11.2 The Kronecker delta

Section 7.7 has introduced the Dirac delta. The discrete analog of the Dirac delta is the *Kronecker delta*[10]

$$\delta_i \equiv \begin{cases}1 & \text{if } i = 0,\\ 0 & \text{otherwise;}\end{cases} \tag{11.16}$$

or

$$\delta_{ij} \equiv \begin{cases}1 & \text{if } i = j,\\ 0 & \text{otherwise.}\end{cases} \tag{11.17}$$

The Kronecker delta enjoys the Dirac-like properties that

$$\sum_{i=-\infty}^{\infty} \delta_i = \sum_{i=-\infty}^{\infty} \delta_{ij} = \sum_{j=-\infty}^{\infty} \delta_{ij} = 1 \tag{11.18}$$

and that

$$\sum_{j=-\infty}^{\infty} \delta_{ij}a_{jk} = a_{ik}, \tag{11.19}$$

the latter of which is the Kronecker sifting property. The Kronecker equations (11.18) and (11.19) parallel the Dirac equations (7.13) and (7.14).

Chs. 11 and 14 will find frequent use for the Kronecker delta. Later, § 15.4.3 will revisit the Kronecker delta in another light.

[9] Recall from § 2.3 that $\prod_k A_k = \cdots A_3A_2A_1$, whereas $\coprod_k A_k = A_1A_2A_3\cdots$.

[10] [66, "Kronecker delta," 15:59, 31 May 2006]

## 11.3 Dimensionality and matrix forms

An $m \times n$ matrix like

$$X = \begin{bmatrix} -4 & 0 \\ 1 & 2 \\ 2 & -1 \end{bmatrix}$$

can be viewed as the $\infty \times \infty$ matrix

$$X = \begin{bmatrix} \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \\ \cdots & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\ \cdots & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\ \cdots & 0 & 0 & -4 & 0 & 0 & 0 & 0 & \cdots \\ \cdots & 0 & 0 & 1 & 2 & 0 & 0 & 0 & \cdots \\ \cdots & 0 & 0 & 2 & -1 & 0 & 0 & 0 & \cdots \\ \cdots & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\ \cdots & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\ & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix},$$

with zeros in the unused cells. As before, $x_{11} = -4$ and $x_{32} = -1$, but now $x_{ij}$ exists for all integral $i$ and $j$; for instance, $x_{(-1)(-1)} = 0$. For such a matrix, indeed for all matrices, the matrix multiplication rule (11.4) generalizes to

$$\begin{aligned} B &= AX, \\ b_{ik} &= \sum_{j=-\infty}^{\infty} a_{ij} x_{jk}. \end{aligned} \tag{11.20}$$

For square matrices whose purpose is to manipulate other matrices or vectors in place, merely padding with zeros often does not suit. Consider for example the square matrix

$$A_3 = \begin{bmatrix} 1 & 0 & 0 \\ 5 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

This $A_3$ is indeed a matrix, but when it acts $A_3 X$ as a row operator on some $3 \times p$ matrix $X$, its effect is to add to $X$'s second row, 5 times the first. Further consider

$$A_4 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 5 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix},$$

which does the same to a $4 \times p$ matrix $X$. We can also define $A_5, A_6, A_7, \ldots$, if we want; but, really, all these express the same operation: "to add to the second row, 5 times the first."

The $\infty \times \infty$ matrix

$$A = \begin{bmatrix}
\ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \\
\cdots & 1 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\
\cdots & 0 & 1 & 0 & 0 & 0 & 0 & 0 & \cdots \\
\cdots & 0 & 0 & 1 & 0 & 0 & 0 & 0 & \cdots \\
\cdots & 0 & 0 & 5 & 1 & 0 & 0 & 0 & \cdots \\
\cdots & 0 & 0 & 0 & 0 & 1 & 0 & 0 & \cdots \\
\cdots & 0 & 0 & 0 & 0 & 0 & 1 & 0 & \cdots \\
\cdots & 0 & 0 & 0 & 0 & 0 & 0 & 1 & \cdots \\
 & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots
\end{bmatrix}$$

expresses the operation generally. As before, $a_{11} = 1$ and $a_{21} = 5$, but now also $a_{(-1)(-1)} = 1$ and $a_{09} = 0$, among others. By running ones infinitely both ways out the *main diagonal,* we guarantee by (11.20) that when $A$ acts $AX$ on a matrix $X$ of any dimensionality whatsoever, $A$ adds to the second row of $X$, 5 times the first—and affects no other row. (But what if $X$ is a $1 \times p$ matrix, and *has* no second row? Then the operation $AX$ creates a new second row, 5 times the first—or rather so fills in $X$'s previously null second row.)

In the infinite-dimensional view, the matrices $A$ and $X$ differ essentially.[11] This section explains, developing some nonstandard formalisms the derivations of later sections and chapters can use.[12]

[11] This particular section happens to use the symbols $A$ and $X$ to represent certain specific matrix forms because such usage flows naturally from the usage $A\mathbf{x} = \mathbf{b}$ of § 11.1. Such usage admittedly proves awkward in other contexts. Traditionally in matrix work and elsewhere in the book, the letter $A$ does not necessarily represent an extended operator as it does here, but rather an arbitrary matrix of no particular form.

[12] The idea of infinite dimensionality is sure to discomfit some readers, who have studied matrices before and are used to thinking of a matrix as having some definite size. There is nothing wrong with thinking of a matrix as having some definite size, only that that view does not suit the present book's development. And really, the idea of an $\infty \times 1$ vector or an $\infty \times \infty$ matrix should not seem so strange. After all, consider the vector $\mathbf{u}$ such that

$$u_\ell = \sin \ell\epsilon,$$

where $0 < \epsilon \ll 1$ and $\ell$ is an integer, which holds all values of the function $\sin\theta$ of a real argument $\theta$. Of course one does not actually write down or store all the elements of an infinite-dimensional vector or matrix, any more than one actually writes down or stores all the bits (or digits) of $2\pi$. Writing them down or storing them is not the point. The point is that infinite dimensionality is all right; that the idea thereof does not threaten to overturn the reader's preëxisting matrix knowledge; that, though the construct seem unfamiliar, no fundamental conceptual barrier rises against it.

Different ways of looking at the same mathematics can be extremely useful to the applied mathematician. The applied mathematical reader who has never heretofore considered

### 11.3.1 The null and dimension-limited matrices

The *null matrix* is just what its name implies:

$$0 = \begin{bmatrix} \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \\ \cdots & 0 & 0 & 0 & 0 & 0 & \cdots \\ \cdots & 0 & 0 & 0 & 0 & 0 & \cdots \\ \cdots & 0 & 0 & 0 & 0 & 0 & \cdots \\ \cdots & 0 & 0 & 0 & 0 & 0 & \cdots \\ \cdots & 0 & 0 & 0 & 0 & 0 & \cdots \\ & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix};$$

or more compactly,

$$[0]_{ij} = 0.$$

Special symbols like $\overline{\overline{0}}$, $\mathbf{0}$ or $O$ are possible for the null matrix, but usually a simple 0 suffices. There are no surprises here; the null matrix brings all the expected properties of a zero, like

$$\begin{aligned} 0 + A &= A, \\ [0][X] &= 0. \end{aligned}$$

The same symbol 0 used for the null scalar (zero) and the null matrix is used for the null vector, too. Whether the scalar 0, the vector 0 and the matrix 0 actually represent different things is a matter of semantics, but the three are interchangeable for most practical purposes in any case. Basically, a zero is a zero is a zero; there's not much else to it.[13]

Now a formality: the ordinary $m \times n$ matrix $X$ can be viewed, infinite-dimensionally, as a variation on the null matrix, inasmuch as $X$ differs from the null matrix only in the $mn$ elements $x_{ij}$, $1 \le i \le m$, $1 \le j \le n$. Though the theoretical dimensionality of $X$ be $\infty \times \infty$, one need record only the $mn$ elements, plus the values of $m$ and $n$, to retain complete information about such a matrix. So the semantics are these: when we call a matrix $X$ an $m \times n$ *matrix,* or more precisely a *dimension-limited matrix* with an $m \times n$

infinite dimensionality in vectors and matrices would be well served to take the opportunity to do so here. As we shall discover in Ch. 12, dimensionality is a poor measure of a matrix's size in any case. What really counts is not a matrix's $m \times n$ dimensionality but rather its *rank.*

[13] Well, of course, there's a lot else to it, when it comes to dividing by zero as in Ch. 4, or to summing an infinity of zeros as in Ch. 7, but those aren't what we were speaking of here.

*active region,* we will mean formally that $X$ is an $\infty \times \infty$ matrix whose elements are all zero outside the $m \times n$ rectangle:

$$x_{ij} = 0 \text{ except where } 1 \le i \le m \text{ and } 1 \le j \le n. \tag{11.21}$$

By these semantics, every $3 \times 2$ matrix (for example) is also a formally a $4 \times 4$ matrix; but a $4 \times 4$ matrix is not in general a $3 \times 2$ matrix.

### 11.3.2 The identity and scalar matrices and the extended operator

The *general identity matrix*—or simply, the *identity matrix*—is

$$I = \begin{bmatrix} \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \\ \cdots & 1 & 0 & 0 & 0 & 0 & \cdots \\ \cdots & 0 & 1 & 0 & 0 & 0 & \cdots \\ \cdots & 0 & 0 & 1 & 0 & 0 & \cdots \\ \cdots & 0 & 0 & 0 & 1 & 0 & \cdots \\ \cdots & 0 & 0 & 0 & 0 & 1 & \cdots \\ & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix},$$

or more compactly,

$$[I]_{ij} = \delta_{ij}, \tag{11.22}$$

where $\delta_{ij}$ is the Kronecker delta of § 11.2. The identity matrix $I$ is a matrix 1, as it were,[14] bringing the essential property one expects of a 1:

$$IX = X = XI. \tag{11.23}$$

The *scalar matrix* is

$$\lambda I = \begin{bmatrix} \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \\ \cdots & \lambda & 0 & 0 & 0 & 0 & \cdots \\ \cdots & 0 & \lambda & 0 & 0 & 0 & \cdots \\ \cdots & 0 & 0 & \lambda & 0 & 0 & \cdots \\ \cdots & 0 & 0 & 0 & \lambda & 0 & \cdots \\ \cdots & 0 & 0 & 0 & 0 & \lambda & \cdots \\ & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix},$$

or more compactly,

$$[\lambda I]_{ij} = \lambda \delta_{ij}, \tag{11.24}$$

[14] In fact you can write it as 1 if you like. That is essentially what it is. The $I$ can be regarded as standing for "identity" or as the Roman numeral I.

If the identity matrix $I$ is a matrix 1, then the scalar matrix $\lambda I$ is a matrix $\lambda$, such that

$$[\lambda I]X = \lambda X = X[\lambda I]. \tag{11.25}$$

The identity matrix is (to state the obvious) just the scalar matrix with $\lambda = 1$.

The *extended operator* $A$ is a variation on the scalar matrix $\lambda I$, $\lambda \neq 0$, inasmuch as $A$ differs from $\lambda I$ only in $p$ specific elements, with $p$ a finite number. Symbolically,

$$\begin{aligned} a_{ij} &= \begin{cases} (\lambda)(\delta_{ij} + \alpha_k) & \text{if } (i,j) = (i_k, j_k),\ 1 \le k \le p, \\ \lambda\delta_{ij} & \text{otherwise;} \end{cases} \\ \lambda &\neq 0. \end{aligned} \tag{11.26}$$

The several $\alpha_k$ control how the extended operator $A$ differs from $\lambda I$. One need record only the several $\alpha_k$ along with their respective addresses $(i_k, j_k)$, plus the scale $\lambda$, to retain complete information about such a matrix. For example, for an extended operator fitting the pattern

$$A = \begin{bmatrix} \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \\ \cdots & \lambda & 0 & 0 & 0 & 0 & \cdots \\ \cdots & \lambda\alpha_1 & \lambda & 0 & 0 & 0 & \cdots \\ \cdots & 0 & 0 & \lambda & \lambda\alpha_2 & 0 & \cdots \\ \cdots & 0 & 0 & 0 & \lambda(1+\alpha_3) & 0 & \cdots \\ \cdots & 0 & 0 & 0 & 0 & \lambda & \cdots \\ & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix},$$

one need record only the values of $\alpha_1$, $\alpha_2$ and $\alpha_3$, the respective addresses $(2,1)$, $(3,4)$ and $(4,4)$, and the value of the scale $\lambda$; this information alone implies the entire $\infty \times \infty$ matrix $A$.

When we call a matrix $A$ an *extended $n \times n$ operator,* or an extended operator with an *$n \times n$ active region,* we will mean formally that $A$ is an $\infty \times \infty$ matrix and is further an extended operator for which

$$1 \le i_k \le n \text{ and } 1 \le j_k \le n \text{ for all } 1 \le k \le p. \tag{11.27}$$

That is, an extended $n \times n$ operator is one whose several $\alpha_k$ all lie within the $n \times n$ square. The $A$ in the example is an extended $4 \times 4$ operator (and also a $5 \times 5$, a $6 \times 6$, etc., but not a $3 \times 3$).

(Often in practice for smaller operators—especially in the typical case that $\lambda = 1$—one finds it easier just to record all the $n \times n$ elements of the active region. This is fine. Large matrix operators however tend to be

*sparse,* meaning that they depart from $\lambda I$ in only a very few of their many elements. It would waste a lot of computer memory explicitly to store all those zeros, so one normally stores just the few elements, instead.)

Implicit in the definition of the extended operator is that the identity matrix $I$ and the scalar matrix $\lambda I$, $\lambda \neq 0$, are extended operators with $0 \times 0$ active regions (and also $1 \times 1$, $2 \times 2$, etc.). If $\lambda = 0$, however, the scalar matrix $\lambda I$ is just the null matrix, which is no extended operator but rather by definition a $0 \times 0$ dimension-limited matrix.

### 11.3.3 The active region

Though maybe obvious, it bears stating explicitly that a product of dimension-limited and/or extended-operational matrices with $n \times n$ active regions itself has an $n \times n$ active region.[15] (Remember that a matrix with an $m' \times n'$ active region also by definition has an $n \times n$ active region if $m' \leq n$ and $n' \leq n$.) If any of the factors has dimension-limited form then so does the product; otherwise the product is an extended operator.[16]

### 11.3.4 Other matrix forms

Besides the dimension-limited form of § 11.3.1 and the extended-operational form of § 11.3.2, other infinite-dimensional matrix forms are certainly possible. One could for example advantageously define a "null sparse" form, recording only nonzero elements and their addresses in an otherwise null matrix; or a "tridiagonal extended" form, bearing repeated entries not only along the main diagonal but also along the diagonals just above and just below. Section 11.9 introduces one worthwhile matrix which fits neither the dimension-limited nor the extended-operational form. Still, the dimension-

[15] The section's earlier subsections formally define the term *active region* with respect to each of the two matrix forms.

[16] If symbolic proof of the subsection's claims is wanted, here it is in outline:

$$\begin{aligned}
a_{ij} &= \lambda_a \delta_{ij} \qquad \text{unless } 1 \leq (i,j) \leq n, \\
b_{ij} &= \lambda_b \delta_{ij} \qquad \text{unless } 1 \leq (i,j) \leq n; \\
[AB]_{ij} &= \sum_k a_{ik} b_{kj} \\
&= \begin{cases} \sum_k (\lambda_a \delta_{ik}) b_{kj} = \lambda_a b_{ij} & \text{unless } 1 \leq i \leq n \\ \sum_k a_{ik} (\lambda_b \delta_{kj}) = \lambda_b a_{ij} & \text{unless } 1 \leq j \leq n \end{cases} \\
&= \lambda_a \lambda_b \delta_{ij} \quad \text{unless } 1 \leq (i,j) \leq n.
\end{aligned}$$

It's probably easier just to sketch the matrices and look at them, though.

limited and extended-operational forms are normally the most useful, and they are the ones we will principally be handling in this book.

One reason to have defined specific infinite-dimensional matrix forms is to show how straightforwardly one can fully represent a practical matrix of an infinity of elements by a modest, finite quantity of information. Further reasons to have defined such forms will soon occur.

### 11.3.5 The rank-$r$ identity matrix

The *rank-r identity matrix* $I_r$ is the dimension-limited matrix for which

$$[I_r]_{ij} = \begin{cases} \delta_{ij} & \text{if } 1 \le i \le r \text{ and/or } 1 \le j \le r, \\ 0 & \text{otherwise,} \end{cases} \tag{11.28}$$

where either the "and" or the "or" can be regarded (it makes no difference). The effect of $I_r$ is that

$$\begin{aligned} I_m X &= X = X I_n, \\ I_m \mathbf{x} &= \mathbf{x}, \end{aligned} \tag{11.29}$$

where $X$ is an $m \times n$ matrix and $\mathbf{x}$, an $m \times 1$ vector. Examples of $I_r$ include

$$I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

(Remember that in the infinite-dimensional view, $I_3$, though a $3 \times 3$ matrix, is formally an $\infty \times \infty$ matrix with zeros in the unused cells. It has only the three ones and fits the $3 \times 3$ dimension-limited form of § 11.3.1. The areas of $I_3$ not shown are all zero, even along the main diagonal.)

The rank $r$ can be any nonnegative integer, even zero (though the rank-zero identity matrix $I_0$ is in fact the null matrix, normally just written 0). If alternate indexing limits are needed (for instance for a computer-indexed identity matrix whose indices run from 0 to $r - 1$), the notation $I_a^b$, where

$$[I_a^b]_{ij} \equiv \begin{cases} \delta_{ij} & \text{if } a \le i \le b \text{ and/or } a \le j \le b, \\ 0 & \text{otherwise,} \end{cases} \tag{11.30}$$

can be used; the rank in this case is $r = b - a + 1$, which is just the count of ones along the matrix's main diagonal.

The name "rank-$r$" implies that $I_r$ has a "rank" of $r$, and indeed it does. For the moment, however, we will discern the attribute of rank only in the rank-$r$ identity matrix itself. Section 12.5 defines *rank* for matrices more generally.

### 11.3.6 The truncation operator

The rank-$r$ identity matrix $I_r$ is also the *truncation operator.* Attacking from the left, as in $I_r A$, it retains the first through $r$th rows of $A$ but cancels other rows. Attacking from the right, as in $AI_r$, it retains the first through $r$th columns. Such truncation is useful symbolically to reduce an extended operator to dimension-limited form.

Whether a matrix $C$ has dimension-limited or extended-operational form (though not necessarily if it has some other form), if it has an $m \times n$ active region[17] and

$$m \leq r,$$
$$n \leq r,$$

then

$$I_r C = I_r C I_r = C I_r. \tag{11.31}$$

For such a matrix, (11.31) says at least two things:

- It is superfluous to truncate both rows and columns; it suffices to truncate one or the other.
- The rank-$r$ identity matrix $I_r$ commutes freely past $C$.

Evidently big identity matrices commute freely where small ones cannot (and the general identity matrix $I = I_\infty$ commutes freely past everything).

### 11.3.7 The elementary vector and the lone-element matrix

The *lone-element matrix* $E_{mn}$ is the matrix with a one in the $mn$th cell and zeros elsewhere:

$$[E_{mn}]_{ij} \equiv \delta_{im}\delta_{jn} = \begin{cases} 1 & \text{if } i = m \text{ and } j = n, \\ 0 & \text{otherwise.} \end{cases} \tag{11.32}$$

By this definition, $C = \sum_{i,j} c_{ij} E_{ij}$ for any matrix $C$. The vector analog of the lone-element matrix is the *elementary vector* $\mathbf{e}_m$, which has a one as the $m$th element:

$$[\mathbf{e}_m]_i \equiv \delta_{im} = \begin{cases} 1 & \text{if } i = m, \\ 0 & \text{otherwise.} \end{cases} \tag{11.33}$$

By this definition, $[I]_{*j} = \mathbf{e}_j$ and $[I]_{i*} = \mathbf{e}_i^T$.

[17] Refer to the definitions of *active region* in §§ 11.3.1 and 11.3.2. That a matrix has an $m \times n$ active region does not necessarily mean that it is all zero outside the $m \times n$ rectangle. (After all, if it were always all zero outside, then there would be little point in applying a truncation operator. There would be nothing there to truncate.)

### 11.3.8 Off-diagonal entries

It is interesting to observe and useful to note that if

$$[C_1]_{i*} = [C_2]_{i*} = \mathbf{e}_i^T,$$

then also

$$[C_1 C_2]_{i*} = \mathbf{e}_i^T; \tag{11.34}$$

and likewise that if

$$[C_1]_{*j} = [C_2]_{*j} = \mathbf{e}_j,$$

then also

$$[C_1 C_2]_{*j} = \mathbf{e}_j. \tag{11.35}$$

The product of matrices has off-diagonal entries in a row or column only if at least one of the factors itself has off-diagonal entries in that row or column. Or, less readably but more precisely, *the $i$th row or $j$th column of the product of matrices can depart from $\mathbf{e}_i^T$ or $\mathbf{e}_j$, respectively, only if the corresponding row or column of at least one of the factors so departs.* The reason is that in (11.34), $C_1$ acts as a row operator on $C_2$; that if $C_1$'s $i$th row is $\mathbf{e}_i^T$, then its action is merely to duplicate $C_2$'s $i$th row, which itself is just $\mathbf{e}_i^T$. Parallel logic naturally applies to (11.35).

## 11.4 The elementary operator

Section 11.1.3 has introduced the general row or column operator. Conventionally denoted $T$, the *elementary operator* is a simple extended row or column operator from sequences of which more complicated extended operators can be built. The elementary operator $T$ comes in three kinds.[18]

- The first is the *interchange elementary*

$$T_{[i\leftrightarrow j]} = I - (E_{ii} + E_{jj}) + (E_{ij} + E_{ji}), \tag{11.36}$$

  which by operating $T_{[i\leftrightarrow j]}A$ or $AT_{[i\leftrightarrow j]}$ respectively interchanges $A$'s $i$th row or column with its $j$th.[19]

[18] In § 11.3, the symbol $A$ specifically represented an extended operator, but here and generally the symbol represents any matrix.

[19] As a matter of definition, some authors [42] forbid $T_{[i\leftrightarrow i]}$ as an elementary operator, where $j = i$, since after all $T_{[i\leftrightarrow i]} = I$; which is to say that the operator doesn't actually do anything. There exist legitimate tactical reasons to forbid (as in § 11.6), but normally this book permits.

- The second is the *scaling elementary*

$$T_{\alpha[i]} = I + (\alpha - 1)E_{ii}, \quad \alpha \neq 0, \tag{11.37}$$

  which by operating $T_{\alpha[i]}A$ or $AT_{\alpha[i]}$ scales (multiplies) $A$'s $i$th row or column, respectively, by the factor $\alpha$.

- The third and last is the *addition elementary*

$$T_{\alpha[ij]} = I + \alpha E_{ij}, \quad i \neq j, \tag{11.38}$$

  which by operating $T_{\alpha[ij]}A$ adds to the $i$th row of $A$, $\alpha$ times the $j$th row; or which by operating $AT_{\alpha[ij]}$ adds to the $j$th column of $A$, $\alpha$ times the $i$th column.

Examples of the elementary operators include

$$T_{[1\leftrightarrow 2]} = \begin{bmatrix} \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \\ \cdots & 0 & 1 & 0 & 0 & 0 & \cdots \\ \cdots & 1 & 0 & 0 & 0 & 0 & \cdots \\ \cdots & 0 & 0 & 1 & 0 & 0 & \cdots \\ \cdots & 0 & 0 & 0 & 1 & 0 & \cdots \\ \cdots & 0 & 0 & 0 & 0 & 1 & \cdots \\ & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix},$$

$$T_{5[4]} = \begin{bmatrix} \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \\ \cdots & 1 & 0 & 0 & 0 & 0 & \cdots \\ \cdots & 0 & 1 & 0 & 0 & 0 & \cdots \\ \cdots & 0 & 0 & 1 & 0 & 0 & \cdots \\ \cdots & 0 & 0 & 0 & 5 & 0 & \cdots \\ \cdots & 0 & 0 & 0 & 0 & 1 & \cdots \\ & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix},$$

$$T_{5[21]} = \begin{bmatrix} \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \\ \cdots & 1 & 0 & 0 & 0 & 0 & \cdots \\ \cdots & 5 & 1 & 0 & 0 & 0 & \cdots \\ \cdots & 0 & 0 & 1 & 0 & 0 & \cdots \\ \cdots & 0 & 0 & 0 & 1 & 0 & \cdots \\ \cdots & 0 & 0 & 0 & 0 & 1 & \cdots \\ & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}.$$

It is good to define a concept aesthetically. One should usually do so when one can; and indeed in this case one might reasonably promote either definition on aesthetic grounds. However, an applied mathematician ought not to let a mere definition entangle him. What matters is the underlying concept. Where the definition does not serve the concept well, the applied mathematician considers whether it were not worth the effort to adapt the definition accordingly.

Note that none of these, and in fact no elementary operator of any kind, differs from $I$ in more than four elements.

### 11.4.1 Properties

Significantly, elementary operators as defined above are always invertible (which is to say, reversible in effect), with

$$\begin{aligned} T_{[i\leftrightarrow j]}^{-1} &= T_{[j\leftrightarrow i]} = T_{[i\leftrightarrow j]}, \\ T_{\alpha[i]}^{-1} &= T_{(1/\alpha)[i]}, \\ T_{\alpha[ij]}^{-1} &= T_{-\alpha[ij]}, \end{aligned} \tag{11.39}$$

being themselves elementary operators such that

$$T^{-1}T = I = TT^{-1} \tag{11.40}$$

in each case.[20] This means that any sequence of elementaries $\prod_k T_k$ can safely be undone by the reverse sequence $\coprod_k T_k^{-1}$:

$$\coprod_k T_k^{-1} \prod_k T_k = I = \prod_k T_k \coprod_k T_k^{-1}. \tag{11.41}$$

The rank-$r$ identity matrix $I_r$ is no elementary operator,[21] nor is the lone-element matrix $E_{mn}$; but the general identity matrix $I$ is indeed an elementary operator. The last can be considered a distinct, fourth kind of elementary operator if desired; but it is probably easier just to regard it as an elementary of any of the first three kinds, since $I = T_{[i\leftrightarrow i]} = T_{1[i]} = T_{0[ij]}$.

From (11.31), we have that

$$I_r T = I_r T I_r = T I_r \quad \text{if } 1 \le i \le r \text{ and } 1 \le j \le r \tag{11.42}$$

for any elementary operator $T$ which operates within the given bounds. Equation (11.42) lets an identity matrix with sufficiently high rank pass through a sequence of elementaries as needed.

In general, the transpose of an elementary row operator is the corresponding elementary column operator. Curiously, the interchange elementary is its own transpose and adjoint:

$$T_{[i\leftrightarrow j]}^{*} = T_{[i\leftrightarrow j]} = T_{[i\leftrightarrow j]}^{T}. \tag{11.43}$$

[20] The addition elementary $T_{\alpha[ii]}$ and the scaling elementary $T_{0[i]}$ are forbidden precisely because they are not generally invertible.

[21] If the statement seems to contradict statements of some other books, it is only a matter of definition. This book finds it convenient to define the elementary operator in infinite-dimensional, extended-operational form. The other books are not wrong; their underlying definitions just differ slightly.

### 11.4.2 Commutation and sorting

Elementary operators often occur in long chains like

$$A = T_{-4[32]}T_{[2\leftrightarrow 3]}T_{(1/5)[3]}T_{(1/2)[31]}T_{5[21]}T_{[1\leftrightarrow 3]},$$

with several elementaries of all kinds intermixed. Some applications demand that the elementaries be sorted and grouped by kind, as

$$A = \left(T_{[2\leftrightarrow 3]}T_{[1\leftrightarrow 3]}\right)\left(T_{-4[21]}T_{(1/0\text{xA})[13]}T_{5[23]}\right)\left(T_{(1/5)[1]}\right)$$

or as

$$A = \left(T_{-4[32]}T_{(1/0\text{xA})[21]}T_{5[31]}\right)\left(T_{(1/5)[2]}\right)\left(T_{[2\leftrightarrow 3]}T_{[1\leftrightarrow 3]}\right),$$

among other possible orderings. Though you probably cannot tell just by looking, the three products above are different orderings of the same elementary chain; they yield the same $A$ and thus represent exactly the same matrix operation. Interesting is that the act of reordering the elementaries has altered some of them into other elementaries of the same kind, but has changed the kind of none of them.

One sorts a chain of elementary operators by repeatedly exchanging adjacent pairs. This of course supposes that one can exchange adjacent pairs, which seems impossible since matrix multiplication is not commutative: $A_1A_2 \neq A_2A_1$. However, at the moment we are dealing in elementary operators only; and for most pairs $T_1$ and $T_2$ of elementary operators, though indeed $T_1T_2 \neq T_2T_1$, it so happens that there exists either a $T_1'$ such that $T_1T_2 = T_2T_1'$ or a $T_2'$ such that $T_1T_2 = T_2'T_1$, where $T_1'$ and $T_2'$ are elementaries of the same kinds respectively as $T_1$ and $T_2$. The attempt sometimes fails when both $T_1$ and $T_2$ are addition elementaries, but all other pairs commute in this way. Significantly, *elementaries of different kinds always commute.* And, though commutation can alter one (never both) of the two elementaries, it changes the kind of neither.

Many qualitatively distinct pairs of elementaries exist; we will list these exhaustively in a moment. First, however, we should like to observe a natural hierarchy among the three kinds of elementary: (i) interchange; (ii) scaling; (iii) addition.

- The interchange elementary is the strongest. Itself subject to alteration only by another interchange elementary, it can alter any elementary by commuting past. When an interchange elementary commutes past another elementary of any kind, what it alters are the other elementary's indices $i$ and/or $j$ (or $m$ and/or $n$, or whatever symbols

happen to represent the indices in question). When two interchange elementaries commute past one another, only one of the two is altered. (Which one? Either. The mathematician chooses.) Refer to Table 11.1.

- Next in strength is the scaling elementary. Only an interchange elementary can alter it, and it in turn can alter only an addition elementary. Scaling elementaries do not alter one another during commutation. When a scaling elementary commutes past an addition elementary, what it alters is the latter's scale $\alpha$ (or $\beta$, or whatever symbol happens to represent the scale in question). Refer to Table 11.2.

- The addition elementary, last and weakest, is subject to alteration by either of the other two, itself having no power to alter any elementary during commutation. A pair of addition elementaries are the only pair that can altogether fail to commute—they fail when the row index of one equals the column index of the other—but when they do commute, neither alters the other. Refer to Table 11.3.

Tables 11.1, 11.2 and 11.3 list all possible pairs of elementary operators, as the reader can check. The only pairs that fail to commute are the last three of Table 11.3.

## 11.5 Inversion and similarity (introduction)

If Tables 11.1, 11.2 and 11.3 exhaustively describe the commutation of one elementary past another elementary, then what can one write of the commutation of an elementary past the general matrix $A$? With some matrix algebra,

$$\begin{aligned} TA &= (TA)(I) = (TA)(T^{-1}T), \\ AT &= (I)(AT) = (TT^{-1})(AT), \end{aligned}$$

one can write that

$$\begin{aligned} TA &= [TAT^{-1}]T, \\ AT &= T[T^{-1}AT], \end{aligned} \tag{11.44}$$

where $T^{-1}$ is given by (11.39). An elementary commuting rightward changes $A$ to $TAT^{-1}$; commuting leftward, to $T^{-1}AT$.

Table 11.1: Inverting, commuting, combining and expanding elementary operators: interchange. In the table, $i \neq j \neq m \neq n$; no two indices are the same. Notice that the effect an interchange elementary $T_{[m\leftrightarrow n]}$ has in passing any other elementary, even another interchange elementary, is simply to replace $m$ by $n$ and $n$ by $m$ among the indices of the other elementary.

$$
\begin{aligned}
T_{[m\leftrightarrow n]} &= T_{[n\leftrightarrow m]} \\
T_{[m\leftrightarrow m]} &= I \\
IT_{[m\leftrightarrow n]} &= T_{[m\leftrightarrow n]}I \\
T_{[m\leftrightarrow n]}T_{[m\leftrightarrow n]} &= T_{[m\leftrightarrow n]}T_{[n\leftrightarrow m]} = T_{[n\leftrightarrow m]}T_{[m\leftrightarrow n]} = I \\
T_{[m\leftrightarrow n]}T_{[i\leftrightarrow n]} &= T_{[i\leftrightarrow n]}T_{[m\leftrightarrow i]} = T_{[i\leftrightarrow m]}T_{[m\leftrightarrow n]} \\
&= \left(T_{[i\leftrightarrow n]}T_{[m\leftrightarrow n]}\right)^2 \\
T_{[m\leftrightarrow n]}T_{[i\leftrightarrow j]} &= T_{[i\leftrightarrow j]}T_{[m\leftrightarrow n]} \\
T_{[m\leftrightarrow n]}T_{\alpha[m]} &= T_{\alpha[n]}T_{[m\leftrightarrow n]} \\
T_{[m\leftrightarrow n]}T_{\alpha[i]} &= T_{\alpha[i]}T_{[m\leftrightarrow n]} \\
T_{[m\leftrightarrow n]}T_{\alpha[ij]} &= T_{\alpha[ij]}T_{[m\leftrightarrow n]} \\
T_{[m\leftrightarrow n]}T_{\alpha[in]} &= T_{\alpha[im]}T_{[m\leftrightarrow n]} \\
T_{[m\leftrightarrow n]}T_{\alpha[mj]} &= T_{\alpha[nj]}T_{[m\leftrightarrow n]} \\
T_{[m\leftrightarrow n]}T_{\alpha[mn]} &= T_{\alpha[nm]}T_{[m\leftrightarrow n]}
\end{aligned}
$$

Table 11.2: Inverting, commuting, combining and expanding elementary operators: scaling. In the table, $i \neq j \neq m \neq n$; no two indices are the same.

$$\begin{aligned}
T_{1[m]} &= I \\
IT_{\beta[m]} &= T_{\beta[m]}I \\
T_{(1/\beta)[m]}T_{\beta[m]} &= I \\
T_{\beta[m]}T_{\alpha[m]} &= T_{\alpha[m]}T_{\beta[m]} = T_{\alpha\beta[m]} \\
T_{\beta[m]}T_{\alpha[i]} &= T_{\alpha[i]}T_{\beta[m]} \\
T_{\beta[m]}T_{\alpha[ij]} &= T_{\alpha[ij]}T_{\beta[m]} \\
T_{\beta[m]}T_{\alpha\beta[im]} &= T_{\alpha[im]}T_{\beta[m]} \\
T_{\beta[m]}T_{\alpha[mj]} &= T_{\alpha\beta[mj]}T_{\beta[m]}
\end{aligned}$$

Table 11.3: Inverting, commuting, combining and expanding elementary operators: addition. In the table, $i \neq j \neq m \neq n$; no two indices are the same. The last three lines give pairs of addition elementaries that do not commute.

$$\begin{aligned}
T_{0[ij]} &= I \\
IT_{\alpha[ij]} &= T_{\alpha[ij]}I \\
T_{-\alpha[ij]}T_{\alpha[ij]} &= I \\
T_{\beta[ij]}T_{\alpha[ij]} &= T_{\alpha[ij]}T_{\beta[ij]} = T_{(\alpha+\beta)[ij]} \\
T_{\beta[mj]}T_{\alpha[ij]} &= T_{\alpha[ij]}T_{\beta[mj]} \\
T_{\beta[in]}T_{\alpha[ij]} &= T_{\alpha[ij]}T_{\beta[in]} \\
T_{\beta[mn]}T_{\alpha[ij]} &= T_{\alpha[ij]}T_{\beta[mn]} \\
T_{\beta[mi]}T_{\alpha[ij]} &= T_{\alpha[ij]}T_{\alpha\beta[mj]}T_{\beta[mi]} \\
T_{\beta[jn]}T_{\alpha[ij]} &= T_{\alpha[ij]}T_{-\alpha\beta[in]}T_{\beta[jn]} \\
T_{\beta[ji]}T_{\alpha[ij]} &\neq T_{\alpha[ij]}T_{\beta[ji]}
\end{aligned}$$

First encountered in § 11.4, the notation $T^{-1}$ means the *inverse* of the elementary operator $T$, such that

$$T^{-1}T = I = TT^{-1}.$$

Matrix inversion is not for elementary operators only, though. Many more general matrices $C$ also have inverses such that

$$C^{-1}C = I = CC^{-1}. \tag{11.45}$$

(Do all matrices have such inverses? No. For example, the null matrix has no such inverse.) The broad question of how to invert a general matrix $C$, we leave for Chs. 12 and 13 to address. For the moment however we should like to observe three simple rules involving matrix inversion.

First, nothing in the logic leading to (11.44) actually requires the matrix $T$ there to be an elementary operator. Any matrix $C$ for which $C^{-1}$ is known can fill the role. Hence,

$$\begin{aligned} CA &= [CAC^{-1}]C, \\ AC &= C[C^{-1}AC]. \end{aligned} \tag{11.46}$$

The transformation $CAC^{-1}$ or $C^{-1}AC$ is called a *similarity transformation.* Sections 12.2 and 14.9 speak further of this.

Second,

$$\begin{aligned} \left(C^T\right)^{-1} &= C^{-T} = \left(C^{-1}\right)^T, \\ \left(C^*\right)^{-1} &= C^{-*} = \left(C^{-1}\right)^*, \end{aligned} \tag{11.47}$$

where $C^{-*}$ is condensed notation for conjugate transposition and inversion in either order and $C^{-T}$ is of like style. Equation (11.47) is a consequence of (11.14), since for conjugate transposition

$$\left(C^{-1}\right)^* C^* = \left[CC^{-1}\right]^* = [I]^* = I = [I]^* = \left[C^{-1}C\right]^* = C^*\left(C^{-1}\right)^*$$

and similarly for nonconjugate transposition.

Third,

$$\left(\prod_k C_k\right)^{-1} = \coprod_k C_k^{-1}. \tag{11.48}$$

This rule emerges upon repeated application of (11.45), which yields

$$\coprod_k C_k^{-1} \prod_k C_k = I = \prod_k C_k \coprod_k C_k^{-1}.$$

Table 11.4: Matrix inversion properties. (The properties work equally for $C^{-1(r)}$ as for $C^{-1}$ if $A$ honors an $r \times r$ active region. The full notation $C^{-1(r)}$ for the rank-$r$ inverse incidentally is not standard, usually is not needed, and normally is not used.)

$$\begin{aligned}
C^{-1}C &= I = CC^{-1} \\
C^{-1(r)}C &= I_r = CC^{-1(r)} \\
\left(C^T\right)^{-1} &= C^{-T} = \left(C^{-1}\right)^T \\
\left(C^*\right)^{-1} &= C^{-*} = \left(C^{-1}\right)^*
\end{aligned}$$

$$\begin{aligned}
CA &= [CAC^{-1}]C \\
AC &= C[C^{-1}AC] \\
\left(\prod_k C_k\right)^{-1} &= \coprod_k C_k^{-1}
\end{aligned}$$

A more limited form of the inverse exists than the infinite-dimensional form of (11.45). This is the rank-$r$ inverse, a matrix $C^{-1(r)}$ such that

$$C^{-1(r)}C = I_r = CC^{-1(r)}. \tag{11.49}$$

The full notation $C^{-1(r)}$ is not standard and usually is not needed, since the context usually implies the rank. When so, one can abbreviate the notation to $C^{-1}$. In either notation, (11.47) and (11.48) apply equally for the rank-$r$ inverse as for the infinite-dimensional inverse. Because of (11.31), eqn. (11.46) too applies for the rank-$r$ inverse if $A$'s active region is limited to $r \times r$. (Section 13.2 uses the rank-$r$ inverse to solve an exactly determined linear system. This is a famous way to use the inverse, with which many or most readers will already be familiar; but before using it so in Ch. 13, we shall first learn how to compute it reliably in Ch. 12.)

Table 11.4 summarizes.

## 11.6 Parity

Consider the sequence of integers or other objects $1, 2, 3, \ldots, n$. By successively interchanging pairs of the objects (any pairs, not just adjacent pairs), one can achieve any desired permutation (§ 4.2.1). For example, beginning

with $1, 2, 3, 4, 5$, one can achieve the permutation $3, 5, 1, 4, 2$ by interchanging first the 1 and 3, then the 2 and 5.

Now contemplate all possible pairs:

$$\begin{array}{ccccc} (1,2) & (1,3) & (1,4) & \cdots & (1,n); \\ & (2,3) & (2,4) & \cdots & (2,n); \\ & & (3,4) & \cdots & (3,n); \\ & & & \ddots & \vdots \\ & & & & (n-1,n). \end{array}$$

In a given permutation (like $3, 5, 1, 4, 2$), some pairs will appear in correct order with respect to one another, while others will appear in incorrect order. (In $3, 5, 1, 4, 2$, the pair $[1, 2]$ appears in correct order in that the larger 2 stands to the right of the smaller 1; but the pair $[1, 3]$ appears in incorrect order in that the larger 3 stands to the *left* of the smaller 1.) If $p$ is the number of pairs which appear in incorrect order (in the example, $p = 6$), and if $p$ is even, then we say that the permutation has *even* or *positive parity;* if odd, then *odd* or *negative parity.*[22]

Now consider: every interchange of adjacent elements must either increment or decrement $p$ by one, reversing parity. Why? Well, think about it. If two elements are adjacent and their order is correct, then interchanging falsifies the order, but only of that pair (no other element interposes, thus the interchange affects the ordering of no other pair). Complementarily, if the order is incorrect, then interchanging rectifies the order. Either way, an adjacent interchange alters $p$ by exactly $\pm 1$, thus reversing parity.

What about nonadjacent elements? Does interchanging a pair of these reverse parity, too? To answer the question, let $u$ and $v$ represent the two elements interchanged, with $a_1, a_2, \ldots, a_m$ the elements lying between. Before the interchange:

$$\ldots, u, a_1, a_2, \ldots, a_{m-1}, a_m, v, \ldots$$

After the interchange:

$$\ldots, v, a_1, a_2, \ldots, a_{m-1}, a_m, u, \ldots$$

The interchange reverses with respect to one another just the pairs

$$\begin{array}{cccccc} (u,a_1) & (u,a_2) & \cdots & (u,a_{m-1}) & (u,a_m) \\ (a_1,v) & (a_2,v) & \cdots & (a_{m-1},v) & (a_m,v) \\ (u,v) & & & & \end{array}$$

[22] For readers who learned arithmetic in another language than English, the *even* integers are $\ldots, -4, -2, 0, 2, 4, 6, \ldots$; the *odd* integers are $\ldots, -3, -1, 1, 3, 5, 7, \ldots$.

The number of pairs reversed is odd. Since each reversal alters $p$ by $\pm 1$, the net change in $p$ apparently also is odd, reversing parity. It seems that regardless of how distant the pair, *interchanging any pair of elements reverses the permutation's parity.*

The sole exception arises when an element is interchanged with itself. This does not change parity, but it does not change anything else, either, so in parity calculations we ignore it.[23] All other interchanges reverse parity.

We discuss parity in this, a chapter on matrices, because parity concerns the elementary interchange operator of § 11.4. The rows or columns of a matrix can be considered elements in a sequence. If so, then the interchange operator $T_{[i\leftrightarrow j]}$, $i \neq j$, acts precisely in the manner described, interchanging rows or columns and thus reversing parity. It follows that if $i_k \neq j_k$ and $q$ is odd, then $\prod_{k=1}^{q} T_{[i_k\leftrightarrow j_k]} \neq I$. However, it is possible that $\prod_{k=1}^{q} T_{[i_k\leftrightarrow j_k]} = I$ if $q$ is even. In any event, even $q$ implies even $p$, which means even (positive) parity; odd $q$ implies odd $p$, which means odd (negative) parity.

We shall have more to say about parity in §§ 11.7.1 and 14.1.

## 11.7 The quasielementary operator

Multiplying sequences of the elementary operators of § 11.4, one can form much more complicated operators, which per (11.41) are always invertible. Such complicated operators are not trivial to analyze, however, so one finds it convenient to define an intermediate class of operators, called in this book the *quasielementary operators,* more complicated than elementary operators but less so than arbitrary matrices.

A quasielementary operator is composed of elementaries only of a single kind. There are thus three kinds of quasielementary—interchange, scaling and addition—to match the three kinds of elementary. With respect to interchange and scaling, any sequences of elementaries of the respective kinds are allowed. With respect to addition, there are some extra rules, explained in § 11.7.3.

The three subsections which follow respectively introduce the three kinds of quasielementary operator.

[23] This is why some authors forbid self-interchanges, as explained in footnote 19.

### 11.7.1 The interchange quasielementary or general interchange operator

Any product $P$ of zero or more interchange elementaries,

$$P = \prod_k T_{[i_k \leftrightarrow j_k]}, \tag{11.50}$$

constitutes an *interchange quasielementary*, *permutation matrix*, *permutor* or *general interchange operator*.[24] An example is

$$P = T_{[2\leftrightarrow 5]} T_{[1\leftrightarrow 3]} = \begin{bmatrix} \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \\ \cdots & 0 & 0 & 1 & 0 & 0 & \cdots \\ \cdots & 0 & 0 & 0 & 0 & 1 & \cdots \\ \cdots & 1 & 0 & 0 & 0 & 0 & \cdots \\ \cdots & 0 & 0 & 0 & 1 & 0 & \cdots \\ \cdots & 0 & 1 & 0 & 0 & 0 & \cdots \\ & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}.$$

This operator resembles $I$ in that it has a single one in each row and in each column, but the ones here do not necessarily run along the main diagonal. The effect of the operator is to shuffle the rows or columns of the matrix it operates on, without altering any of the rows or columns it shuffles.

By (11.41), (11.39), (11.43) and (11.15), the inverse of the general interchange operator is

$$\begin{aligned} P^{-1} &= \left( \prod_k T_{[i_k \leftrightarrow j_k]} \right)^{-1} = \coprod_k T^{-1}_{[i_k \leftrightarrow j_k]} \\ &= \coprod_k T_{[i_k \leftrightarrow j_k]} \\ &= \coprod_k T^*_{[i_k \leftrightarrow j_k]} = \left( \prod_k T_{[i_k \leftrightarrow j_k]} \right)^* \\ &= P^* = P^T \end{aligned} \tag{11.51}$$

(where $P^* = P^T$ because $P$ has only real elements). The inverse, transpose and adjoint of the general interchange operator are thus the same:

$$P^T P = P^* P = I = P P^* = P P^T. \tag{11.52}$$

[24] The letter $P$ here recalls the verb "to permute."

A significant attribute of the general interchange operator $P$ is its parity: positive or even parity if the number of interchange elementaries $T_{[i_k \leftrightarrow j_k]}$ which compose it is even; negative or odd parity if the number is odd. This works precisely as described in § 11.6. For the purpose of parity determination, only interchange elementaries $T_{[i_k \leftrightarrow j_k]}$ for which $i_k \neq j_k$ are counted; any $T_{[i \leftrightarrow i]} = I$ noninterchanges are ignored. Thus the example's $P$ above has even parity (two interchanges), as does $I$ itself (zero interchanges), but $T_{[i \leftrightarrow j]}$ alone (one interchange) has odd parity if $i \neq j$. As we shall see in § 14.1, the positive (even) and negative (odd) parities sometimes lend actual positive and negative senses to the matrices they describe. The parity of the general interchange operator $P$ concerns us for this reason.

Parity, incidentally, is a property of the matrix $P$ itself, not just of the operation $P$ represents. No interchange quasielementary $P$ has positive parity as a row operator but negative as a column operator. The reason is that, regardless of whether one ultimately means to use $P$ as a row or column operator, the matrix is nonetheless composable as a definite sequence of interchange elementaries. It is the number of interchanges, not the use, which determines $P$'s parity.

### 11.7.2 The scaling quasielementary or general scaling operator

Like the interchange quasielementary $P$ of § 11.7.1, the *scaling quasielementary, diagonal matrix* or *general scaling operator* $D$ consists of a product of zero or more elementary operators, in this case elementary scaling operators:[25]

$$D = \prod_{i=-\infty}^{\infty} T_{\alpha_i[i]} = \coprod_{i=-\infty}^{\infty} T_{\alpha_i[i]} = \sum_{i=-\infty}^{\infty} \alpha_i E_{ii} = \begin{bmatrix} \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \\ \cdots & * & 0 & 0 & 0 & 0 & \cdots \\ \cdots & 0 & * & 0 & 0 & 0 & \cdots \\ \cdots & 0 & 0 & * & 0 & 0 & \cdots \\ \cdots & 0 & 0 & 0 & * & 0 & \cdots \\ \cdots & 0 & 0 & 0 & 0 & * & \cdots \\ & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix} \tag{11.53}$$

(of course it might be that $\alpha_i = 1$, hence that $T_{\alpha_i[i]} = I$, for some, most or even all $i$; however, $\alpha_i = 0$ is forbidden by the definition of the scaling

[25] The letter $D$ here recalls the adjective "diagonal."

elementary). An example is

$$D = T_{-5[4]}T_{4[2]}T_{7[1]} = \begin{bmatrix} \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \\ \cdots & 7 & 0 & 0 & 0 & 0 & \cdots \\ \cdots & 0 & 4 & 0 & 0 & 0 & \cdots \\ \cdots & 0 & 0 & 1 & 0 & 0 & \cdots \\ \cdots & 0 & 0 & 0 & -5 & 0 & \cdots \\ \cdots & 0 & 0 & 0 & 0 & 1 & \cdots \\ & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}.$$

This operator resembles $I$ in that all its entries run down the main diagonal; but these entries, though never zeros, are not necessarily ones, either. They are nonzero scaling factors. The effect of the operator is to scale the rows or columns of the matrix it operates on.

The general scaling operator is a particularly simple matrix. Its inverse is evidently

$$D^{-1} = \coprod_{i=-\infty}^{\infty} T_{(1/\alpha_i)[i]} = \prod_{i=-\infty}^{\infty} T_{(1/\alpha_i)[i]} = \sum_{i=-\infty}^{\infty} \frac{E_{ii}}{\alpha_i}, \tag{11.54}$$

where each element down the main diagonal is individually inverted.

A superset of the general scaling operator is the *diagonal matrix,* defined less restrictively that $[A]_{ij} = 0$ for $i \neq j$, where zeros along the main diagonal are allowed. The conventional notation

$$[\text{diag}\{\mathbf{x}\}]_{ij} \equiv \delta_{ij}x_i = \delta_{ij}x_j, \tag{11.55}$$

$$\text{diag}\{\mathbf{x}\} = \begin{bmatrix} \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \\ \cdots & x_1 & 0 & 0 & 0 & 0 & \cdots \\ \cdots & 0 & x_2 & 0 & 0 & 0 & \cdots \\ \cdots & 0 & 0 & x_3 & 0 & 0 & \cdots \\ \cdots & 0 & 0 & 0 & x_4 & 0 & \cdots \\ \cdots & 0 & 0 & 0 & 0 & x_5 & \cdots \\ & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix},$$

converts a vector $\mathbf{x}$ into a diagonal matrix. The diagonal matrix in general is not invertible and is no quasielementary operator, but is sometimes useful nevertheless.

### 11.7.3 Addition quasielementaries

Any product of interchange elementaries (§ 11.7.1), any product of scaling elementaries (§ 11.7.2), qualifies as a quasielementary operator. Not so, any

product of addition elementaries. To qualify as a quasielementary, a product of elementary addition operators must meet some additional restrictions.

Four types of addition quasielementary are defined:[26]

- the *downward multitarget row addition operator,*[27]

$$\begin{aligned} L_{[j]} &= \prod_{i=j+1}^{\infty} T_{\alpha_{ij}[ij]} = \coprod_{i=j+1}^{\infty} T_{\alpha_{ij}[ij]} \\ &= I + \sum_{i=j+1}^{\infty} \alpha_{ij} E_{ij} \\ &= \begin{bmatrix} \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \\ \cdots & 1 & 0 & 0 & 0 & 0 & \cdots \\ \cdots & 0 & 1 & 0 & 0 & 0 & \cdots \\ \cdots & 0 & 0 & 1 & 0 & 0 & \cdots \\ \cdots & 0 & 0 & * & 1 & 0 & \cdots \\ \cdots & 0 & 0 & * & 0 & 1 & \cdots \\ & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}, \end{aligned} \tag{11.56}$$

whose inverse is

$$\begin{aligned} L_{[j]}^{-1} &= \coprod_{i=j+1}^{\infty} T_{-\alpha_{ij}[ij]} = \prod_{i=j+1}^{\infty} T_{-\alpha_{ij}[ij]} \\ &= I - \sum_{i=j+1}^{\infty} \alpha_{ij} E_{ij} = 2I - L_{[j]}; \end{aligned} \tag{11.57}$$

[26] In this subsection the explanations are briefer than in the last two, but the pattern is similar. The reader can fill in the details.

[27] The letter $L$ here recalls the adjective "lower."

- the *upward multitarget row addition operator,*[28]

$$\begin{aligned} U_{[j]} &= \coprod_{i=-\infty}^{j-1} T_{\alpha_{ij}[ij]} = \prod_{i=-\infty}^{j-1} T_{\alpha_{ij}[ij]} \\ &= I + \sum_{i=-\infty}^{j-1} \alpha_{ij} E_{ij} \\ &= \begin{bmatrix} \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \\ \cdots & 1 & 0 & * & 0 & 0 & \cdots \\ \cdots & 0 & 1 & * & 0 & 0 & \cdots \\ \cdots & 0 & 0 & 1 & 0 & 0 & \cdots \\ \cdots & 0 & 0 & 0 & 1 & 0 & \cdots \\ \cdots & 0 & 0 & 0 & 0 & 1 & \cdots \\ & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}, \end{aligned} \tag{11.58}$$

  whose inverse is

$$\begin{aligned} U_{[j]}^{-1} &= \prod_{i=-\infty}^{j-1} T_{-\alpha_{ij}[ij]} = \coprod_{i=-\infty}^{j-1} T_{-\alpha_{ij}[ij]} \\ &= I - \sum_{i=-\infty}^{j-1} \alpha_{ij} E_{ij} = 2I - U_{[j]}; \end{aligned} \tag{11.59}$$

- the *rightward multitarget column addition operator,* which is the transpose $L_{[j]}^T$ of the downward operator; and

- the *leftward multitarget column addition operator,* which is the transpose $U_{[j]}^T$ of the upward operator.

## 11.8 The unit triangular matrix

Yet more complicated than the quasielementary of § 11.7 is the *unit triangular matrix,* with which we draw this necessary but tedious chapter toward

[28] The letter $U$ here recalls the adjective "upper."

a long close:

$$L = I + \sum_{i=-\infty}^{\infty} \sum_{j=-\infty}^{i-1} \alpha_{ij} E_{ij} = I + \sum_{j=-\infty}^{\infty} \sum_{i=j+1}^{\infty} \alpha_{ij} E_{ij} \tag{11.60}$$

$$= \begin{bmatrix} \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \\ \cdots & 1 & 0 & 0 & 0 & 0 & \cdots \\ \cdots & * & 1 & 0 & 0 & 0 & \cdots \\ \cdots & * & * & 1 & 0 & 0 & \cdots \\ \cdots & * & * & * & 1 & 0 & \cdots \\ \cdots & * & * & * & * & 1 & \cdots \\ & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix};$$

$$U = I + \sum_{i=-\infty}^{\infty} \sum_{j=i+1}^{\infty} \alpha_{ij} E_{ij} = I + \sum_{j=-\infty}^{\infty} \sum_{i=-\infty}^{j-1} \alpha_{ij} E_{ij} \tag{11.61}$$

$$= \begin{bmatrix} \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \\ \cdots & 1 & * & * & * & * & \cdots \\ \cdots & 0 & 1 & * & * & * & \cdots \\ \cdots & 0 & 0 & 1 & * & * & \cdots \\ \cdots & 0 & 0 & 0 & 1 & * & \cdots \\ \cdots & 0 & 0 & 0 & 0 & 1 & \cdots \\ & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}.$$

The former is a *unit lower triangular matrix;* the latter, a *unit upper triangular matrix.* The unit triangular matrix is a generalized addition quasielementary, which adds not only to multiple targets but also from multiple sources—but in one direction only: downward or leftward for $L$ or $U^T$ (or $U^*$); upward or rightward for $U$ or $L^T$ (or $L^*$).

The general *triangular matrix* $L_S$ or $U_S$, which by definition can have any values along its main diagonal, is sometimes of interest, as in the Schur decomposition of § 14.10.[29] The *strictly triangular matrix* $L - I$ or $U - I$ is likewise sometimes of interest, as in Table 11.5.[30] However, such matrices cannot in general be expressed as products of elementary operators and this section does not treat them.

This section presents and derives the basic properties of the unit triangular matrix.

[29] The subscript $S$ here stands for Schur. Other books typically use the symbols $L$ and $U$ for the general triangular matrix of Schur, but this book distinguishes by the subscript.

[30] [66, "Schur decomposition," 00:32, 30 Aug. 2007]

### 11.8.1 Construction

To make a unit triangular matrix is straightforward:

$$\begin{aligned} L &= \coprod_{j=-\infty}^{\infty} L_{[j]}; \\ U &= \prod_{j=-\infty}^{\infty} U_{[j]}. \end{aligned} \tag{11.62}$$

So long as the multiplication is done in the order indicated,[31] then conveniently,

$$\begin{aligned} \Big[L\Big]_{ij} &= \Big[L_{[j]}\Big]_{ij}, \\ \Big[U\Big]_{ij} &= \Big[U_{[j]}\Big]_{ij}, \end{aligned} \tag{11.63}$$

which is to say that the entries of $L$ and $U$ are respectively nothing more than the relevant entries of the several $L_{[j]}$ and $U_{[j]}$. Equation (11.63) enables one to use (11.62) immediately and directly, without calculation, to build any unit triangular matrix desired.

The correctness of (11.63) is most easily seen if the several $L_{[j]}$ and $U_{[j]}$ are regarded as column operators acting sequentially on $I$:

$$\begin{aligned} L &= (I)\left(\coprod_{j=-\infty}^{\infty} L_{[j]}\right); \\ U &= (I)\left(\prod_{j=-\infty}^{\infty} U_{[j]}\right). \end{aligned}$$

The reader can construct an inductive proof symbolically on this basis without too much difficulty if desired, but just thinking about how $L_{[j]}$ adds columns leftward and $U_{[j]}$, rightward, then considering the order in which the several $L_{[j]}$ and $U_{[j]}$ act, (11.63) follows at once.

[31] Recall again from § 2.3 that $\prod_k A_k = \cdots A_3 A_2 A_1$, whereas $\coprod_k A_k = A_1 A_2 A_3 \cdots$. This means that $(\prod_k A_k)(C)$ applies first $A_1$, then $A_2$, $A_3$ and so on, as row operators to $C$; whereas $(C)(\coprod_k A_k)$ applies first $A_1$, then $A_2$, $A_3$ and so on, as column operators to $C$. The symbols $\prod$ and $\coprod$ as this book uses them can thus be thought of respectively as row and column sequencers.

### 11.8.2 The product of like unit triangular matrices

The product of like unit triangular matrices,

$$\begin{aligned} L_1 L_2 &= L, \\ U_1 U_2 &= U, \end{aligned} \tag{11.64}$$

is another unit triangular matrix of the same type. The proof for unit lower and unit upper triangular matrices is the same. In the unit lower triangular case, one starts from a form of the definition of a unit lower triangular matrix:

$$[L_1]_{ij} \text{ or } [L_2]_{ij} = \begin{cases} 0 & \text{if } i < j, \\ 1 & \text{if } i = j. \end{cases}$$

Then,

$$[L_1 L_2]_{ij} = \sum_{m=-\infty}^{\infty} [L_1]_{im} [L_2]_{mj}.$$

But as we have just observed, $[L_1]_{im}$ is null when $i < m$, and $[L_2]_{mj}$ is null when $m < j$. Therefore,

$$[L_1 L_2]_{ij} = \begin{cases} 0 & \text{if } i < j, \\ \sum_{m=j}^{i} [L_1]_{im} [L_2]_{mj} & \text{if } i \geq j. \end{cases}$$

Inasmuch as this is true, nothing prevents us from weakening the statement to read

$$[L_1 L_2]_{ij} = \begin{cases} 0 & \text{if } i < j, \\ \sum_{m=j}^{i} [L_1]_{im} [L_2]_{mj} & \text{if } i = j. \end{cases}$$

But this is just

$$[L_1 L_2]_{ij} = \begin{cases} 0 & \text{if } i < j, \\ [L_1]_{ij} [L_2]_{ij} = [L_1]_{ii} [L_2]_{ii} = (1)(1) = 1 & \text{if } i = j, \end{cases}$$

which again is the very definition of a unit lower triangular matrix. Hence (11.64).

### 11.8.3 Inversion

Inasmuch as any unit triangular matrix can be constructed from addition quasielementaries by (11.62), inasmuch as (11.63) supplies the specific quasielementaries, and inasmuch as (11.57) or (11.59) gives the inverse of each

such quasielementary, one can always invert a unit triangular matrix easily by

$$\begin{aligned} L^{-1} &= \prod_{j=-\infty}^{\infty} L_{[j]}^{-1}, \\ U^{-1} &= \coprod_{j=-\infty}^{\infty} U_{[j]}^{-1}. \end{aligned} \tag{11.65}$$

In view of (11.64), therefore, *the inverse of a unit lower triangular matrix is another unit lower triangular matrix; and the inverse of a unit upper triangular matrix, another unit upper triangular matrix.*

It is plain to see but still interesting to note that—unlike the inverse—the adjoint or transpose of a unit lower triangular matrix is a unit upper triangular matrix; and that the adjoint or transpose of a unit upper triangular matrix is a unit lower triangular matrix. The adjoint reverses the sense of the triangle.

### 11.8.4 The parallel unit triangular matrix

If a unit triangular matrix fits the special, restricted form

$$\begin{aligned} L_{\parallel}^{\{k\}} &= I + \sum_{j=-\infty}^{k} \sum_{i=k+1}^{\infty} \alpha_{ij} E_{ij} \\ &= \begin{bmatrix} \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \\ \cdots & 1 & 0 & 0 & 0 & 0 & \cdots \\ \cdots & 0 & 1 & 0 & 0 & 0 & \cdots \\ \cdots & 0 & 0 & 1 & 0 & 0 & \cdots \\ \cdots & * & * & * & 1 & 0 & \cdots \\ \cdots & * & * & * & 0 & 1 & \cdots \\ & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix} \end{aligned} \tag{11.66}$$

or

$$U_{\parallel}^{\{k\}} = I + \sum_{j=k}^{\infty} \sum_{i=-\infty}^{k-1} \alpha_{ij} E_{ij} \tag{11.67}$$

$$= \begin{bmatrix} \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \\ \cdots & 1 & 0 & * & * & * & \cdots \\ \cdots & 0 & 1 & * & * & * & \cdots \\ \cdots & 0 & 0 & 1 & 0 & 0 & \cdots \\ \cdots & 0 & 0 & 0 & 1 & 0 & \cdots \\ \cdots & 0 & 0 & 0 & 0 & 1 & \cdots \\ & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix},$$

confining its nonzero elements to a rectangle within the triangle as shown, then it is a *parallel unit triangular matrix* and has some special properties the general unit triangular matrix lacks.

The general unit lower triangular matrix $L$ acting $LA$ on a matrix $A$ adds the rows of $A$ downward. The parallel unit lower triangular matrix $L_{\parallel}^{\{k\}}$ acting $L_{\parallel}^{\{k\}}A$ also adds rows downward, but with the useful restriction that it makes no row of $A$ both source and target. The addition is *from* $A$'s rows through the $k$th, *to* $A$'s $(k+1)$th row onward. A horizontal frontier separates source from target, which thus march in $A$ as separate squads.

Similar observations naturally apply with respect to the parallel unit upper triangular matrix $U_{\parallel}^{\{k\}}$, which acting $U_{\parallel}^{\{k\}}A$ adds rows upward, and also with respect to $L_{\parallel}^{\{k\}T}$ and $U_{\parallel}^{\{k\}T}$, which acting $AL_{\parallel}^{\{k\}T}$ and $AU_{\parallel}^{\{k\}T}$ add columns respectively rightward and leftward (remembering that $L_{\parallel}^{\{k\}T}$ is no unit lower but a unit upper triangular matrix; that $U_{\parallel}^{\{k\}T}$ is the lower). Each separates source from target in the matrix $A$ it operates on.

The reason we care about the separation of source from target is that, in matrix arithmetic generally, where source and target are not separate but remain intermixed, the sequence matters in which rows or columns are added. That is, in general,

$$T_{\alpha_1[i_1 j_1]} T_{\alpha_2[i_2 j_2]} \neq I + \alpha_1 E_{i_1 j_1} + \alpha_2 E_{i_2 j_2} \neq T_{\alpha_2[i_2 j_2]} T_{\alpha_1[i_1 j_1]}.$$

It makes a difference whether the one addition comes before, during or after the other—but only because the target of the one addition might be the source of the other. The danger is that $i_1 = j_2$ or $i_2 = j_1$. Remove this danger, and the sequence ceases to matter (refer to Table 11.3).

That is exactly what the parallel unit triangular matrix does: it separates source from target and thus removes the danger. It is for this reason that

the parallel unit triangular matrix brings the useful property that

$$\begin{aligned}
L_{\parallel}^{\{k\}} &= I + \sum_{j=-\infty}^{k} \sum_{i=k+1}^{\infty} \alpha_{ij} E_{ij} \\
&= \coprod_{j=-\infty}^{k} \prod_{i=k+1}^{\infty} T_{\alpha_{ij}[ij]} = \coprod_{j=-\infty}^{k} \coprod_{i=k+1}^{\infty} T_{\alpha_{ij}[ij]} \\
&= \prod_{j=-\infty}^{k} \prod_{i=k+1}^{\infty} T_{\alpha_{ij}[ij]} = \prod_{j=-\infty}^{k} \coprod_{i=k+1}^{\infty} T_{\alpha_{ij}[ij]} \\
&= \prod_{i=k+1}^{\infty} \coprod_{j=-\infty}^{k} T_{\alpha_{ij}[ij]} = \coprod_{i=k+1}^{\infty} \coprod_{j=-\infty}^{k} T_{\alpha_{ij}[ij]} \\
&= \prod_{i=k+1}^{\infty} \prod_{j=-\infty}^{k} T_{\alpha_{ij}[ij]} = \coprod_{i=k+1}^{\infty} \prod_{j=-\infty}^{k} T_{\alpha_{ij}[ij]}, \\
U_{\parallel}^{\{k\}} &= I + \sum_{j=k}^{\infty} \sum_{i=-\infty}^{k-1} \alpha_{ij} E_{ij} \\
&= \prod_{j=k}^{\infty} \coprod_{i=-\infty}^{k-1} T_{\alpha_{ij}[ij]} = \cdots ,
\end{aligned} \tag{11.68}$$

which says that one can build a parallel unit triangular matrix equally well in any sequence—in contrast to the case of the general unit triangular matrix, whose construction per (11.62) one must sequence carefully. (Though eqn. 11.68 does not show them, even more sequences are possible. You can scramble the factors' ordering any random way you like. The multiplication is fully commutative.) Under such conditions, the inverse of the parallel unit

triangular matrix is particularly simple:[32]

$$\begin{aligned}
L_{\|}^{\{k\}\,-1} &= I - \sum_{j=-\infty}^{k} \sum_{i=k+1}^{\infty} \alpha_{ij} E_{ij} = 2I - L_{\|}^{\{k\}} \\
&= \prod_{j=-\infty}^{k} \coprod_{i=k+1}^{\infty} T_{-\alpha_{ij}[ij]} = \cdots , \\
U_{\|}^{\{k\}\,-1} &= I - \sum_{j=k}^{\infty} \sum_{i=-\infty}^{k-1} \alpha_{ij} E_{ij} = 2I - U_{\|}^{\{k\}} \\
&= \coprod_{j=k}^{\infty} \prod_{i=-\infty}^{k-1} T_{-\alpha_{ij}[ij]} = \cdots ,
\end{aligned} \tag{11.69}$$

where again the elementaries can be multiplied in any order. Pictorially,

$$L_{\|}^{\{k\}\,-1} = \begin{bmatrix}
\ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \\
\cdots & 1 & 0 & 0 & 0 & 0 & \cdots \\
\cdots & 0 & 1 & 0 & 0 & 0 & \cdots \\
\cdots & 0 & 0 & 1 & 0 & 0 & \cdots \\
\cdots & -* & -* & -* & 1 & 0 & \cdots \\
\cdots & -* & -* & -* & 0 & 1 & \cdots \\
 & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots
\end{bmatrix},$$

$$U_{\|}^{\{k\}\,-1} = \begin{bmatrix}
\ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \\
\cdots & 1 & 0 & -* & -* & -* & \cdots \\
\cdots & 0 & 1 & -* & -* & -* & \cdots \\
\cdots & 0 & 0 & 1 & 0 & 0 & \cdots \\
\cdots & 0 & 0 & 0 & 1 & 0 & \cdots \\
\cdots & 0 & 0 & 0 & 0 & 1 & \cdots \\
 & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots
\end{bmatrix}.$$

The inverse of a parallel unit triangular matrix is just the matrix itself, only with each element off the main diagonal negated. Table 11.5 records a few properties that come immediately of the last observation and from the parallel unit triangular matrix's basic layout.

[32] There is some odd parochiality at play in applied mathematics when one calls such collections of symbols as (11.69) "particularly simple." Nevertheless, in the present context the idea (11.69) represents is indeed simple: that one can multiply constituent elementaries in any order and still reach the same parallel unit triangular matrix; that the elementaries in this case do not interfere.

Table 11.5: Properties of the parallel unit triangular matrix. (In the table, the notation $I_a^b$ represents the generalized dimension-limited indentity matrix or truncator of eqn. 11.30. Note that the inverses $L_\parallel^{\{k\}\,-1} = L_\parallel^{\{k\}\prime}$ and $U_\parallel^{\{k\}\,-1} = U_\parallel^{\{k\}\prime}$ are parallel unit triangular matrices themselves, such that the table's properties hold for them, too.)

$$\frac{L_\parallel^{\{k\}} + L_\parallel^{\{k\}\,-1}}{2} = I = \frac{U_\parallel^{\{k\}} + U_\parallel^{\{k\}\,-1}}{2}$$

$$\begin{aligned} I_{k+1}^{\infty} L_\parallel^{\{k\}} I_{-\infty}^{k} &= L_\parallel^{\{k\}} - I = I_{k+1}^{\infty}(L_\parallel^{\{k\}} - I) I_{-\infty}^{k} \\ I_{-\infty}^{k-1} U_\parallel^{\{k\}} I_{k}^{\infty} &= U_\parallel^{\{k\}} - I = I_{-\infty}^{k-1}(U_\parallel^{\{k\}} - I) I_{k}^{\infty} \end{aligned}$$

If $L_\parallel^{\{k\}}$ honors an $n \times n$ active region, then
$(I_n - I_k) L_\parallel^{\{k\}} I_k = L_\parallel^{\{k\}} - I = (I_n - I_k)(L_\parallel^{\{k\}} - I) I_k$
and $(I - I_n)(L_\parallel^{\{k\}} - I) = 0 = (L_\parallel^{\{k\}} - I)(I - I_n)$.

If $U_\parallel^{\{k\}}$ honors an $n \times n$ active region, then
$I_{k-1} U_\parallel^{\{k\}} (I_n - I_{k-1}) = U_\parallel^{\{k\}} - I = I_{k-1}(U_\parallel^{\{k\}} - I)(I_n - I_{k-1})$
and $(I - I_n)(U_\parallel^{\{k\}} - I) = 0 = (U_\parallel^{\{k\}} - I)(I - I_n)$.

### 11.8.5 The partial unit triangular matrix

Besides the notation $L$ and $U$ for the general unit lower and unit upper triangular matrices and the notation $L_{\parallel}^{\{k\}}$ and $U_{\parallel}^{\{k\}}$ for the parallel unit lower and unit upper triangular matrices, we shall find it useful to introduce the additional notation

$$
\begin{aligned}
L^{[k]} &= I + \sum_{j=k}^{\infty} \sum_{i=j+1}^{\infty} \alpha_{ij} E_{ij} \qquad (11.70) \\
&= \begin{bmatrix}
\ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \\
\cdots & 1 & 0 & 0 & 0 & 0 & \cdots \\
\cdots & 0 & 1 & 0 & 0 & 0 & \cdots \\
\cdots & 0 & 0 & 1 & 0 & 0 & \cdots \\
\cdots & 0 & 0 & * & 1 & 0 & \cdots \\
\cdots & 0 & 0 & * & * & 1 & \cdots \\
 & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots
\end{bmatrix}, \\
U^{[k]} &= I + \sum_{j=-\infty}^{k} \sum_{i=-\infty}^{j-1} \alpha_{ij} E_{ij} \qquad (11.71) \\
&= \begin{bmatrix}
\ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \\
\cdots & 1 & * & * & 0 & 0 & \cdots \\
\cdots & 0 & 1 & * & 0 & 0 & \cdots \\
\cdots & 0 & 0 & 1 & 0 & 0 & \cdots \\
\cdots & 0 & 0 & 0 & 1 & 0 & \cdots \\
\cdots & 0 & 0 & 0 & 0 & 1 & \cdots \\
 & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots
\end{bmatrix}
\end{aligned}
$$

for unit triangular matrices whose off-diagonal content is confined to a narrow wedge and

$$L^{\{k\}} = I + \sum_{j=-\infty}^{k} \sum_{i=j+1}^{\infty} \alpha_{ij} E_{ij} \tag{11.72}$$

$$= \begin{bmatrix} \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \\ \cdots & 1 & 0 & 0 & 0 & 0 & \cdots \\ \cdots & * & 1 & 0 & 0 & 0 & \cdots \\ \cdots & * & * & 1 & 0 & 0 & \cdots \\ \cdots & * & * & * & 1 & 0 & \cdots \\ \cdots & * & * & * & 0 & 1 & \cdots \\ & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix},$$

$$U^{\{k\}} = I + \sum_{j=k}^{\infty} \sum_{i=-\infty}^{j-1} \alpha_{ij} E_{ij} \tag{11.73}$$

$$= \begin{bmatrix} \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \\ \cdots & 1 & 0 & * & * & * & \cdots \\ \cdots & 0 & 1 & * & * & * & \cdots \\ \cdots & 0 & 0 & 1 & * & * & \cdots \\ \cdots & 0 & 0 & 0 & 1 & * & \cdots \\ \cdots & 0 & 0 & 0 & 0 & 1 & \cdots \\ & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

for the supplementary forms.[33] Such notation is not standard in the literature, but it serves a purpose in this book and is introduced here for this reason. If names are needed for $L^{[k]}$, $U^{[k]}$, $L^{\{k\}}$ and $U^{\{k\}}$, the former pair can be called *minor partial unit triangular matrices,* and the latter pair, *major partial unit triangular matrices.* Whether minor or major, the partial unit triangular matrix is a matrix which leftward or rightward of the $k$th column resembles $I$. Of course partial unit triangular matrices which resemble $I$ above or below the $k$th *row* are equally possible, and can be denoted $L^{[k]T}$, $U^{[k]T}$, $L^{\{k\}T}$ and $U^{\{k\}T}$.

Observe that the parallel unit triangular matrices $L_{\parallel}^{\{k\}}$ and $U_{\parallel}^{\{k\}}$ of § 11.8.4 are in fact also major partial unit triangular matrices, as the notation suggests.

[33] The notation is arguably imperfect in that $L^{\{k\}} + L^{[k]} - I \neq L$ but rather that $L^{\{k\}} + L^{[k+1]} - I = L$. The conventional notation $\sum_{k=a}^{b} f(k) + \sum_{k=b}^{c} f(k) \neq \sum_{k=a}^{c} f(k)$ suffers the same arguable imperfection.

## 11.9 The shift operator

Not all useful matrices fit the dimension-limited and extended-operational forms of § 11.3. An exception is the *shift operator* $H_k$, defined that

$$[H_k]_{ij} = \delta_{i(j+k)}. \tag{11.74}$$

For example,

$$H_2 = \begin{bmatrix} & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \\ \cdots & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\ \cdots & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\ \cdots & 1 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\ \cdots & 0 & 1 & 0 & 0 & 0 & 0 & 0 & \cdots \\ \cdots & 0 & 0 & 1 & 0 & 0 & 0 & 0 & \cdots \\ \cdots & 0 & 0 & 0 & 1 & 0 & 0 & 0 & \cdots \\ \cdots & 0 & 0 & 0 & 0 & 1 & 0 & 0 & \cdots \\ & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \end{bmatrix}.$$

Operating $H_k A$, $H_k$ shifts $A$'s rows downward $k$ steps. Operating $AH_k$, $H_k$ shifts $A$'s columns leftward $k$ steps. Inasmuch as the shift operator shifts all rows or columns of the matrix it operates on, its active region is $\infty \times \infty$ in extent. Obviously, the shift operator's inverse, transpose and adjoint are the same:

$$\begin{aligned} H_k^T H_k = H_k^* H_k = I = H_k H_k^* = H_k H_k^T, \\ H_k^{-1} = H_k^T = H_k^* = H_{-k}. \end{aligned} \tag{11.75}$$

Further obvious but useful identities include that

$$\begin{aligned} (I_\ell - I_k) H_k = H_k I_{\ell-k}, \\ H_{-k}(I_\ell - I_k) = I_{\ell-k} H_{-k}. \end{aligned} \tag{11.76}$$

## 11.10 The Jacobian derivative

Chapter 4 has introduced the derivative of a function with respect to a scalar variable. One can also take the derivative of a function with respect to a vector variable, and the function itself can be vector-valued. The derivative is

$$\left[\frac{d\mathbf{f}}{d\mathbf{x}}\right]_{ij} = \frac{\partial f_i}{\partial x_j}. \tag{11.77}$$

For instance, if $\mathbf{x}$ has three elements and $\mathbf{f}$ has two, then

$$\frac{d\mathbf{f}}{d\mathbf{x}} = \begin{bmatrix} \dfrac{\partial f_1}{\partial x_1} & \dfrac{\partial f_1}{\partial x_2} & \dfrac{\partial f_1}{\partial x_3} \\ \dfrac{\partial f_2}{\partial x_1} & \dfrac{\partial f_2}{\partial x_2} & \dfrac{\partial f_2}{\partial x_3} \end{bmatrix}.$$

This is called the *Jacobian derivative,* the *Jacobian matrix,* or just the *Jacobian.*[34] Each of its columns is the derivative with respect to one element of $\mathbf{x}$.

The Jacobian derivative of a vector with respect to itself is

$$\frac{d\mathbf{x}}{d\mathbf{x}} = I. \tag{11.78}$$

The derivative is not $I_n$ as one might think, because, even if $\mathbf{x}$ has only $n$ elements, still, one could vary $x_{n+1}$ in principle, and $\partial x_{n+1}/\partial x_{n+1} \neq 0$.

The Jacobian derivative obeys the derivative product rule (4.25) in the form[35]

$$\begin{aligned}
\frac{d}{d\mathbf{x}}\left(\mathbf{g}^T A \mathbf{f}\right) &= \left[\mathbf{g}^T A \left(\frac{d\mathbf{f}}{d\mathbf{x}}\right)\right] + \left[\left(\frac{d\mathbf{g}}{d\mathbf{x}}\right)^T A \mathbf{f}\right]^T, \\
\frac{d}{d\mathbf{x}}\left(\mathbf{g}^* A \mathbf{f}\right) &= \left[\mathbf{g}^* A \left(\frac{d\mathbf{f}}{d\mathbf{x}}\right)\right] + \left[\left(\frac{d\mathbf{g}}{d\mathbf{x}}\right)^* A \mathbf{f}\right]^T,
\end{aligned} \tag{11.79}$$

valid for any constant matrix $A$—as is seen by applying the definition (4.19) of the derivative, which here is

$$\frac{\partial\left(\mathbf{g}^* A \mathbf{f}\right)}{\partial x_j} = \lim_{\partial x_j \to 0} \frac{(\mathbf{g} + \partial\mathbf{g}/2)^* A (\mathbf{f} + \partial\mathbf{f}/2) - (\mathbf{g} - \partial\mathbf{g}/2)^* A (\mathbf{f} - \partial\mathbf{f}/2)}{\partial x_j},$$

and simplifying.

The shift operator of § 11.9 and the Jacobian derivative of this section complete the family of matrix rudiments we shall need to begin to do increasingly interesting things with matrices in Chs. 13 and 14. Before doing interesting things, however, we must treat two more foundational matrix matters. The two are the Gauss-Jordan decomposition and the matter of matrix rank, which will be the subjects of Ch. 12, next.

[34] [66, "Jacobian," 00:50, 15 Sept. 2007]

[35] Notice that the last term on (11.79)'s second line is transposed, not adjointed.

# Chapter 12

# Matrix rank and the Gauss-Jordan decomposition

Chapter 11 has brought the matrix and its rudiments, the latter including

- lone-element matrix $E$ (§ 11.3.7),
- the null matrix 0 (§ 11.3.1),
- the rank-$r$ identity matrix $I_r$ (§ 11.3.5),
- the general identity matrix $I$ and the scalar matrix $\lambda I$ (§ 11.3.2),
- the elementary operator $T$ (§ 11.4),
- the quasielementary operator $P$, $D$, $L_{[k]}$ or $U_{[k]}$ (§ 11.7), and
- the unit triangular matrix $L$ or $U$ (§ 11.8).

Such rudimentary forms have useful properties, as we have seen. The general matrix $A$ does not necessarily have any of these properties, but it turns out that one can factor any matrix whatsoever into a product of rudiments which do have the properties, and that several orderly procedures are known to do so. The simplest of these, and indeed one of the more useful, is the Gauss-Jordan decomposition. This chapter introduces it.

Section 11.3 has deëmphasized the concept of matrix dimensionality $m \times n$, supplying in its place the new concept of matrix rank. However, that section has actually defined rank only for the rank-$r$ identity matrix $I_r$. In fact all matrices have rank. This chapter explains.

Before treating the Gauss-Jordan decomposition and the matter of matrix rank as such, however, we shall find it helpful to prepare two preliminaries thereto: (i) the matter of the linear independence of vectors; and (ii) the elementary similarity transformation. The chapter begins with these.

Except in § 12.2, the chapter demands more rigor than one likes in such a book as this. However, it is hard to see how to avoid the rigor here, and logically the chapter cannot be omitted. We will drive through the chapter in as few pages as can be managed, and then onward to the more interesting matrix topics of Chs. 13 and 14.

## 12.1 Linear independence

Linear independence is a significant possible property of a set of vectors—whether the set be the several columns of a matrix, the several rows, or some other vectors—the property being defined as follows. A vector is *linearly independent* if its role cannot be served by the other vectors in the set. More formally, the $n$ vectors of the set $\{\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3, \mathbf{a}_4, \mathbf{a}_5, \ldots, \mathbf{a}_n\}$ are linearly independent if and only if none of them can be expressed as a linear combination—a weighted sum—of the others. That is, the several $\mathbf{a}_k$ are linearly independent iff

$$\alpha_1\mathbf{a}_1 + \alpha_2\mathbf{a}_2 + \alpha_3\mathbf{a}_3 + \cdots + \alpha_n\mathbf{a}_n \neq 0 \tag{12.1}$$

for all nontrivial $\alpha_k$, where "nontrivial $\alpha_k$" means the several $\alpha_k$, at least one of which is nonzero (*trivial* $\alpha_k$, by contrast, would be $\alpha_1 = \alpha_2 = \alpha_3 = \cdots = \alpha_n = 0$). Vectors which can combine nontrivially to reach the null vector are by definition *linearly dependent.*

Linear independence is a property of vectors. Technically the property applies to scalars, too, inasmuch as a scalar resembles a one-element vector—so, any nonzero scalar alone is linearly independent—but there is no such thing as a linearly independent pair of scalars, because one of the pair can always be expressed as a complex multiple of the other. Significantly but less obviously, there is also no such thing as a linearly independent set which includes the null vector; (12.1) forbids it. Paradoxically, even the single-member, $n = 1$ set consisting only of $\mathbf{a}_1 = 0$ is, strictly speaking, not linearly independent.

For consistency of definition, we regard the empty, $n = 0$ set as linearly independent, on the technical ground that the only possible linear combination of the empty set is trivial.[1]

[1]This is the kind of thinking which typically governs mathematical edge cases. One

If a linear combination of several independent vectors $\mathbf{a}_k$ forms a vector $\mathbf{b}$, then one might ask: can there exist a different linear combination of the same vectors $\mathbf{a}_k$ which also forms $\mathbf{b}$? That is, if

$$\beta_1\mathbf{a}_1 + \beta_2\mathbf{a}_2 + \beta_3\mathbf{a}_3 + \cdots + \beta_n\mathbf{a}_n = \mathbf{b},$$

where the several $\mathbf{a}_k$ satisfy (12.1), then is

$$\beta'_1\mathbf{a}_1 + \beta'_2\mathbf{a}_2 + \beta'_3\mathbf{a}_3 + \cdots + \beta'_n\mathbf{a}_n = \mathbf{b}$$

possible? To answer the question, suppose that it were possible. The difference of the two equations then would be

$$(\beta'_1 - \beta_1)\mathbf{a}_1 + (\beta'_2 - \beta_2)\mathbf{a}_2 + (\beta'_3 - \beta_3)\mathbf{a}_3 + \cdots + (\beta'_n - \beta_n)\mathbf{a}_n = 0.$$

According to (12.1), this could only be so if the coefficients in the last equation where trivial—that is, only if $\beta'_1 - \beta_1 = 0$, $\beta'_2 - \beta_2 = 0$, $\beta'_3 - \beta_3 = 0$, …, $\beta'_n - \beta_n = 0$. But this says no less than that the two linear combinations, which we had supposed to differ, were in fact one and the same. One concludes therefore that, *if a vector* $\mathbf{b}$ *can be expressed as a linear combination of several linearly independent vectors* $\mathbf{a}_k$*, then it cannot be expressed as any other combination of the same vectors.* The combination is unique.

Linear independence can apply in any dimensionality, but it helps to visualize the concept geometrically in three dimensions, using the three-dimensional geometrical vectors of § 3.3. Two such vectors are independent so long as they do not lie along the same line. A third such vector is independent of the first two so long as it does not lie in their common plane. A fourth such vector (unless it points off into some unvisualizable fourth dimension) cannot possibly then be independent of the three.

We discuss the linear independence of vectors in this, a chapter on matrices, because (§ 11.1) a matrix is essentially a sequence of vectors—either of column vectors or of row vectors, depending on one's point of view. As we shall see in § 12.5, the important property of matrix *rank* depends on the number of linearly independent columns or rows a matrix has.

---

could define the empty set to be linearly dependent if one really wanted to, but what then of the observation that adding a vector to a linearly dependent set never renders the set independent? Surely in this light it is preferable just to define the empty set as independent in the first place. Similar thinking makes $0! = 1$, $\sum_{k=0}^{-1} a_k z^k = 0$, and 2 not 1 the least prime, among other examples.

## 12.2 The elementary similarity transformation

Section 11.5 and its (11.46) have introduced the *similarity transformation* $CAC^{-1}$ or $C^{-1}AC$, which arises when an operator $C$ commutes respectively rightward or leftward past a matrix $A$. The similarity transformation has several interesting properties, some of which we are now prepared to discuss, particularly in the case in which the operator happens to be an elementary, $C = T$. In this case, the several rules of Table 12.1 obtain.

Most of the table's rules are fairly obvious if the meaning of the symbols is understood, though to grasp some of the rules it helps to sketch the relevant matrices on a sheet of paper. Of course rigorous symbolic proofs can be constructed after the pattern of § 11.8.2, but they reveal little or nothing sketching the matrices does not. The symbols $P$, $D$, $L$ and $U$ of course represent the quasielementaries and unit triangular matrices of §§ 11.7 and 11.8. The symbols $P'$, $D'$, $L'$ and $U'$ also represent quasielementaries and unit triangular matrices, only not necessarily the same ones $P$, $D$, $L$ and $U$ do.

The rules permit one to commute some but not all elementaries past a quasielementary or unit triangular matrix, without fundamentally altering the character of the quasielementary or unit triangular matrix, and sometimes without altering the matrix at all. The rules find use among other places in the Gauss-Jordan decomposition of § 12.3.

## 12.3 The Gauss-Jordan decomposition

The *Gauss-Jordan decomposition* of an arbitrary, dimension-limited, $m \times n$ matrix $A$ is[2]

$$\begin{aligned} A = G_> I_r G_< &= PDLUI_rKS, \\ G_< &\equiv KS, \\ G_> &\equiv PDLU, \end{aligned} \tag{12.2}$$

where

- $P$ and $S$ are general interchange operators (§ 11.7.1);

[2]Most introductory linear algebra texts this writer has met call the Gauss-Jordan decomposition instead the "$LU$ decomposition" and include fewer factors in it, typically merging $D$ into $L$ and omitting $K$ and $S$. They also omit $I_r$, since their matrices have pre-defined dimensionality. Perhaps the reader will agree that the decomposition is cleaner as presented here.

Table 12.1: Some elementary similarity transformations.

$$
\begin{aligned}
T_{[i\leftrightarrow j]} I T_{[i\leftrightarrow j]} &= I \\
T_{[i\leftrightarrow j]} P T_{[i\leftrightarrow j]} &= P' \\
T_{[i\leftrightarrow j]} D T_{[i\leftrightarrow j]} &= D' = D + ([D]_{jj} - [D]_{ii})\, E_{ii} + ([D]_{ii} - [D]_{jj})\, E_{jj} \\
T_{[i\leftrightarrow j]} D T_{[i\leftrightarrow j]} &= D \qquad \text{if } [D]_{ii} = [D]_{jj} \\
T_{[i\leftrightarrow j]} L^{[k]} T_{[i\leftrightarrow j]} &= L^{[k]} \quad \text{if } i < k \text{ and } j < k \\
T_{[i\leftrightarrow j]} U^{[k]} T_{[i\leftrightarrow j]} &= U^{[k]} \quad \text{if } i > k \text{ and } j > k \\
T_{[i\leftrightarrow j]} L^{\{k\}} T_{[i\leftrightarrow j]} &= L^{\{k\}\prime} \quad \text{if } i > k \text{ and } j > k \\
T_{[i\leftrightarrow j]} U^{\{k\}} T_{[i\leftrightarrow j]} &= U^{\{k\}\prime} \quad \text{if } i < k \text{ and } j < k \\
T_{[i\leftrightarrow j]} L_{\|}^{\{k\}} T_{[i\leftrightarrow j]} &= L_{\|}^{\{k\}\prime} \quad \text{if } i > k \text{ and } j > k \\
T_{[i\leftrightarrow j]} U_{\|}^{\{k\}} T_{[i\leftrightarrow j]} &= U_{\|}^{\{k\}\prime} \quad \text{if } i < k \text{ and } j < k \\
T_{\alpha[i]} I T_{(1/\alpha)[i]} &= I \\
T_{\alpha[i]} D T_{(1/\alpha)[i]} &= D \\
T_{\alpha[i]} A T_{(1/\alpha)[i]} &= A' \qquad \text{where } A \text{ is any of} \\
&\qquad\qquad L, U, L^{[k]}, U^{[k]}, L^{\{k\}}, U^{\{k\}}, L_{\|}^{\{k\}}, U_{\|}^{\{k\}} \\
T_{\alpha[ij]} I T_{-\alpha[ij]} &= I \\
T_{\alpha[ij]} D T_{-\alpha[ij]} &= D + ([D]_{jj} - [D]_{ii})\, \alpha E_{ij} \neq D' \\
T_{\alpha[ij]} D T_{-\alpha[ij]} &= D \qquad \text{if } [D]_{ii} = [D]_{jj} \\
T_{\alpha[ij]} L T_{-\alpha[ij]} &= L' \qquad \text{if } i > j \\
T_{\alpha[ij]} U T_{-\alpha[ij]} &= U' \qquad \text{if } i < j
\end{aligned}
$$

- $D$ is a general scaling operator (§ 11.7.2);
- $L$ and $U$ are respectively unit lower and unit upper triangular matrices (§ 11.8);
- $K = L_{\parallel}^{\{r\}T}$ is the transpose of a parallel unit lower triangular matrix, being thus a parallel unit upper triangular matrix (§ 11.8.4);
- $G_>$ and $G_<$ are composites[3] as defined by (12.2); and
- $r$ is an unspecified rank.

The Gauss-Jordan decomposition is also called the *Gauss-Jordan factorization.*

Whether all possible matrices $A$ have a Gauss-Jordan decomposition (they do, in fact) is a matter this section addresses. However—at least for matrices which do have one—because $G_>$ and $G_<$ are composed of invertible factors, one can left-multiply the equation $A = G_> I_r G_<$ by $G_>^{-1}$ and right-multiply it by $G_<^{-1}$ to obtain

$$\begin{aligned} U^{-1}L^{-1}D^{-1}P^{-1}AS^{-1}K^{-1} = G_>^{-1}AG_<^{-1} &= I_r, \\ S^{-1}K^{-1} &= G_<^{-1}, \\ U^{-1}L^{-1}D^{-1}P^{-1} &= G_>^{-1}, \end{aligned} \tag{12.3}$$

the Gauss-Jordan's complementary form.

### 12.3.1 Motive

Equation (12.2) seems inscrutable. The equation itself is easy enough to read, but just as there are many ways to factor a scalar (0xC = [4][3] = $[2]^2[3]$ = [2][6], for example), there are likewise many ways to factor a matrix. Why choose this particular way?

There are indeed many ways. We shall meet some of the others in §§ 13.11, 14.6, 14.10 and 14.12. The Gauss-Jordan decomposition we meet here however has both significant theoretical properties and useful practical applications, and in any case needs less advanced preparation to appreciate than the others, and (at least as developed in this book) precedes the others logically. It emerges naturally when one posits a pair of square, $n \times n$

[3] One can pronounce $G_>$ and $G_<$ respectively as "$G$ acting rightward" and "$G$ acting leftward." The letter $G$ itself can be regarded as standing for "Gauss-Jordan," but admittedly it is chosen as much because otherwise we were running out of available Roman capitals!

matrices, $A$ and $A^{-1}$, for which $A^{-1}A = I_n$, where $A$ is known and $A^{-1}$ is to be determined. (The $A^{-1}$ here is the $A^{-1(n)}$ of eqn. 11.49. However, it is only supposed here that $A^{-1}A = I_n$; it is not *yet* claimed that $AA^{-1} = I_n$.)

To determine $A^{-1}$ is not an entirely trivial problem. The matrix $A^{-1}$ such that $A^{-1}A = I_n$ may or may not exist (usually it does exist if $A$ is square, but even then it may not, as we shall soon see), and even if it does exist, how to determine it is not immediately obvious. And still, if one can determine $A^{-1}$, that is only for square $A$; what if $A$ is not square? In the present subsection however we are not trying to prove anything, only to motivate, so for the moment let us suppose a square $A$ for which $A^{-1}$ does exist, and let us seek $A^{-1}$ by left-multiplying $A$ by a sequence $\prod T$ of elementary row operators, each of which makes the matrix more nearly resemble $I_n$. When $I_n$ is finally achieved, then we shall have that

$$\left(\prod T\right)(A) = I_n,$$

or, left-multiplying by $I_n$ and observing that $I_n^2 = I_n$,

$$(I_n)\left(\prod T\right)(A) = I_n,$$

which implies that

$$A^{-1} = (I_n)\left(\prod T\right).$$

The product of elementaries which transforms $A$ to $I_n$, truncated (§ 11.3.6) to $n \times n$ dimensionality, itself constitutes $A^{-1}$. This observation is what motivates the Gauss-Jordan decomposition.

By successive steps,[4] a concrete example:

$$\begin{aligned}
A &= \begin{bmatrix} 2 & -4 \\ 3 & -1 \end{bmatrix}, \\
\begin{bmatrix} \frac{1}{2} & 0 \\ 0 & 1 \end{bmatrix} A &= \begin{bmatrix} 1 & -2 \\ 3 & -1 \end{bmatrix}, \\
\begin{bmatrix} 1 & 0 \\ -3 & 1 \end{bmatrix}\begin{bmatrix} \frac{1}{2} & 0 \\ 0 & 1 \end{bmatrix} A &= \begin{bmatrix} 1 & -2 \\ 0 & 5 \end{bmatrix}, \\
\begin{bmatrix} 1 & 0 \\ 0 & \frac{1}{5} \end{bmatrix}\begin{bmatrix} 1 & 0 \\ -3 & 1 \end{bmatrix}\begin{bmatrix} \frac{1}{2} & 0 \\ 0 & 1 \end{bmatrix} A &= \begin{bmatrix} 1 & -2 \\ 0 & 1 \end{bmatrix}, \\
\begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & \frac{1}{5} \end{bmatrix}\begin{bmatrix} 1 & 0 \\ -3 & 1 \end{bmatrix}\begin{bmatrix} \frac{1}{2} & 0 \\ 0 & 1 \end{bmatrix} A &= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \\
\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & \frac{1}{5} \end{bmatrix}\begin{bmatrix} 1 & 0 \\ -3 & 1 \end{bmatrix}\begin{bmatrix} \frac{1}{2} & 0 \\ 0 & 1 \end{bmatrix} A &= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.
\end{aligned}$$

[4] Theoretically, all elementary operators including the ones here have extended-operational form (§ 11.3.2), but all those $\cdots$ ellipses clutter the page too much. Only the $2 \times 2$ active regions are shown here.

Hence,

$$A^{-1} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & \frac{1}{5} \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -3 & 1 \end{bmatrix} \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} -\frac{1}{\mathrm{A}} & \frac{2}{5} \\ -\frac{3}{\mathrm{A}} & \frac{1}{5} \end{bmatrix}.$$

Using the elementary commutation identity that $T_{\beta[m]}T_{\alpha[mj]} = T_{\alpha\beta[mj]}T_{\beta[m]}$, from Table 11.2, to group like operators, we have that

$$A^{-1} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -\frac{3}{5} & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & \frac{1}{5} \end{bmatrix} \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} -\frac{1}{\mathrm{A}} & \frac{2}{5} \\ -\frac{3}{\mathrm{A}} & \frac{1}{5} \end{bmatrix};$$

or, multiplying the two scaling elementaries to merge them into a single general scaling operator (§ 11.7.2),

$$A^{-1} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -\frac{3}{5} & 1 \end{bmatrix} \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{5} \end{bmatrix} = \begin{bmatrix} -\frac{1}{\mathrm{A}} & \frac{2}{5} \\ -\frac{3}{\mathrm{A}} & \frac{1}{5} \end{bmatrix}.$$

The last equation is written symbolically as

$$A^{-1} = I_2 U^{-1} L^{-1} D^{-1},$$

from which

$$A = DLUI_2 = \begin{bmatrix} 2 & 0 \\ 0 & 5 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ \frac{3}{5} & 1 \end{bmatrix} \begin{bmatrix} 1 & -2 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 2 & -4 \\ 3 & -1 \end{bmatrix}.$$

Now, admittedly, the equation $A = DLUI_2$ is not (12.2)—or rather, it is (12.2), but only in the special case that $r = 2$ and $P = S = K = I$—which begs the question: why do we need the factors $P$, $S$ and $K$ in the first place? The answer regarding $P$ and $S$ is that these factors respectively gather row and column interchange elementaries, of which the example given has used none but which other examples sometimes need or want, particularly to avoid dividing by zero when they encounter a zero in an inconvenient cell of the matrix (the reader might try reducing $A = [0\ 1; 1\ 0]$ to $I_2$, for instance; a row or column interchange is needed here). Regarding $K$, this factor comes into play when $A$ has broad rectangular rather than square shape, and also sometimes when one of the rows of $A$ happens to be a linear combination of the others. The last point, we are not quite ready to detail yet, but at present we are only motivating not proving, so if the reader will accept the other factors and suspend judgment on $K$ until the actual need for it emerges in § 12.3.3, step 12, then we will proceed on this basis.

### 12.3.2 Method

The Gauss-Jordan decomposition of a matrix $A$ is not discovered at one stroke but rather is gradually built up, elementary by elementary. It begins with the equation

$$A = IIIIAII,$$

where the six $I$ hold the places of the six Gauss-Jordan factors $P$, $D$, $L$, $U$, $K$ and $S$ of (12.2). By successive elementary operations, the $A$ on the right is gradually transformed into $I_r$, while the six $I$ are gradually transformed into the six Gauss-Jordan factors. The decomposition thus ends with the equation

$$A = PDLUI_rKS,$$

which is (12.2). In between, while the several matrices are gradually being transformed, the equation is represented as

$$A = \tilde{P}\tilde{D}\tilde{L}\tilde{U}\tilde{I}\tilde{K}\tilde{S}, \tag{12.4}$$

where the initial value of $\tilde{I}$ is $A$ and the initial values of $\tilde{P}$, $\tilde{D}$, etc., are all $I$.

Each step of the transformation goes as follows. The matrix $\tilde{I}$ is left- or right-multiplied by an elementary operator $T$. To compensate, one of the six factors is right- or left-multiplied by $T^{-1}$. Intervening factors are multiplied by both $T$ and $T^{-1}$, which multiplication constitutes an elementary similarity transformation as described in § 12.2. For example,

$$A = \tilde{P}\left(\tilde{D}T_{(1/\alpha)[i]}\right)\left(T_{\alpha[i]}\tilde{L}T_{(1/\alpha)[i]}\right)\left(T_{\alpha[i]}\tilde{U}T_{(1/\alpha)[i]}\right)\left(T_{\alpha[i]}\tilde{I}\right)\tilde{K}\tilde{S},$$

which is just (12.4), inasmuch as the adjacent elementaries cancel one another; then,

$$\begin{aligned}
\tilde{I} &\leftarrow T_{\alpha[i]}\tilde{I}, \\
\tilde{U} &\leftarrow T_{\alpha[i]}\tilde{U}T_{(1/\alpha)[i]}, \\
\tilde{L} &\leftarrow T_{\alpha[i]}\tilde{L}T_{(1/\alpha)[i]}, \\
\tilde{D} &\leftarrow \tilde{D}T_{(1/\alpha)[i]},
\end{aligned}$$

thus associating the operation with the appropriate factor—in this case, $\tilde{D}$. Such elementary row and column operations are repeated until $\tilde{I} = I_r$, at which point (12.4) has become the Gauss-Jordan decomposition (12.2).

### 12.3.3 The algorithm

Having motivated the Gauss-Jordan decomposition in § 12.3.1 and having proposed a basic method to pursue it in § 12.3.2, we shall now establish a definite, orderly, failproof algorithm to achieve it. Broadly, the algorithm

- copies $A$ into the variable working matrix $\tilde{I}$ (step 1 below),
- reduces $\tilde{I}$ by suitable row (and maybe column) operations to unit upper triangular form (steps 2 through 7),
- establishes a rank $r$ (step 8), and
- reduces the now unit triangular $\tilde{I}$ further to the rank-$r$ identity matrix $I_r$ (steps 9 through 13).

Specifically, the algorithm decrees the following steps. (The steps as written include many parenthetical remarks—so many that some steps seem to consist more of parenthetical remarks than of actual algorithm. The remarks are unnecessary to execute the algorithm's steps as such. They are however necessary to explain and to justify the algorithm's steps to the reader.)

1. Begin by initializing

$$\tilde{P} \leftarrow I,\ \tilde{D} \leftarrow I,\ \tilde{L} \leftarrow I,\ \tilde{U} \leftarrow I,\ \tilde{K} \leftarrow I,\ \tilde{S} \leftarrow I,$$
$$\tilde{I} \leftarrow A,$$
$$i \leftarrow 1,$$

   where $\tilde{I}$ holds the part of $A$ remaining to be decomposed, where $i$ is a row index, and where the others are the variable working matrices of (12.4). (The eventual goal will be to factor all of $\tilde{I}$ away, leaving $\tilde{I} = I_r$, though the precise value of $r$ will not be known until step 8. Since $A$ by definition is a dimension-limited $m \times n$ matrix, one naturally need not store $A$ beyond the $m \times n$ active region. What is less clear until one has read the whole algorithm, but nevertheless true, is that one also need not store the dimension-limited $\tilde{I}$ beyond the $m \times n$ active region. The other six variable working matrices each have extended-operational form, but they also confine their activity to well-defined regions: $m \times m$ for $\tilde{P}$, $\tilde{D}$, $\tilde{L}$ and $\tilde{U}$; $n \times n$ for $\tilde{K}$ and $\tilde{S}$. One need store none of the matrices beyond these bounds.)

2. (Besides arriving at this point from step 1 above, the algorithm also reënters here from step 7 below. From step 1, $\tilde{I} = A$ and $\tilde{L} = I$, so

this step 2 though logical seems unneeded. The need grows clear once one has read through step 7.) Observe that neither the $i$th row of $\tilde{I}$ nor any row below it has an entry left of the $i$th column, that $\tilde{I}$ is all-zero below-leftward of and directly leftward of (though not directly below) the *pivot* element $\tilde{\imath}_{ii}$.[5] Observe also that above the $i$th row, the matrix has proper unit upper triangular form (§ 11.8). Regarding the other factors, notice that $\tilde{L}$ enjoys the major partial unit triangular form $L^{\{i-1\}}$ (§ 11.8.5) and that $\tilde{d}_{kk} = 1$ for all $k \geq i$. Pictorially,

$$\tilde{D} = \begin{bmatrix} \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \\ \cdots & * & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\ \cdots & 0 & * & 0 & 0 & 0 & 0 & 0 & \cdots \\ \cdots & 0 & 0 & * & 0 & 0 & 0 & 0 & \cdots \\ \cdots & 0 & 0 & 0 & 1 & 0 & 0 & 0 & \cdots \\ \cdots & 0 & 0 & 0 & 0 & 1 & 0 & 0 & \cdots \\ \cdots & 0 & 0 & 0 & 0 & 0 & 1 & 0 & \cdots \\ \cdots & 0 & 0 & 0 & 0 & 0 & 0 & 1 & \cdots \\ & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix},$$

$$\tilde{L} = L^{\{i-1\}} = \begin{bmatrix} \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \\ \cdots & 1 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\ \cdots & * & 1 & 0 & 0 & 0 & 0 & 0 & \cdots \\ \cdots & * & * & 1 & 0 & 0 & 0 & 0 & \cdots \\ \cdots & * & * & * & 1 & 0 & 0 & 0 & \cdots \\ \cdots & * & * & * & 0 & 1 & 0 & 0 & \cdots \\ \cdots & * & * & * & 0 & 0 & 1 & 0 & \cdots \\ \cdots & * & * & * & 0 & 0 & 0 & 1 & \cdots \\ & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix},$$

$$\tilde{I} = \begin{bmatrix} \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \\ \cdots & 1 & * & * & * & * & * & * & \cdots \\ \cdots & 0 & 1 & * & * & * & * & * & \cdots \\ \cdots & 0 & 0 & 1 & * & * & * & * & \cdots \\ \cdots & 0 & 0 & 0 & * & * & * & * & \cdots \\ \cdots & 0 & 0 & 0 & * & * & * & * & \cdots \\ \cdots & 0 & 0 & 0 & * & * & * & * & \cdots \\ \cdots & 0 & 0 & 0 & * & * & * & * & \cdots \\ & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix},$$

where the $i$th row and $i$th column are depicted at center.

[5] The notation $\tilde{\imath}_{ii}$ looks interesting, but this is accidental. The $\tilde{\imath}$ relates not to the doubled, subscribed index $ii$ but to $\tilde{I}$. The notation $\tilde{\imath}_{ii}$ thus means $[\tilde{I}]_{ii}$—in other words, it means the current $ii$th element of the variable working matrix $\tilde{I}$.

3. Choose a nonzero element $\tilde{\imath}_{pq} \neq 0$ on or below the pivot row, where $p \geq i$ and $q \geq i$. (The easiest choice may simply be $\tilde{\imath}_{ii}$, where $p = q = i$, if $\tilde{\imath}_{ii} \neq 0$; but any nonzero element from the $i$th row downward can in general be chosen. Beginning students of the Gauss-Jordan or $LU$ decomposition are conventionally taught to choose first the least possible $q$ then the least possible $p$. When one has no reason to choose otherwise, that is as good a choice as any. There is however no actual need to choose so. In fact alternate choices can sometimes improve practical numerical accuracy.[6,7] Theoretically nonetheless, when doing exact arithmetic, the choice is quite arbitrary, so long as $\tilde{\imath}_{pq} \neq 0$.) If no nonzero element is available—if all remaining rows $p \geq i$ are now null—then skip directly to step 8.

4. Observing that (12.4) can be expanded to read

$$\begin{aligned} A &= \left(\tilde{P}T_{[p\leftrightarrow i]}\right)\left(T_{[p\leftrightarrow i]}\tilde{D}T_{[p\leftrightarrow i]}\right)\left(T_{[p\leftrightarrow i]}\tilde{L}T_{[p\leftrightarrow i]}\right)\left(T_{[p\leftrightarrow i]}\tilde{U}T_{[p\leftrightarrow i]}\right) \\ &\quad \times \left(T_{[p\leftrightarrow i]}\tilde{I}T_{[i\leftrightarrow q]}\right)\left(T_{[i\leftrightarrow q]}\tilde{K}T_{[i\leftrightarrow q]}\right)\left(T_{[i\leftrightarrow q]}\tilde{S}\right) \\ &= \left(\tilde{P}T_{[p\leftrightarrow i]}\right)\tilde{D}\left(T_{[p\leftrightarrow i]}\tilde{L}T_{[p\leftrightarrow i]}\right)\tilde{U} \\ &\quad \times \left(T_{[p\leftrightarrow i]}\tilde{I}T_{[i\leftrightarrow q]}\right)\tilde{K}\left(T_{[i\leftrightarrow q]}\tilde{S}\right), \end{aligned}$$

[6] A typical Intel or AMD x86-class computer processor represents a C/C++ `double`-type floating-point number, $x = 2^p b$, in 0x40 bits of computer memory. Of the 0x40 bits, 0x34 are for the number's mantissa $2.0 \leq b < 4.0$ (not $1.0 \leq b < 2.0$ as one might expect), 0xB are for the number's exponent $-0\text{x}3\text{FF} \leq p \leq 0\text{x}3\text{FE}$, and one is for the number's $\pm$ sign. (The mantissa's high-order bit, which is always 1, is implied not stored, thus is one neither of the 0x34 nor of the 0x40 bits.) The out-of-bounds exponents $p = -0\text{x}400$ and $p = 0\text{x}3\text{FF}$ serve specially respectively to encode 0 and $\infty$. All this is standard computing practice. Such a floating-point representation is easily accurate enough for most practical purposes, but of course it is not generally exact. [33, § 1-4.2.2]

[7] The Gauss-Jordan's floating-point errors come mainly from dividing by small pivots. Such errors are naturally avoided by avoiding small pivots, at least until as late in the algorithm as possible. Smallness however is relative: a small pivot in a row and a column each populated by even smaller elements is unlikely to cause as much error as is a large pivot in a row and a column each populated by even larger elements.

To choose a pivot, any of several heuristics are reasonable. The following heuristic if programmed intelligently might not be too computationally expensive: Define the pivot-smallness metric

$$\tilde{\eta}_{pq}^2 \equiv \frac{2\tilde{\imath}_{pq}^* \tilde{\imath}_{pq}}{\sum_{p'=i}^{m} \tilde{\imath}_{p'q}^* \tilde{\imath}_{p'q} + \sum_{q'=i}^{n} \tilde{\imath}_{pq'}^* \tilde{\imath}_{pq'}}.$$

Choose the $p$ and $q$ of least $\tilde{\eta}_{pq}^2$. If two are equally least, then choose first the lesser column index $q$, then if necessary the lesser row index $p$.

let

$$\begin{aligned}\tilde{P} &\leftarrow \tilde{P}T_{[p\leftrightarrow i]},\\ \tilde{L} &\leftarrow T_{[p\leftrightarrow i]}\tilde{L}T_{[p\leftrightarrow i]},\\ \tilde{I} &\leftarrow T_{[p\leftrightarrow i]}\tilde{I}T_{[i\leftrightarrow q]},\\ \tilde{S} &\leftarrow T_{[i\leftrightarrow q]}\tilde{S},\end{aligned}$$

thus interchanging the $p$th with the $i$th row and the $q$th with the $i$th column, to bring the chosen element to the pivot position. (Refer to Table 12.1 for the similarity transformations. The $\tilde{U}$ and $\tilde{K}$ transformations disappear because at this stage of the algorithm, still $\tilde{U} = \tilde{K} = I$. The $\tilde{D}$ transformation disappears because $p \geq i$ and because $\tilde{d}_{kk} = 1$ for all $k \geq i$. Regarding the $\tilde{L}$ transformation, it does not disappear, but $\tilde{L}$ has major partial unit triangular form $L^{\{i-1\}}$, which form according to Table 12.1 it retains since $i - 1 < i \leq p$.)

5. Observing that (12.4) can be expanded to read

$$\begin{aligned}A &= \tilde{P}\left(\tilde{D}T_{\tilde{\imath}_{ii}[i]}\right)\left(T_{(1/\tilde{\imath}_{ii})[i]}\tilde{L}T_{\tilde{\imath}_{ii}[i]}\right)\left(T_{(1/\tilde{\imath}_{ii})[i]}\tilde{U}T_{\tilde{\imath}_{ii}[i]}\right)\\ &\quad\times\left(T_{(1/\tilde{\imath}_{ii})[i]}\tilde{I}\right)\tilde{K}\tilde{S}\\ &= \tilde{P}\left(\tilde{D}T_{\tilde{\imath}_{ii}[i]}\right)\left(T_{(1/\tilde{\imath}_{ii})[i]}\tilde{L}T_{\tilde{\imath}_{ii}[i]}\right)\tilde{U}\left(T_{(1/\tilde{\imath}_{ii})[i]}\tilde{I}\right)\tilde{K}\tilde{S},\end{aligned}$$

normalize the new $\tilde{\imath}_{ii}$ pivot by letting

$$\begin{aligned}\tilde{D} &\leftarrow \tilde{D}T_{\tilde{\imath}_{ii}[i]},\\ \tilde{L} &\leftarrow T_{(1/\tilde{\imath}_{ii})[i]}\tilde{L}T_{\tilde{\imath}_{ii}[i]},\\ \tilde{I} &\leftarrow T_{(1/\tilde{\imath}_{ii})[i]}\tilde{I}.\end{aligned}$$

This forces $\tilde{\imath}_{ii} = 1$. It also changes the value of $\tilde{d}_{ii}$. Pictorially after

this step,

$$\tilde{D} = \begin{bmatrix} \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \\ \cdots & * & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\ \cdots & 0 & * & 0 & 0 & 0 & 0 & 0 & \cdots \\ \cdots & 0 & 0 & * & 0 & 0 & 0 & 0 & \cdots \\ \cdots & 0 & 0 & 0 & * & 0 & 0 & 0 & \cdots \\ \cdots & 0 & 0 & 0 & 0 & 1 & 0 & 0 & \cdots \\ \cdots & 0 & 0 & 0 & 0 & 0 & 1 & 0 & \cdots \\ \cdots & 0 & 0 & 0 & 0 & 0 & 0 & 1 & \cdots \\ & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix},$$

$$\tilde{I} = \begin{bmatrix} \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \\ \cdots & 1 & * & * & * & * & * & * & \cdots \\ \cdots & 0 & 1 & * & * & * & * & * & \cdots \\ \cdots & 0 & 0 & 1 & * & * & * & * & \cdots \\ \cdots & 0 & 0 & 0 & 1 & * & * & * & \cdots \\ \cdots & 0 & 0 & 0 & * & * & * & * & \cdots \\ \cdots & 0 & 0 & 0 & * & * & * & * & \cdots \\ \cdots & 0 & 0 & 0 & * & * & * & * & \cdots \\ & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}.$$

(Though the step changes $\tilde{L}$, too, again it leaves $\tilde{L}$ in the major partial unit triangular form $L^{\{i-1\}}$, because $i - 1 < i$. Refer to Table 12.1.)

6. Observing that (12.4) can be expanded to read

$$\begin{aligned} A &= \tilde{P}\tilde{D}\left(\tilde{L}T_{\tilde{\imath}_{pi}[pi]}\right)\left(T_{-\tilde{\imath}_{pi}[pi]}\tilde{U}T_{\tilde{\imath}_{pi}[pi]}\right)\left(T_{-\tilde{\imath}_{pi}[pi]}\tilde{I}\right)\tilde{K}\tilde{S} \\ &= \tilde{P}\tilde{D}\left(\tilde{L}T_{\tilde{\imath}_{pi}[pi]}\right)\tilde{U}\left(T_{-\tilde{\imath}_{pi}[pi]}\tilde{I}\right)\tilde{K}\tilde{S}, \end{aligned}$$

clear $\tilde{I}$'s $i$th column below the pivot by letting

$$\tilde{L} \leftarrow \left(\tilde{L}\right)\left(\coprod_{p=i+1}^{m} T_{\tilde{\imath}_{pi}[pi]}\right),$$

$$\tilde{I} \leftarrow \left(\prod_{p=i+1}^{m} T_{-\tilde{\imath}_{pi}[pi]}\right)\left(\tilde{I}\right).$$

This forces $\tilde{\imath}_{ip} = 0$ for all $p > i$. It also fills in $\tilde{L}$'s $i$th column below the pivot, advancing that matrix from the $L^{\{i-1\}}$ form to the $L^{\{i\}}$ form.

Pictorially,

$$\tilde{L} = L^{\{i\}} = \begin{bmatrix} \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \\ \cdots & 1 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\ \cdots & * & 1 & 0 & 0 & 0 & 0 & 0 & \cdots \\ \cdots & * & * & 1 & 0 & 0 & 0 & 0 & \cdots \\ \cdots & * & * & * & 1 & 0 & 0 & 0 & \cdots \\ \cdots & * & * & * & * & 1 & 0 & 0 & \cdots \\ \cdots & * & * & * & * & 0 & 1 & 0 & \cdots \\ \cdots & * & * & * & * & 0 & 0 & 1 & \cdots \\ & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix},$$

$$\tilde{I} = \begin{bmatrix} \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \\ \cdots & 1 & * & * & * & * & * & * & \cdots \\ \cdots & 0 & 1 & * & * & * & * & * & \cdots \\ \cdots & 0 & 0 & 1 & * & * & * & * & \cdots \\ \cdots & 0 & 0 & 0 & 1 & * & * & * & \cdots \\ \cdots & 0 & 0 & 0 & 0 & * & * & * & \cdots \\ \cdots & 0 & 0 & 0 & 0 & * & * & * & \cdots \\ \cdots & 0 & 0 & 0 & 0 & * & * & * & \cdots \\ & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}.$$

(Note that it is not necessary actually to apply the addition elementaries here one by one. Together they easily form an addition quasielementary $L_{[i]}$, thus can be applied all at once. See § 11.7.3.)

7. Increment

$$i \leftarrow i + 1.$$

Go back to step 2.

8. Decrement

$$i \leftarrow i - 1$$

to undo the last instance of step 7 (even if there never was an instance of step 7), thus letting $i$ point to the matrix's last nonzero row. After decrementing, let the rank

$$r \equiv i.$$

Notice that, certainly, $r \leq m$ and $r \leq n$.

9. (Besides arriving at this point from step 8 above, the algorithm also reënters here from step 11 below.) If $i = 0$, then skip directly to step 12.

10. Observing that (12.4) can be expanded to read

$$A = \tilde{P}\tilde{D}\tilde{L}\left(\tilde{U}T_{\tilde{i}_{pi}[pi]}\right)\left(T_{-\tilde{i}_{pi}[pi]}\tilde{I}\right)\tilde{K}\tilde{S},$$

clear $\tilde{I}$'s $i$th column above the pivot by letting

$$\tilde{U} \leftarrow \left(\tilde{U}\right)\left(\prod_{p=1}^{i-1} T_{\tilde{i}_{pi}[pi]}\right),$$
$$\tilde{I} \leftarrow \left(\coprod_{p=1}^{i-1} T_{-\tilde{i}_{pi}[pi]}\right)\left(\tilde{I}\right).$$

This forces $\tilde{i}_{ip} = 0$ for all $p \neq i$. It also fills in $\tilde{U}$'s $i$th column above the pivot, advancing that matrix from the $U^{\{i+1\}}$ form to the $U^{\{i\}}$ form. Pictorially,

$$\tilde{U} = U^{\{i\}} = \begin{bmatrix} \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \\ \cdots & 1 & 0 & 0 & * & * & * & * & \cdots \\ \cdots & 0 & 1 & 0 & * & * & * & * & \cdots \\ \cdots & 0 & 0 & 1 & * & * & * & * & \cdots \\ \cdots & 0 & 0 & 0 & 1 & * & * & * & \cdots \\ \cdots & 0 & 0 & 0 & 0 & 1 & * & * & \cdots \\ \cdots & 0 & 0 & 0 & 0 & 0 & 1 & * & \cdots \\ \cdots & 0 & 0 & 0 & 0 & 0 & 0 & 1 & \cdots \\ & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix},$$

$$\tilde{I} = \begin{bmatrix} \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \\ \cdots & 1 & * & * & 0 & 0 & 0 & 0 & \cdots \\ \cdots & 0 & 1 & * & 0 & 0 & 0 & 0 & \cdots \\ \cdots & 0 & 0 & 1 & 0 & 0 & 0 & 0 & \cdots \\ \cdots & 0 & 0 & 0 & 1 & 0 & 0 & 0 & \cdots \\ \cdots & 0 & 0 & 0 & 0 & 1 & 0 & 0 & \cdots \\ \cdots & 0 & 0 & 0 & 0 & 0 & 1 & 0 & \cdots \\ \cdots & 0 & 0 & 0 & 0 & 0 & 0 & 1 & \cdots \\ & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}.$$

(As in step 6, here again it is not necessary actually to apply the addition elementaries one by one. Together they easily form an addition quasielementary $U_{[i]}$. See § 11.7.3.)

11. Decrement $i \leftarrow i - 1$. Go back to step 9.

12. Notice that $\tilde{I}$ now has the form of a rank-$r$ identity matrix, except with $n-r$ extra columns dressing its right edge (often $r=n$ however; then there are no extra columns). Pictorially,

$$\tilde{I} = \begin{bmatrix} \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \\ \cdots & 1 & 0 & 0 & 0 & * & * & * & \cdots \\ \cdots & 0 & 1 & 0 & 0 & * & * & * & \cdots \\ \cdots & 0 & 0 & 1 & 0 & * & * & * & \cdots \\ \cdots & 0 & 0 & 0 & 1 & * & * & * & \cdots \\ \cdots & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\ \cdots & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\ \cdots & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\ & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}.$$

Observing that (12.4) can be expanded to read

$$A = \tilde{P}\tilde{D}\tilde{L}\tilde{U}\left(\tilde{I}T_{-\tilde{\imath}_{pq}[pq]}\right)\left(T_{\tilde{\imath}_{pq}[pq]}\tilde{K}\right)\tilde{S},$$

use the now conveniently elementarized columns of $\tilde{I}$'s main body to suppress the extra columns on its right edge by

$$\tilde{I} \leftarrow \left(\tilde{I}\right)\left(\prod_{q=r+1}^{n}\coprod_{p=1}^{r} T_{-\tilde{\imath}_{pq}[pq]}\right),$$
$$\tilde{K} \leftarrow \left(\coprod_{q=r+1}^{n}\prod_{p=1}^{r} T_{\tilde{\imath}_{pq}[pq]}\right)\left(\tilde{K}\right).$$

(Actually, entering this step, it was that $\tilde{K}=I$, so in fact $\tilde{K}$ becomes just the product above. As in steps 6 and 10, here again it is not necessary actually to apply the addition elementaries one by one. Together they easily form a parallel unit upper—not lower—triangular matrix $L_{\|}^{\{r\}T}$. See § 11.8.4.)

13. Notice now that $\tilde{I}=I_r$. Let

$$P \equiv \tilde{P},\ D \equiv \tilde{D},\ L \equiv \tilde{L},\ U \equiv \tilde{U},\ K \equiv \tilde{K},\ S \equiv \tilde{S}.$$

End.

Never stalling, the algorithm cannot fail to achieve $\tilde{I}=I_r$ and thus a complete Gauss-Jordan decomposition of the form (12.2), though what value

the rank $r$ might turn out to have is not normally known to us in advance. (We have not yet proven, but will in § 12.5, that the algorithm always produces the same $I_r$, the same rank $r \geq 0$, regardless of which pivots $\tilde{\imath}_{pq} \neq 0$ one happens to choose in step 3 along the way. We can safely ignore this unproven fact however for the immediate moment.)

### 12.3.4 Rank and independent rows

Observe that the Gauss-Jordan algorithm of § 12.3.3 operates always within the bounds of the original $m \times n$ matrix $A$. Therefore, necessarily,

$$\begin{aligned} r &\leq m, \\ r &\leq n. \end{aligned} \tag{12.5}$$

The rank $r$ exceeds the number neither of the matrix's rows nor of its columns. This is unsurprising. Indeed the narrative of the algorithm's step 8 has already noticed the fact.

Observe also however that *the rank always fully reaches $r = m$ if the rows of the original matrix $A$ are linearly independent.* The reason for this observation is that the rank can fall short, $r < m$, only if step 3 finds a null row $i \leq m$; but step 3 can find such a null row only if step 6 has created one (or if there were a null row in the original matrix $A$; but according to § 12.1, such null rows never were linearly independent in the first place). How do we know that step 6 can never create a null row? We know this because the action of step 6 is to add multiples only of *current and earlier* pivot rows to rows in $\tilde{I}$ which have not yet been on pivot.[8] According to (12.1), such action has no power to cancel the independent rows it targets.

[8] If the truth of the sentence's assertion regarding the action of step 6 seems nonobvious, one can drown the assertion rigorously in symbols to prove it, but before going to that extreme consider: The action of steps 3 and 4 is to choose a pivot row $p \geq i$ and to shift it upward to the $i$th position. The action of step 6 then is to add multiples of the chosen pivot row downward only—that is, only to rows which have not yet been on pivot. This being so, steps 3 and 4 in the second iteration find no unmixed rows available to choose as second pivot, but find only rows which already include multiples of the first pivot row. Step 6 in the second iteration therefore adds downward multiples of the second pivot row, *which already includes a multiple of the first pivot row.* Step 6 in the $i$th iteration adds downward multiples of the $i$th pivot row, which already includes multiples of the first through $(i-1)$th. So it comes to pass that multiples only of current and earlier pivot rows are added to rows which have not yet been on pivot. To no row is ever added, directly or indirectly, a multiple of itself—until step 10, which does not belong to the algorithm's main loop and has nothing to do with the availability of nonzero rows to step 3.

### 12.3.5 Inverting the factors

Inverting the six Gauss-Jordan factors is easy. Sections 11.7 and 11.8 have shown how. One need not however go even to that much trouble. Each of the six factors—$P$, $D$, $L$, $U$, $K$ and $S$—is composed of a sequence $\prod T$ of elementary operators. Each of the six inverse factors—$P^{-1}$, $D^{-1}$, $L^{-1}$, $U^{-1}$, $K^{-1}$ and $S^{-1}$—is therefore composed of the *reverse* sequence $\coprod T^{-1}$ of *inverse* elementary operators. Refer to (11.41). If one merely records the sequence of elementaries used to build each of the six factors—if one reverses each sequence, inverts each elementary, and multiplies—then the six inverse factors result.

And, in fact, it isn't even that hard. One actually need not record the individual elementaries; one can invert, multiply and forget them in stream. This means starting the algorithm from step 1 with six extra variable working matrices (besides the seven already there):

$$\tilde{P}^{-1} \leftarrow I;\ \tilde{D}^{-1} \leftarrow I;\ \tilde{L}^{-1} \leftarrow I;\ \tilde{U}^{-1} \leftarrow I;\ \tilde{K}^{-1} \leftarrow I;\ \tilde{S}^{-1} \leftarrow I.$$

(There is no $\tilde{I}^{-1}$, not because it would not be useful, but because its initial value would be[9] $A^{-1(r)}$, unknown at algorithm's start.) Then, for each operation on any of $\tilde{P}$, $\tilde{D}$, $\tilde{L}$, $\tilde{U}$, $\tilde{K}$ or $\tilde{S}$, one operates inversely on the corresponding inverse matrix. For example, in step 5,

$$\begin{aligned}
\tilde{D} &\leftarrow \tilde{D}T_{\tilde{\imath}_{ii}[i]}, & \tilde{D}^{-1} &\leftarrow T_{(1/\tilde{\imath}_{ii})[i]}\tilde{D}^{-1}, \\
\tilde{L} &\leftarrow T_{(1/\tilde{\imath}_{ii})[i]}\tilde{L}T_{\tilde{\imath}_{ii}[i]}, & \tilde{L}^{-1} &\leftarrow T_{(1/\tilde{\imath}_{ii})[i]}\tilde{L}^{-1}T_{\tilde{\imath}_{ii}[i]}, \\
\tilde{I} &\leftarrow T_{(1/\tilde{\imath}_{ii})[i]}\tilde{I}.
\end{aligned}$$

With this simple extension, the algorithm yields all the factors not only of the Gauss-Jordan decomposition (12.2) but simultaneously also of the Gauss-Jordan's complementary form (12.3).

### 12.3.6 Truncating the factors

None of the six factors of (12.2) actually needs to retain its entire extended-operational form (§ 11.3.2). The four factors on the left, row operators, act wholly by their $m \times m$ squares; the two on the right, column operators, by their $n \times n$. Indeed, neither $I_r$ nor $A$ has anything but zeros outside the $m \times n$ rectangle, anyway, so there is nothing for the six operators to act upon beyond those bounds in any event. We can truncate all six operators to dimension-limited forms (§ 11.3.1) for this reason if we want.

[9] Section 11.5 explains the notation.

To truncate the six operators formally, we left-multiply (12.2) by $I_m$ and right-multiply it by $I_n$, obtaining

$$I_m A I_n = I_m P D L U I_r K S I_n.$$

According to § 11.3.6, the $I_m$ and $I_n$ respectively truncate rows and columns, actions which have no effect on $A$ since it is already a dimension-limited $m \times n$ matrix. By successive steps, then,

$$\begin{aligned} A &= I_m P D L U I_r K S I_n \\ &= I_m^7 P D L U I_r^2 K S I_n^3; \end{aligned}$$

and finally, by using (11.31) or (11.42) repeatedly,

$$A = (I_m P I_m)(I_m D I_m)(I_m L I_m)(I_m U I_r)(I_r K I_n)(I_n S I_n), \tag{12.6}$$

where the dimensionalities of the six factors on the equation's right side are respectively $m \times m$, $m \times m$, $m \times m$, $m \times r$, $r \times n$ and $n \times n$. Equation (12.6) expresses any dimension-limited rectangular matrix $A$ as the product of six particularly simple, dimension-limited rectangular factors.

By similar reasoning from (12.2),

$$A = (I_m G_> I_r)(I_r G_< I_n), \tag{12.7}$$

where the dimensionalities of the two factors are $m \times r$ and $r \times n$.

The book will seldom point it out again explicitly, but one can straightforwardly truncate not only the Gauss-Jordan factors but most other factors and operators, too, by the method of this subsection.[10]

[10] Comes the objection, "Black, why do you make it more complicated than it needs to be? For what reason must all your matrices have infinite dimensionality, anyway? They don't do it that way in my other linear algebra book."

It is a fair question. The answer is that this book is a book of applied mathematical theory; and theoretically in the author's view, infinite-dimensional matrices are significantly neater to handle. To append a null row or a null column to a dimension-limited matrix is to alter the matrix in no essential way, nor is there any real difference between $T_{5[21]}$ when it row-operates on a $3 \times p$ matrix and the same elementary when it row-operates on a $4 \times p$. The relevant theoretical constructs ought to reflect such insights. Hence infinite dimensionality.

Anyway, a matrix displaying an infinite field of zeros resembles a shipment delivering an infinite supply of nothing; one need not be too impressed with either. The two matrix forms of § 11.3 manifest the sense that a matrix can represent a linear transformation, whose rank matters; or a reversible row or column operation, whose rank does not. The extended-operational form, having infinite rank, serves the latter case. In either case, however, the dimensionality $m \times n$ of the matrix is a distraction. It is the rank $r$, if any, that counts.

Table 12.2: A few properties of the Gauss-Jordan factors.

$$P^* = P^{-1} = P^T$$
$$S^* = S^{-1} = S^T$$
$$P^{-*} = P = P^{-T}$$
$$S^{-*} = S = S^{-T}$$
$$\frac{K + K^{-1}}{2} = I$$

$$\begin{aligned}
I_r K (I_n - I_r) &= K - I &&= I_r (K - I)(I_n - I_r) \\
I_r K^{-1} (I_n - I_r) &= K^{-1} - I &&= I_r (K^{-1} - I)(I_n - I_r) \\
(I - I_n)(K - I) &= 0 &&= (K - I)(I - I_n) \\
(I - I_n)(K^{-1} - I) &= 0 &&= (K^{-1} - I)(I - I_n)
\end{aligned}$$

### 12.3.7 Properties of the factors

One would expect such neatly formed operators as the factors of the Gauss-Jordan to enjoy some useful special properties. Indeed they do. Table 12.2 lists a few. The table's properties formally come from (11.52) and Table 11.5; but, if one firmly grasps the matrix forms involved and comprehends the notation (neither of which is trivial to do), if one understands that the operator $(I_n - I_r)$ is a truncator that selects columns $r+1$ through $n$ of the matrix it operates leftward upon, and if one sketches the relevant factors schematically with a pencil, then the properties are plainly seen without reference to Ch. 11 as such.

The table's properties regarding $P$ and $S$ express a general advantage all permutors share. The table's properties regarding $K$ are admittedly less significant, included mostly only because § 13.3 will need them. Still, even the $K$ properties are always true. They might find other uses.

Further properties of the several Gauss-Jordan factors can be gleaned from the respectively relevant subsections of §§ 11.7 and 11.8.

### 12.3.8 Marginalizing the factor $I_n$

If $A$ happens to be a square, $n \times n$ matrix and if it develops that the rank $r = n$, then one can take advantage of (11.31) to rewrite the Gauss-Jordan

decomposition (12.2) in the form

$$PDLUKSI_n = A = I_nPDLUKS, \tag{12.8}$$

thus marginalizing the factor $I_n$. This is to express the Gauss-Jordan solely in row operations or solely in column operations. It does not change the algorithm and it does not alter the factors; it merely reorders the factors after the algorithm has determined them. It fails however if $A$ is rectangular or $r < n$.

### 12.3.9 Decomposing an extended operator

Once one has derived the Gauss-Jordan decomposition, to extend it to decompose a reversible, $n \times n$ extended operator $A$ (where per § 11.3.2 $A$ outside the $n \times n$ active region resembles the infinite-dimensional identity matrix $I$) is trivial. One merely writes

$$A = PDLUKS,$$

wherein the $I_r$ has become an $I$. Or, equivalently, one decomposes the $n \times n$ dimension-limited matrix $I_nA = I_nAI_n = AI_n$ as

$$AI_n = PDLUI_nKS = PDLUKSI_n,$$

from which, inasmuch as all the factors present but $I_n$ are $n \times n$ extended operators, the preceding equation results.

One can decompose only reversible extended operators so. The Gauss-Jordan fails on irreversible extended operators, for which the rank of the truncated form $AI_n$ is $r < n$. See § 12.5.

This subsection's equations remain unnumbered because they say little new. Their only point, really, is that what an operator does outside some appropriately delimited active region is seldom interesting, because the vector on which the operator ultimately acts is probably null there in any event. In such a context it may not matter whether one truncates the operator. Indeed this was also the point of § 12.3.6 and, if you like, of (11.31), too.[11]

[11] If "it may not matter," as the narrative says, then one might just put all matrices in dimension-limited form. Many books do. To put them all in dimension-limited form however brings at least three effects the book you are reading prefers to avoid. First, it leaves shift-and-truncate operations hard to express cleanly (refer to §§ 11.3.6 and 11.9 and, as a typical example of the usage, eqn. 13.7). Second, it confuses the otherwise natural extension of discrete vectors into continuous functions. Third, it leaves one to consider the ranks of reversible operators like $T_{[1\leftrightarrow 2]}$ that naturally should have no rank. The last

Regarding the present section as a whole, the Gauss-Jordan decomposition is a significant achievement. It is not the only matrix decomposition—further interesting decompositions include the Gram-Schmidt of § 13.11, the diagonal of § 14.6, the Schur of § 14.10 and the singular-value of § 14.12, among others—but the Gauss-Jordan nonetheless reliably factors an arbitrary $m \times n$ matrix $A$, which we had not known how to handle very well, into a product of unit triangular matrices and quasielementaries, which we do. We will put the Gauss-Jordan to good use in Ch. 13. However, before closing the present chapter we should like finally, squarely to define and to establish the concept of matrix rank, not only for $I_r$ but for all matrices. To do that, we shall first need one more preliminary: the technique of vector replacement.

## 12.4 Vector replacement

Consider a set of $m+1$ (not necessarily independent) vectors

$$\{\mathbf{u}, \mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_m\}.$$

As a definition, the *space* these vectors *address* consists of all linear combinations of the set's several vectors. That is, the space consists of all vectors $\mathbf{b}$ formable as

$$\beta_o\mathbf{u} + \beta_1\mathbf{a}_1 + \beta_2\mathbf{a}_2 + \cdots + \beta_m\mathbf{a}_m = \mathbf{b}. \tag{12.9}$$

Now consider a specific vector $\mathbf{v}$ in the space,

$$\psi_o\mathbf{u} + \psi_1\mathbf{a}_1 + \psi_2\mathbf{a}_2 + \cdots + \psi_m\mathbf{a}_m = \mathbf{v}, \tag{12.10}$$

for which

$$\psi_o \neq 0.$$

Solving (12.10) for $\mathbf{u}$, we find that

$$\frac{1}{\psi_o}\mathbf{v} - \frac{\psi_1}{\psi_o}\mathbf{a}_1 - \frac{\psi_2}{\psi_o}\mathbf{a}_2 - \cdots - \frac{\psi_m}{\psi_o}\mathbf{a}_m = \mathbf{u}.$$

of the three is arguably most significant: matrix rank is such an important attribute that one prefers to impute it only to those operators about which it actually says something interesting.

Nevertheless, the extended-operational matrix form is hardly more than a formality. All it says is that the extended operator unobtrusively leaves untouched anything it happens to find outside its operational domain, whereas a dimension-limited operator would have truncated whatever it found there. Since what is found outside the operational domain is often uninteresting, this may be a distinction without a difference, which one can safely ignore.

With the change of variables

$$\begin{aligned}
\phi_o &\leftarrow \frac{1}{\psi_o}, \\
\phi_1 &\leftarrow -\frac{\psi_1}{\psi_o}, \\
\phi_2 &\leftarrow -\frac{\psi_2}{\psi_o}, \\
&\vdots \\
\phi_m &\leftarrow -\frac{\psi_m}{\psi_o},
\end{aligned}$$

for which, quite symmetrically, it happens that

$$\begin{aligned}
\psi_o &= \frac{1}{\phi_o}, \\
\psi_1 &= -\frac{\phi_1}{\phi_o}, \\
\psi_2 &= -\frac{\phi_2}{\phi_o}, \\
&\vdots \\
\psi_m &= -\frac{\phi_m}{\phi_o},
\end{aligned}$$

the solution is

$$\phi_o \mathbf{v} + \phi_1 \mathbf{a}_1 + \phi_2 \mathbf{a}_2 + \cdots + \phi_m \mathbf{a}_m = \mathbf{u}. \tag{12.11}$$

Equation (12.11) has identical form to (12.10), only with the symbols $\mathbf{u} \leftrightarrow \mathbf{v}$ and $\psi \leftrightarrow \phi$ swapped. Since $\phi_o = 1/\psi_o$, assuming that $\psi_o$ is finite it even appears that

$$\phi_o \neq 0;$$

so, the symmetry is complete. Table 12.3 summarizes.

Now further consider an arbitrary vector $\mathbf{b}$ which lies in the space addressed by the vectors

$$\{\mathbf{u}, \mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_m\}.$$

Does the same $\mathbf{b}$ also lie in the space addressed by the vectors

$$\{\mathbf{v}, \mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_m\}?$$

Table 12.3: The symmetrical equations of § 12.4.

$$\begin{array}{rcl|rcl}
\psi_o\mathbf{u}+\psi_1\mathbf{a}_1+\psi_2\mathbf{a}_2 & & & \phi_o\mathbf{v}+\phi_1\mathbf{a}_1+\phi_2\mathbf{a}_2 & & \\
\quad +\cdots+\psi_m\mathbf{a}_m & = & \mathbf{v} & \quad +\cdots+\phi_m\mathbf{a}_m & = & \mathbf{u} \\
0 \neq \dfrac{1}{\psi_o} & = & \phi_o & 0 \neq \dfrac{1}{\phi_o} & = & \psi_o \\
-\dfrac{\psi_1}{\psi_o} & = & \phi_1 & -\dfrac{\phi_1}{\phi_o} & = & \psi_1 \\
-\dfrac{\psi_2}{\psi_o} & = & \phi_2 & -\dfrac{\phi_2}{\phi_o} & = & \psi_2 \\
\vdots & & & \vdots & & \\
-\dfrac{\psi_m}{\psi_o} & = & \phi_m & -\dfrac{\phi_m}{\phi_o} & = & \psi_m
\end{array}$$

To show that it does, we substitute into (12.9) the expression for $\mathbf{u}$ from (12.11), obtaining the form

$$(\beta_o)(\phi_o\mathbf{v}+\phi_1\mathbf{a}_1+\phi_2\mathbf{a}_2+\cdots+\phi_m\mathbf{a}_m)+\beta_1\mathbf{a}_1+\beta_2\mathbf{a}_2+\cdots+\beta_m\mathbf{a}_m=\mathbf{b}.$$

Collecting terms, this is

$$\beta_o\phi_o\mathbf{v}+(\beta_o\phi_1+\beta_1)\mathbf{a}_1+(\beta_o\phi_2+\beta_2)\mathbf{a}_2+\cdots+(\beta_o\phi_m+\beta_m)\mathbf{a}_m=\mathbf{b},$$

in which we see that, yes, $\mathbf{b}$ does indeed also lie in the latter space. Naturally the problem's $\mathbf{u} \leftrightarrow \mathbf{v}$ symmetry then guarantees the converse, that an arbitrary vector $\mathbf{b}$ which lies in the latter space also lies in the former. Therefore, a vector $\mathbf{b}$ must lie in both spaces or neither, never just in one or the other. The two spaces are, in fact, one and the same.

This leads to the following useful conclusion. Given a set of vectors

$$\{\mathbf{u}, \mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_m\},$$

one can safely replace the $\mathbf{u}$ by a new vector $\mathbf{v}$, obtaining the new set

$$\{\mathbf{v}, \mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_m\},$$

provided that the replacement vector $\mathbf{v}$ includes at least a little of the replaced vector $\mathbf{u}$ ($\psi_o \neq 0$ in eqn. 12.10) and that $\mathbf{v}$ is otherwise an honest

linear combination of the several vectors of the original set, untainted by foreign contribution. *Such vector replacement does not in any way alter the space addressed.* The new space is exactly the same as the old.

As a corollary, *if the vectors of the original set happen to be linearly independent (§ 12.1), then the vectors of the new set are linearly independent, too;* for, if it were that

$$\gamma_o \mathbf{v} + \gamma_1 \mathbf{a}_1 + \gamma_2 \mathbf{a}_2 + \cdots + \gamma_m \mathbf{a}_m = 0$$

for nontrivial $\gamma_o$ and $\gamma_k$, then either $\gamma_o = 0$—impossible since that would make the several $\mathbf{a}_k$ themselves linearly dependent—or $\gamma_o \neq 0$, in which case $\mathbf{v}$ would be a linear combination of the several $\mathbf{a}_k$ alone. But if $\mathbf{v}$ were a linear combination of the several $\mathbf{a}_k$ alone, then (12.10) would still also explicitly make $\mathbf{v}$ a linear combination of the same $\mathbf{a}_k$ plus a nonzero multiple of $\mathbf{u}$. Yet both combinations cannot be, because according to § 12.1, two distinct combinations of independent vectors can never target the same $\mathbf{v}$. The contradiction proves false the assumption which gave rise to it: that the vectors of the new set were linearly dependent. Hence the vectors of the new set are equally as independent as the vectors of the old.

## 12.5 Rank

Sections 11.3.5 and 11.3.6 have introduced the rank-$r$ identity matrix $I_r$, where the integer $r$ is the number of ones the matrix has along its main diagonal. Other matrices have rank, too. Commonly, an $n \times n$ matrix has rank $r = n$, but consider the matrix

$$\begin{bmatrix} 5 & 1 & 6 \\ 3 & 6 & 9 \\ 2 & 4 & 6 \end{bmatrix}.$$

The third column of this matrix is the sum of the first and second columns. Also, the third row is just two-thirds the second. Either way, by columns or by rows, the matrix has only two independent vectors. The rank of this $3 \times 3$ matrix is not $r = 3$ but $r = 2$.

This section establishes properly the important concept of matrix *rank.* The section demonstrates that every matrix has a definite, unambiguous rank, and shows how this rank can be calculated.

To forestall potential confusion in the matter, we should immediately observe that—like the rest of this chapter but unlike some other parts of the book—this section explicitly trades in exact numbers. If a matrix element

here is 5, then it is exactly 5; if 0, then exactly 0. Many real-world matrices, of course—especially matrices populated by measured data—can never truly be exact, but that is not the point here. Here, the numbers are exact.[12]

### 12.5.1 A logical maneuver

In § 12.5.2 we will execute a pretty logical maneuver, which one might name, "the end justifies the means."[13] When embedded within a larger logical construct as in § 12.5.2, the maneuver if unexpected can confuse, so this subsection is to prepare the reader to expect the maneuver.

The logical maneuver follows this basic pattern.

> If $P_1$ then $Q$. If $P_2$ then $Q$. If $P_3$ then $Q$. Which of $P_1$, $P_2$ and $P_3$ are true is not known, but what is known is that *at least one* of the three is true: $P_1$ or $P_2$ or $P_3$. Therefore, although one can draw no valid conclusion regarding any one of the three predicates $P_1$, $P_2$ or $P_3$, one can still conclude that their common object $Q$ is true.

One valid way to prove $Q$, then, would be to suppose $P_1$ and show that it led to $Q$, then alternately to suppose $P_2$ and show that it separately led to $Q$, then again to suppose $P_3$ and show that it also led to $Q$. The final step would be to show somehow that $P_1$, $P_2$ and $P_3$ could not possibly all be false at once. Herein, the *means* is to assert several individually suspicious claims, none of which one actually means to prove. The *end* which justifies the means is the conclusion $Q$, which thereby one can and does prove.

It is a subtle maneuver. Once the reader feels that he grasps its logic, he can proceed to the next subsection where the maneuver is put to use.

[12] It is false to suppose that because applied mathematics *permits* imprecise quantities, like $3.0 \pm 0.1$ inches for the length of your thumb, it also requires them. On the contrary, the length of your thumb may indeed be $3.0 \pm 0.1$ inches, but surely no triangle has $3.0 \pm 0.1$ sides! A triangle has exactly three sides. The ratio of a circle's circumference to its radius is exactly $2\pi$. The author has exactly one brother. A construction contract might require the builder to finish within exactly 180 days (though the actual construction time might be an inexact $t = 172.6 \pm 0.2$ days), and so on. Exact quantities are every bit as valid in applied mathematics as imprecise ones are. Where the distinction matters, it is the applied mathematician's responsibility to distinguish between the two kinds of quantity.

[13] The maneuver's name rings a bit sinister, does it not? The author recommends no such maneuver in social or family life! Logically here however it helps the math.

### 12.5.2 The impossibility of identity-matrix promotion

Consider the matrix equation

$$AI_rB = I_s. \tag{12.12}$$

If $r \geq s$, then it is trivial to find matrices $A$ and $B$ for which (12.12) holds: $A = I_s = B$. If

$$r < s,$$

however, it is not so easy. In fact it is impossible. This subsection proves the impossibility. It shows that *one cannot by any row and column operations, reversible or otherwise, ever transform an identity matrix into another identity matrix of greater rank (§ 11.3.5).*

Equation (12.12) can be written in the form

$$(AI_r)B = I_s, \tag{12.13}$$

where, because $I_r$ attacking from the right is the column truncation operator (§ 11.3.6), the product $AI_r$ is a matrix with an unspecified number of rows but only $r$ columns—or, more precisely, with no more than $r$ nonzero columns. Viewed this way, per § 11.1.3, $B$ operates on the $r$ columns of $AI_r$ to produce the $s$ columns of $I_s$.

The $r$ columns of $AI_r$ are nothing more than the first through $r$th columns of $A$. Let the symbols $\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3, \mathbf{a}_4, \mathbf{a}_5, \ldots, \mathbf{a}_r$ denote these columns. The $s$ columns of $I_s$, then, are nothing more than the elementary vectors $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3, \mathbf{e}_4, \mathbf{e}_5, \ldots, \mathbf{e}_s$ (§ 11.3.7). The claim (12.13) makes is thus that the several vectors $\mathbf{a}_k$ together address each of the several elementary vectors $\mathbf{e}_j$—that is, that a linear combination[14]

$$b_{1j}\mathbf{a}_1 + b_{2j}\mathbf{a}_2 + b_{3j}\mathbf{a}_3 + \cdots + b_{rj}\mathbf{a}_r = \mathbf{e}_j \tag{12.14}$$

exists for each $\mathbf{e}_j$, $1 \leq j \leq s$.

The claim (12.14) will turn out to be false because there are too many $\mathbf{e}_j$, but to prove this, *we shall assume for the moment that the claim were true.* The proof then is by contradiction,[15] and it runs as follows.

[14] Observe that unlike as in § 12.1, here we have not necessarily assumed that the several $\mathbf{a}_k$ are linearly independent.

[15] As the reader will have observed by this point in the book, the technique—also called *reductio ad absurdum*—is the usual mathematical technique to prove impossibility. One assumes the falsehood to be true, then reasons toward a contradiction which proves the assumption false. Section 6.1.1 among others has already illustrated the technique, but the technique's use here is more sophisticated.

Consider the elementary vector $\mathbf{e}_1$. For $j = 1$, (12.14) is

$$b_{11}\mathbf{a}_1 + b_{21}\mathbf{a}_2 + b_{31}\mathbf{a}_3 + \cdots + b_{r1}\mathbf{a}_r = \mathbf{e}_1,$$

which says that the elementary vector $\mathbf{e}_1$ is a linear combination of the several vectors

$$\{\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3, \mathbf{a}_4, \mathbf{a}_5, \ldots, \mathbf{a}_r\}.$$

Because $\mathbf{e}_1$ is a linear combination, according to § 12.4 one can safely replace any of the vectors in the set by $\mathbf{e}_1$ without altering the space addressed. For example, replacing $\mathbf{a}_1$ by $\mathbf{e}_1$,

$$\{\mathbf{e}_1, \mathbf{a}_2, \mathbf{a}_3, \mathbf{a}_4, \mathbf{a}_5, \ldots, \mathbf{a}_r\}.$$

The only restriction per § 12.4 is that $\mathbf{e}_1$ contain at least a little of the vector $\mathbf{a}_k$ it replaces—that $b_{k1} \neq 0$. Of course there is no guarantee specifically that $b_{11} \neq 0$, so for $\mathbf{e}_1$ to replace $\mathbf{a}_1$ might not be allowed. However, inasmuch as $\mathbf{e}_1$ is nonzero, then according to (12.14) at least one of the several $b_{k1}$ also is nonzero; and if $b_{k1}$ is nonzero then $\mathbf{e}_1$ can replace $\mathbf{a}_k$. Some of the $\mathbf{a}_k$ might indeed be forbidden, but never all; there is always at least one $\mathbf{a}_k$ which $\mathbf{e}_1$ can replace. (For example, if $\mathbf{a}_1$ were forbidden because $b_{11} = 0$, then $\mathbf{a}_3$ might be available instead because $b_{31} \neq 0$. In this case the new set would be $\{\mathbf{a}_1, \mathbf{a}_2, \mathbf{e}_1, \mathbf{a}_4, \mathbf{a}_5, \ldots, \mathbf{a}_r\}$.)

Here comes the hard part. Here is where the logical maneuver of § 12.5.1 comes in. The book to this point has established no general method to tell which of the several $\mathbf{a}_k$ the elementary vector $\mathbf{e}_1$ actually contains (§ 13.2 gives the method, but that section depends logically on this one, so we cannot properly appeal to it here). According to (12.14), the vector $\mathbf{e}_1$ might contain some of the several $\mathbf{a}_k$ or all of them, *but surely it contains at least one of them.* Therefore, even though it is illicit to replace an $\mathbf{a}_k$ by an $\mathbf{e}_1$ which contains none of it, even though we have no idea which of the several $\mathbf{a}_k$ the vector $\mathbf{e}_1$ contains, even though replacing the wrong $\mathbf{a}_k$ logically invalidates any conclusion which flows from the replacement, still we can proceed with the proof—provided only that, in the end, we shall find that the illicit choice of replacement and the licit choice had led alike to the same, identical conclusion. If we do so find then—in the end—the logic will demand of us only an assurance that some licit choice had existed at the time the choice was or might have been made. The logic will never ask, even in retrospect, which specific choice had been the licit one, for only the complete absence of licit choices can threaten the present maneuver.

The claim (12.14) guarantees at least one licit choice. Whether as the maneuver also demands, all the choices, licit and illicit, lead ultimately alike to the same, identical conclusion remains to be determined.

Now consider the elementary vector $\mathbf{e}_2$. According to (12.14), $\mathbf{e}_2$ lies in the space addressed by the original set

$$\{\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3, \mathbf{a}_4, \mathbf{a}_5, \ldots, \mathbf{a}_r\}.$$

Therefore as we have seen, $\mathbf{e}_2$ also lies in the space addressed by the new set

$$\{\mathbf{e}_1, \mathbf{a}_2, \mathbf{a}_3, \mathbf{a}_4, \mathbf{a}_5, \ldots, \mathbf{a}_r\}$$

(or $\{\mathbf{a}_1, \mathbf{a}_2, \mathbf{e}_1, \mathbf{a}_4, \mathbf{a}_5, \ldots, \mathbf{a}_r\}$, or whatever the new set happens to be). That is, not only do coefficients $b_{k2}$ exist such that

$$b_{12}\mathbf{a}_1 + b_{22}\mathbf{a}_2 + b_{32}\mathbf{a}_3 + \cdots + b_{r2}\mathbf{a}_r = \mathbf{e}_2,$$

but also coefficients $\beta_{k2}$ exist such that

$$\beta_{12}\mathbf{e}_1 + \beta_{22}\mathbf{a}_2 + \beta_{32}\mathbf{a}_3 + \cdots + \beta_{r2}\mathbf{a}_r = \mathbf{e}_2.$$

Again it is impossible for all the coefficients $\beta_{k2}$ to be zero but, moreover, it is impossible for $\beta_{12}$ to be the sole nonzero coefficient, for (as should seem plain to the reader who grasps the concept of the elementary vector, § 11.3.7) no elementary vector can ever be a linear combination of other elementary vectors alone! The linear combination which forms $\mathbf{e}_2$ evidently includes a nonzero multiple *of at least one of the remaining* $\mathbf{a}_k$. At least one of the $\beta_{k2}$ attached to an $\mathbf{a}_k$ (not $\beta_{12}$, which is attached to $\mathbf{e}_1$) must be nonzero. Therefore by the same reasoning as before, we now choose an $\mathbf{a}_k$ with a nonzero coefficient $\beta_{k2} \neq 0$ and replace it by $\mathbf{e}_2$, obtaining an even newer set of vectors like

$$\{\mathbf{e}_1, \mathbf{a}_2, \mathbf{a}_3, \mathbf{e}_2, \mathbf{a}_5, \ldots, \mathbf{a}_r\}.$$

This newer set addresses precisely the same space as the previous set, thus also as the original set.

And so it goes, replacing one $\mathbf{a}_k$ by an $\mathbf{e}_j$ at a time, until all the $\mathbf{a}_k$ are gone and our set has become

$$\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3, \mathbf{e}_4, \mathbf{e}_5, \ldots, \mathbf{e}_r\},$$

which, as we have reasoned, addresses precisely the same space as did the original set

$$\{\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3, \mathbf{a}_4, \mathbf{a}_5, \ldots, \mathbf{a}_r\}.$$

And this is the one, identical conclusion the maneuver of § 12.5.1 has demanded. All intermediate choices, by various paths licit and illicit, ultimately have led alike to the single conclusion of this paragraph, which thereby is properly established.

Admittedly, such subtle logic may not be easy to discern. Here is another, slightly different light by which to illuminate the question. Suppose again that, by making exactly $r$ replacements, we wish to convert the set

$$\{\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3, \mathbf{a}_4, \mathbf{a}_5, \ldots, \mathbf{a}_r\}$$

into

$$\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3, \mathbf{e}_4, \mathbf{e}_5, \ldots, \mathbf{e}_r\},$$

assuming again per § 12.4 that the several vectors $\mathbf{a}_k$ of the original set, taken together, address each of the elementary vectors $\mathbf{e}_j$, $1 \leq j \leq s$, $r < s$. Suppose further again that we wish per § 12.4 to convert the set without altering the space the set addresses. To reach our goal, first we will put the elementary vector $\mathbf{e}_1$ in the place of one of the several vectors $\mathbf{a}_k$, then we will put $\mathbf{e}_2$ in the place of *one of the remaining* $\mathbf{a}_k$; and so on until, at last, we put $\mathbf{e}_r$ in the place of *the last remaining* $\mathbf{a}_k$. We will give the several $\mathbf{e}_j$, $1 \leq j \leq r$, in their proper order but we might take the several $\mathbf{a}_k$ in any of $r!$ distinct sequences: for instance, in the case that $r = 3$, we might take the several $\mathbf{a}_k$ in any of the $3! = 6$ distinct sequences

$$(\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3);\ (\mathbf{a}_1, \mathbf{a}_3, \mathbf{a}_2);\ (\mathbf{a}_2, \mathbf{a}_1, \mathbf{a}_3);\ (\mathbf{a}_2, \mathbf{a}_3, \mathbf{a}_1);\ (\mathbf{a}_3, \mathbf{a}_1, \mathbf{a}_2);\ (\mathbf{a}_3, \mathbf{a}_2, \mathbf{a}_1);$$

except however that we might (or might not) find certain sequences blockaded in the event. Blockaded? Well, consider for example the sequence $(\mathbf{a}_2, \mathbf{a}_3, \mathbf{a}_1)$, and suppose that $\mathbf{e}_1 = 0\mathbf{a}_1 + 4\mathbf{a}_2 - 2\mathbf{a}_3$ and that $\mathbf{e}_2 = 5\mathbf{a}_1 - (1/2)\mathbf{e}_1 + 0\mathbf{a}_3$ (noticing that the latter already has $\mathbf{e}_1$ on the right instead of $\mathbf{a}_2$): in this case the sequence $(\mathbf{a}_2, \mathbf{a}_3, \mathbf{a}_1)$ is blockaded—which is to say, forbidden—because, once $\mathbf{e}_1$ has replaced $\mathbf{a}_2$, since $\mathbf{e}_2$ then contains none of $\mathbf{a}_3$, $\mathbf{e}_2$ cannot according to § 12.4 replace $\mathbf{a}_3$. [Actually, in this example, both sequences beginning $(\mathbf{a}_1, \ldots)$ are blockaded, too, because the turn of $\mathbf{e}_1$ comes first and, at that time, $\mathbf{e}_1$ contains none of $\mathbf{a}_1$.] Clear? No? Too many subscripts? Well, there's nothing for it: if you wish to understand then you will simply have to trace all the subscripts out with your pencil; the example cannot be made any simpler. Now, although some sequences might be blockaded, no unblockaded sequence can run to a dead end, so to speak. After each unblockaded replacement another replacement will always be possible. The reason is as before: that, according to § 12.4, so long as each elementary

vector $\mathbf{e}_j$ in turn contains some of the vector it replaces, the replacement cannot alter the space the set addresses; that the space by initial assumption includes all the elementary vectors; that each elementary vector in turn must therefore be found contain at least one of the vectors then in the set; that no elementary vector can be composed solely of other elementary vectors; and, consequently, that each elementary vector in turn must be found to contain at least one of the set's then remaining $\mathbf{a}_k$. The logic though slightly complicated is nonethemore escapable. The conclusion is that we can indeed convert $\{\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3, \mathbf{a}_4, \mathbf{a}_5, \ldots, \mathbf{a}_r\}$ into $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3, \mathbf{e}_4, \mathbf{e}_5, \ldots, \mathbf{e}_r\}$, step by step, without altering the space addressed.

The conclusion leaves us with a problem, however. There remain more $\mathbf{e}_j$, $1 \leq j \leq s$, than there are $\mathbf{a}_k$, $1 \leq k \leq r$, because, as we have stipulated, $r < s$. Some elementary vectors $\mathbf{e}_j$, $r < j \leq s$, are evidently left over. Back at the beginning of the section, the claim (12.14) made was that

$$\{\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3, \mathbf{a}_4, \mathbf{a}_5, \ldots, \mathbf{a}_r\}$$

together addressed each of the several elementary vectors $\mathbf{e}_j$. But as we have seen, this amounts to a claim that

$$\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3, \mathbf{e}_4, \mathbf{e}_5, \ldots, \mathbf{e}_r\}$$

together addressed each of the several elementary vectors $\mathbf{e}_j$. Plainly this is impossible with respect to the left-over $\mathbf{e}_j$, $r < j \leq s$. The contradiction proves false the claim which gave rise to it. The false claim: that the several $\mathbf{a}_k$, $1 \leq k \leq r$, addressed all the $\mathbf{e}_j$, $1 \leq j \leq s$, even when $r < s$.

Equation (12.13), which is just (12.12) written differently, asserts that $B$ is a column operator which does precisely what we have just shown impossible: to combine the $r$ columns of $AI_r$ to yield the $s$ columns of $I_s$, the latter of which are just the elementary vectors $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3, \mathbf{e}_4, \mathbf{e}_5, \ldots, \mathbf{e}_s$. Hence finally we conclude that no matrices $A$ and $B$ exist which satisfy (12.12) when $r < s$. In other words, we conclude that *although row and column operations can demote identity matrices in rank, they can never promote them.* The promotion of identity matrices is impossible.

### 12.5.3 General matrix rank and its uniqueness

Step 8 of the Gauss-Jordan algorithm (§ 12.3.3) discovers a rank $r$ for any matrix $A$. One should like to think that this rank $r$ were a definite property of the matrix itself rather than some unreliable artifact of the algorithm, but until now we have lacked the background theory to prove it. Now we have the theory. Here is the proof.

The proof begins with a formal definition of the quantity whose uniqueness we are trying to prove.

- The rank $r$ of an identity matrix $I_r$ is the number of ones along its main diagonal. (This is from § 11.3.5.)
- The rank $r$ of a general matrix $A$ is the rank of an identity matrix $I_r$ to which $A$ can be reduced by *reversible* row and column operations.

A matrix $A$ has rank $r$ if and only if matrices $B_>$ and $B_<$ exist such that

$$\begin{aligned} B_> A B_< &= I_r, \\ A &= B_>^{-1} I_r B_<^{-1}, \\ B_>^{-1} B_> &= I = B_> B_>^{-1}, \\ B_<^{-1} B_< &= I = B_< B_<^{-1}. \end{aligned} \tag{12.15}$$

The question is whether in (12.15) only a single rank $r$ is possible.

To answer the question, we suppose that another rank were possible, that $A$ had not only rank $r$ but also rank $s$. Then,

$$\begin{aligned} A &= B_>^{-1} I_r B_<^{-1}, \\ A &= G_>^{-1} I_s G_<^{-1}. \end{aligned}$$

Combining these equations,

$$B_>^{-1} I_r B_<^{-1} = G_>^{-1} I_s G_<^{-1}.$$

Solving first for $I_r$, then for $I_s$,

$$\begin{aligned} (B_> G_>^{-1}) I_s (G_<^{-1} B_<) &= I_r, \\ (G_> B_>^{-1}) I_r (B_<^{-1} G_<) &= I_s. \end{aligned}$$

Were it that $r \neq s$, then one of these two equations would constitute the demotion of an identity matrix and the other, a promotion. But according to § 12.5.2 and its (12.12), promotion is impossible. Therefore $r \neq s$ is also impossible, and

$$r = s$$

is guaranteed. No matrix has two different ranks. *Matrix rank is unique.*

This finding has two immediate implications:

- Reversible row and/or column operations exist to change any matrix of rank $r$ to *any other matrix* of the same rank. The reason is that, according to (12.15), reversible operations exist to change both matrices to $I_r$ and back.

- No reversible operation can change a matrix's rank.

The discovery that every matrix has a single, unambiguous rank and the establishment of a failproof algorithm—the Gauss-Jordan—to ascertain that rank have not been easy to achieve, but they are important achievements nonetheless, worth the effort thereto. The reason these achievements matter is that the mere dimensionality of a matrix is a chimerical measure of the matrix's true size—as for instance for the $3 \times 3$ example matrix at the head of the section. Matrix rank by contrast is an entirely solid, dependable measure. We will rely on it often.

Section 12.5.8 comments further.

### 12.5.4 The full-rank matrix

According to (12.5), the rank $r$ of a matrix can exceed the number neither of the matrix's rows nor of its columns. The greatest rank possible for an $m \times n$ matrix is the lesser of $m$ and $n$. A *full-rank* matrix, then, is defined to be an $m \times n$ matrix with maximum rank $r = m$ or $r = n$—or, if $m = n$, both. A matrix of less than full rank is a *degenerate* matrix.

Consider a tall $m \times n$ matrix $C$, $m \geq n$, one of whose $n$ columns is a linear combination (§ 12.1) of the others. One could by definition target the dependent column with addition elementaries, using multiples of the other columns to wipe the dependent column out. Having zeroed the dependent column, one could then interchange it over to the matrix's extreme right, effectively throwing the column away, shrinking the matrix to $m \times (n - 1)$ dimensionality. Shrinking the matrix necessarily also shrinks the bound on the matrix's rank to $r \leq n - 1$—which is to say, to $r < n$. But the shrink, done by reversible column operations, is itself reversible, by which § 12.5.3 binds the rank of the original, $m \times n$ matrix $C$ likewise to $r < n$. The matrix $C$, one of whose columns is a linear combination of the others, is necessarily degenerate for this reason.

Now consider a tall matrix $A$ with the same $m \times n$ dimensionality, but with a full $n$ independent columns. The transpose $A^T$ of such a matrix has a full $n$ independent *rows.* One of the conclusions of § 12.3.4 was that a matrix of independent rows always has rank equal to the number of rows. Since $A^T$ is such a matrix, its rank is a full $r = n$. But formally, what this says is that there exist operators $B_<^T$ and $B_>^T$ such that $I_n = B_<^T A^T B_>^T$, the transpose of which equation is $B_> A B_< = I_n$—which in turn says that not only $A^T$, but also $A$ itself, has full rank $r = n$.

Parallel reasoning rules the rows of broad matrices, $m \leq n$, of course. To square matrices, $m = n$, both lines of reasoning apply.

Gathering findings, we have that

- a tall $m \times n$ matrix, $m \geq n$, has full rank if and only if its columns are linearly independent;
- a broad $m \times n$ matrix, $m \leq n$, has full rank if and only if its rows are linearly independent;
- a square $n \times n$ matrix, $m = n$, has full rank if and only if its columns and/or its rows are linearly independent; and
- a square matrix has both independent columns and independent rows, or neither; never just one or the other.

To say that a matrix has *full column rank* is to say that it is tall or square and has full rank $r = n \leq m$. To say that a matrix has *full row rank* is to say that it is broad or square and has full rank $r = m \leq n$. Only a square matrix can have full column rank and full row rank at the same time, because a tall or broad matrix cannot but include, respectively, more columns or more rows than $I_r$.

### 12.5.5 Underdetermined and overdetermined linear systems (introduction)

The last paragraph of § 12.5.4 provokes yet further terminology. A linear system $A\mathbf{x} = \mathbf{b}$ is *underdetermined* if $A$ lacks full column rank—that is, if $r < n$—because inasmuch as some of $A$'s columns then depend linearly on the others such a system maps multiple $n$-element vectors $\mathbf{x}$ to the same $m$-element vector $\mathbf{b}$, meaning that knowledge of $\mathbf{b}$ does not suffice to determine $\mathbf{x}$ uniquely. Complementarily, a linear system $A\mathbf{x} = \mathbf{b}$ is *overdetermined* if $A$ lacks full row rank—that is, if $r < m$. If $A$ lacks both, then the system is paradoxically both underdetermined and overdetermined and is thereby *degenerate*. If $A$ happily has both, then the system is *exactly determined*.

Section 13.2 solves the exactly determined linear system. Section 13.4 solves the nonoverdetermined linear system. Section 13.6 analyzes the unsolvable overdetermined linear system among others. Further generalities await Ch. 13; but, regarding the overdetermined system specifically, the present subsection would observe at least the few following facts.

*An overdetermined linear system* $A\mathbf{x} = \mathbf{b}$ *cannot have a solution for every possible* $m$*-element driving vector* $\mathbf{b}$. The truth of this claim can be

seen by decomposing the system's matrix $A$ by Gauss-Jordan and then left-multiplying the decomposed system by $G_>^{-1}$ to reach the form

$$I_r G_< \mathbf{x} = G_>^{-1} \mathbf{b}.$$

If the $m$-element vector $\mathbf{c} \equiv G_>^{-1}\mathbf{b}$, then $I_r G_< \mathbf{x} = \mathbf{c}$, which is impossible unless the last $m - r$ elements of $\mathbf{c}$ happen to be zero. But since $G_>$ is invertible, each $\mathbf{b}$ corresponds to a unique $\mathbf{c}$ and vice versa; so, if $\mathbf{b}$ is an unrestricted $m$-element vector then so also is $\mathbf{c}$, which verifies the claim.

Complementarily, *a nonoverdetermined linear system $A\mathbf{x} = \mathbf{b}$ does have a solution for every possible $m$-element driving vector $\mathbf{b}$*. This is so because in this case the last $m - r$ elements of $\mathbf{c}$ do happen to be zero; or, better stated, because $\mathbf{c}$ in this case has no nonzeros among its last $m - r$ elements, because it *has* no last $m - r$ elements, for the trivial reason that $r = m$.

It is an analytical error, and an easy one innocently to commit, to require that

$$A\mathbf{x} = \mathbf{b}$$

for unrestricted $\mathbf{b}$ when $A$ lacks full row rank. The error is easy to commit because the equation looks right, because such an equation is indeed valid over a broad domain of $\mathbf{b}$ and might very well have been written correctly in that context, only not in the context of unrestricted $\mathbf{b}$. Analysis including such an error can lead to subtly absurd conclusions. It is never such an analytical error however to require that

$$A\mathbf{x} = 0$$

because, whatever other solutions such a system might have, it has at least the solution $\mathbf{x} = 0$.

### 12.5.6 The full-rank factorization

One sometimes finds dimension-limited matrices of less than full rank inconvenient to handle. However, every dimension-limited, $m \times n$ matrix of rank $r$ can be expressed as the product of two full-rank matrices, one $m \times r$ and the other $r \times n$, both also of rank $r$:

$$A = BC. \tag{12.16}$$

The truncated Gauss-Jordan (12.7) constitutes one such *full-rank factorization:* $B = I_m G_> I_r$, $C = I_r G_< I_n$, good for any matrix. Other full-rank

factorizations are possible, however, including among others the truncated Gram-Schmidt (13.56). The full-rank factorization is not unique.[16]

Of course, if an $m \times n$ matrix already has full rank $r = m$ or $r = n$, then the full-rank factorization is trivial: $A = I_m A$ or $A = A I_n$.

Section 13.6.4 uses the full-rank factorization.

### 12.5.7 Full column rank and the Gauss-Jordan factors $K$ and $S$

The Gauss-Jordan decomposition (12.2),

$$A = PDLUI_r KS,$$

of a tall or square $m \times n$ matrix $A$ of full column rank $r = n \leq m$ always finds the factor $K = I$, regardless of the pivots one chooses during the Gauss-Jordan algorithm's step 3. If one happens always to choose $q = i$ as pivot column then not only $K = I$ but $S = I$, too.

That $K = I$ is seen by the algorithm's step 12, which creates $K$. Step 12 nulls the spare columns $q > r$ that dress $\tilde{I}$'s right, but in this case $\tilde{I}$ has only $r$ columns and therefore has no spare columns to null. Hence step 12 does nothing and $K = I$.

That $S = I$ comes immediately of choosing $q = i$ for pivot column during each iterative instance of the algorithm's step 3. But, one must ask, can one choose so? What if column $q = i$ were unusable? That is, what if the only nonzero elements remaining in $\tilde{I}$'s $i$th column stood above the main diagonal, unavailable for step 4 to bring to pivot? Well, *were* it so, then one would indeed have to choose $q \neq i$ to swap the unusable column away rightward, but see: nothing in the algorithm later fills such a column's zeros with anything else—they remain zeros—so swapping the column away rightward could only delay the crisis. The column would remain unusable. Eventually the column would reappear on pivot when no usable column rightward remained available to swap it with, which contrary to our assumption would mean precisely that $r < n$. Such contradiction can only imply that if $r = n$ then no unusable column can ever appear. One need not swap. We conclude that though one might voluntarily choose $q \neq i$ during the algorithm's step 3, the algorithm cannot force one to do so if $r = n$. Yet if one always does choose $q = i$, as the full-column-rank matrix $A$ evidently leaves one free to do, then indeed $S = I$.

[16] [6, § 3.3][50, "Moore-Penrose generalized inverse"]

Theoretically, the Gauss-Jordan decomposition (12.2) includes the factors $K$ and $S$ precisely to handle matrices with more columns than rank. Matrices of full column rank $r = n$, common in applications, by definition have no such problem. Therefore, the Gauss-Jordan decomposition theoretically needs no $K$ or $S$ for such matrices, which fact lets us abbreviate the decomposition for such matrices to read

$$A = PDLUI_n. \tag{12.17}$$

Observe however that just because one theoretically can set $S = I$ does not mean that one actually should. The column permutor $S$ exists to be used, after all—especially numerically to avoid small pivots during early invocations of the algorithm's step 5. Equation (12.17) is not mandatory but optional for a matrix $A$ of full column rank (though still $r = n$ and thus $K = I$ for such a matrix, even when the unabbreviated eqn. 12.2 is used). There are however times when it is nice to know that one theoretically could, if doing exact arithmetic, set $S = I$ if one wanted to.

Since $PDLU$ acts as a row operator, (12.17) implies that each row of the full-rank matrix $A$ lies in the space the rows of $I_n$ address. This is obvious and boring, but interesting is the converse implication of (12.17)'s complementary form,

$$U^{-1}L^{-1}D^{-1}P^{-1}A = I_n,$$

that each row of $I_n$ lies in the space the rows of $A$ address. The rows of $I_n$ and the rows of $A$ evidently address the same space. One can moreover say the same of $A$'s columns, since $B = A^T$ has full rank just as $A$ does. In the whole, *if a matrix $A$ is square and has full rank $r = n$, then $A$'s columns together, $A$'s rows together, $I_n$'s columns together and $I_n$'s rows together each address the same, complete $n$-dimensional space.*

### 12.5.8 The significance of rank uniqueness

The result of § 12.5.3, that matrix rank is unique, is an extremely important matrix theorem. It constitutes the chapter's chief result, which we have spent so many pages to attain. Without this theorem, the very concept of matrix rank must remain in doubt, along with all that attends to the concept. The theorem is the rock upon which the general theory of the matrix is built.

The concept underlying the theorem promotes the useful sensibility that a matrix's rank, much more than its mere dimensionality or the extent of its

active region, represents the matrix's true size. Dimensionality can deceive, after all. For example, the honest $2 \times 2$ matrix

$$\begin{bmatrix} 5 & 1 \\ 3 & 6 \end{bmatrix}$$

has two independent rows or, alternately, two independent columns, and, hence, rank $r = 2$. One can easily construct a phony $3 \times 3$ matrix from the honest $2 \times 2$, however, simply by applying some $3 \times 3$ row and column elementaries:

$$T_{(2/3)[32]} \begin{bmatrix} 5 & 1 \\ 3 & 6 \end{bmatrix} T_{1[13]} T_{1[23]} = \begin{bmatrix} 5 & 1 & 6 \\ 3 & 6 & 9 \\ 2 & 4 & 6 \end{bmatrix}.$$

The $3 \times 3$ matrix on the equation's right is the one we met at the head of the section. It *looks* like a rank-three matrix, but really has only two independent columns and two independent rows. Its true rank is $r = 2$. We have here caught a matrix impostor pretending to be bigger than it really is.[17]

Now, admittedly, adjectives like "honest" and "phony," terms like "imposter," are a bit hyperbolic. The last paragraph has used them to convey the subjective sense of the matter, but of course there is nothing mathematically improper or illegal about a matrix of less than full rank, so long as the true rank is correctly recognized. When one models a physical phenomenon by a set of equations, one sometimes is dismayed to discover that one of the equations, thought to be independent, is really just a useless combination of the others. This can happen in matrix work, too. The rank of a matrix helps one to recognize how many truly independent vectors, dimensions or equations one actually has available to work with, rather than how many seem available at first glance. That is the sense of matrix rank.

[17] An applied mathematician with some matrix experience actually probably recognizes this particular $3 \times 3$ matrix as a fraud on sight, but it is a very simple example. No one can just look at some arbitrary matrix and instantly perceive its true rank. Consider for instance the $5 \times 5$ matrix (in hexadecimal notation)

$$\begin{bmatrix} 12 & 9 & 3 & 1 & 0 \\ \frac{3}{2} & \frac{\mathrm{F}}{2} & \frac{15}{2} & 2 & 12 \\ \mathrm{D} & 9 & -19 & -\frac{\mathrm{E}}{3} & -6 \\ -2 & 0 & 6 & 1 & 5 \\ 1 & -4 & 4 & 1 & -8 \end{bmatrix}.$$

As the reader can verify by the Gauss-Jordan algorithm, the matrix's rank is not $r = 5$ but $r = 4$.

# Chapter 13

# Inversion and orthonormalization

The undeniably tedious Chs. 11 and 12 have piled the matrix theory deep while affording scant practical reward. Building upon the two tedious chapters, this chapter brings the first rewarding matrix work.

One might be forgiven for forgetting after so many pages of abstract theory that the matrix afforded any reward or had any use at all. Uses however it has. Sections 11.1.1 and 12.5.5 have already broached[1] the matrix's most basic use, the primary subject of this chapter, to represent a system of $m$ linear scalar equations in $n$ unknowns neatly as

$$A\mathbf{x} = \mathbf{b}$$

and to solve the whole system at once by inverting the matrix $A$ that characterizes it.

Now, before we go on, we want to confess that such a use alone, on the surface of it—though interesting—might not have justified the whole uncomfortable bulk of Chs. 11 and 12. We already knew how to solve a simultaneous system of linear scalar equations in principle without recourse to the formality of a matrix, after all, as in the last step to derive (3.9) as far back as Ch. 3. Why should we have suffered two bulky chapters, if only to prepare to do here something we already knew how to do?

The question is a fair one, but admits at least four answers. First, the matrix neatly solves a linear system not only for a particular driving vector $\mathbf{b}$ but for all possible driving vectors $\mathbf{b}$ at one stroke, as this chapter

[1] The reader who has skipped Ch. 12 might at least review § 12.5.5.

explains. Second and yet more impressively, the matrix allows § 13.6 to introduce the *pseudoinverse* to approximate the solution to an unsolvable linear system and, moreover, to do so both optimally and efficiently, whereas such overdetermined systems arise commonly in applications. Third, to solve the linear system neatly is only the primary and most straightforward use of the matrix, not its only use: the even more interesting eigenvalue and its incidents await Ch. 14. Fourth, specific applications aside, one should never underestimate the blunt practical benefit of reducing an arbitrarily large grid of scalars to a single symbol $A$, which one can then manipulate by known algebraic rules. Most students first learning the matrix have wondered at this stage whether it were worth all the tedium; so, if the reader now wonders, then he stands in good company. The matrix finally begins to show its worth here.

The chapter opens in § 13.1 by inverting the square matrix to solve the exactly determined, $n \times n$ linear system in § 13.2. It continues in § 13.3 by computing the rectangular matrix's kernel to solve the nonoverdetermined, $m \times n$ linear system in § 13.4. In § 13.6, it brings forth the aforementioned pseudoinverse, which rightly approximates the solution to the unsolvable overdetermined linear system. After briefly revisiting the Newton-Raphson iteration in § 13.7, it concludes by introducing the concept and practice of vector orthonormalization in §§ 13.8 through 13.12.

## 13.1 Inverting the square matrix

Consider an $n \times n$ square matrix $A$ of full rank $r = n$. Suppose that extended operators $G_>$, $G_<$, $G_>^{-1}$ and $G_<^{-1}$ can be found, each with an $n \times n$ active

region (§ 11.3.2), such that[2]

$$\begin{aligned} G_>^{-1}G_> &= I = G_>G_>^{-1}, \\ G_<^{-1}G_< &= I = G_<G_<^{-1}, \\ A &= G_>I_nG_<. \end{aligned} \tag{13.1}$$

Observing from (11.31) that

$$\begin{aligned} I_nA &= A &&= AI_n, \\ I_nG_<^{-1}G_>^{-1} &= G_<^{-1}I_nG_>^{-1} &&= G_<^{-1}G_>^{-1}I_n, \end{aligned}$$

we find by successive steps that

$$\begin{aligned} A &= G_>I_nG_<, \\ I_nA &= G_>G_<I_n, \\ G_<^{-1}G_>^{-1}I_nA &= I_n, \\ (G_<^{-1}I_nG_>^{-1})(A) &= I_n; \end{aligned}$$

[2]The symbology and associated terminology might disorient a reader who had skipped Chs. 11 and 12. In this book, the symbol $I$ theoretically represents an $\infty \times \infty$ identity matrix. Outside the $m \times m$ or $n \times n$ square, the operators $G_>$ and $G_<$ each resemble the $\infty \times \infty$ identity matrix $I$, which means that the operators affect respectively only the first $m$ rows or $n$ columns of the thing they operate on. (In the present section it happens that $m = n$ because the matrix $A$ of interest is square, but this footnote uses both symbols because generally $m \neq n$.)

The symbol $I_r$ contrarily represents an identity matrix of only $r$ ones, though it too can be viewed as an $\infty \times \infty$ matrix with zeros in the unused regions. If interpreted as an $\infty \times \infty$ matrix, the matrix $A$ of the $m \times n$ system $A\mathbf{x} = \mathbf{b}$ has nonzero content only within the $m \times n$ rectangle.

None of this is complicated, really. Its purpose is merely to separate the essential features of a reversible operation like $G_>$ or $G_<$ from the dimensionality of the vector or matrix on which the operation happens to operate. The definitions do however necessarily, slightly diverge from definitions the reader may have been used to seeing in other books. In this book, one can legally multiply any two matrices, because all matrices are theoretically $\infty \times \infty$, anyway (though whether it makes any sense in a given circumstance to multiply mismatched matrices is another question; sometimes it does make sense, as in eqns. 13.24 and 14.49, but more often it does not—which naturally is why the other books tend to forbid such multiplication).

To the extent that the definitions confuse, the reader might briefly review the earlier chapters, especially § 11.3.

or alternately that

$$\begin{aligned} A &= G_> I_n G_<, \\ AI_n &= I_n G_> G_<, \\ AI_n G_<^{-1} G_>^{-1} &= I_n, \\ (A)(G_<^{-1} I_n G_>^{-1}) &= I_n. \end{aligned}$$

Either way, we have that

$$\begin{aligned} A^{-1}A = I_n = AA^{-1}, \\ A^{-1} \equiv G_<^{-1} I_n G_>^{-1}. \end{aligned} \tag{13.2}$$

Of course, for this to work, $G_>$, $G_<$, $G_>^{-1}$ and $G_<^{-1}$ must exist, be known and honor $n \times n$ active regions, which might seem a practical hurdle. However, (12.2), (12.3) and the body of § 12.3 have shown exactly how to find just such a $G_>$, $G_<$, $G_>^{-1}$ and $G_<^{-1}$ for any square matrix $A$ of full rank, without exception; so, there is no trouble here. The factors do exist, and indeed we know how to find them.

Equation (13.2) features the important matrix $A^{-1}$, the *rank-n inverse* of $A$. We have not yet much studied the rank-$n$ inverse, but we have defined it in (11.49), where we gave it the fuller, nonstandard notation $A^{-1(n)}$. When naming the rank-$n$ inverse in words one usually says simply, "the inverse," because the rank is implied by the size of the square active region of the matrix inverted; but the rank-$n$ inverse from (11.49) is not quite the infinite-dimensional inverse from (11.45), which is what $G_>^{-1}$ and $G_<^{-1}$ are. According to (13.2), the product of $A^{-1}$ and $A$—or, written more fully, the product of $A^{-1(n)}$ and $A$—is, not $I$, but $I_n$.

Properties that emerge from (13.2) include the following.

- Like $A$, the rank-$n$ inverse $A^{-1}$ (more fully written $A^{-1(n)}$) too is an $n \times n$ square matrix of full rank $r = n$.
- Since $A$ is square and has full rank (§ 12.5.4), its rows and, separately, its columns are linearly independent, so it has only the one, unique inverse $A^{-1}$. No other rank-$n$ inverse of $A$ exists.
- On the other hand, inasmuch as $A$ is square and has full rank, it does per (13.2) indeed have an inverse $A^{-1}$. The rank-$n$ inverse exists.
- If $B = A^{-1}$ then $B^{-1} = A$. That is, $A$ is itself the rank-$n$ inverse of $A^{-1}$. The matrices $A$ and $A^{-1}$ thus form an exclusive, reciprocal pair.

- If $B$ is an $n \times n$ square matrix and either $BA = I_n$ or $AB = I_n$, then both equalities in fact hold; thus, $B = A^{-1}$. One can have neither equality without the other.

- Only a square, $n \times n$ matrix of full rank $r = n$ has a rank-$n$ inverse. A matrix $A'$ which is not square, or whose rank falls short of a full $r = n$, is not invertible in the rank-$n$ sense of (13.2).

That $A^{-1}$ is an $n \times n$ square matrix of full rank and that $A$ is itself the inverse of $A^{-1}$ proceed from the definition (13.2) of $A^{-1}$ plus § 12.5.3's finding that reversible operations like $G_>^{-1}$ and $G_<^{-1}$ cannot change $I_n$'s rank. That the inverse exists is plain, inasmuch as the Gauss-Jordan decomposition plus (13.2) reliably calculate it. That the inverse is unique begins from § 12.5.4's observation that the columns (like the rows) of $A$ are linearly independent because $A$ is square and has full rank. From this beginning and the fact that $I_n = AA^{-1}$, it follows that $[A^{-1}]_{*1}$ represents[3] the one and only possible combination of $A$'s columns which achieves $\mathbf{e}_1$, that $[A^{-1}]_{*2}$ represents the one and only possible combination of $A$'s columns which achieves $\mathbf{e}_2$, and so on through $\mathbf{e}_n$. One could observe likewise respecting the independent rows of $A$. Either way, $A^{-1}$ is unique. Moreover, no other $n \times n$ matrix $B \neq A^{-1}$ satisfies *either* requirement of (13.2)—that $BA = I_n$ or that $AB = I_n$—much less both.

It is not claimed that the matrix factors $G_>$ and $G_<$ themselves are unique, incidentally. On the contrary, many different pairs of matrix factors $G_>$ and $G_<$ can yield $A = G_> I_n G_<$, no less than that many different pairs of scalar factors $\gamma_>$ and $\gamma_<$ can yield $\alpha = \gamma_> 1 \gamma_<$. Though the Gauss-Jordan decomposition is a convenient means to $G_>$ and $G_<$, it is hardly the only means, and any proper $G_>$ and $G_<$ found by any means will serve so long as they satisfy (13.1). What are unique are not the factors but the $A$ and $A^{-1}$ they produce.

What of the degenerate $n \times n$ square matrix $A'$, of rank $r < n$? Rank promotion is impossible as §§ 12.5.2 and 12.5.3 have shown, so in the sense of (13.2) such a matrix has no inverse; for, if it had, then $A'^{-1}$ would by definition represent a row or column operation which impossibly promoted $A'$ to the full rank $r = n$ of $I_n$. Indeed, in that it has no inverse such a degenerate matrix closely resembles the scalar 0, which has no reciprocal. Mathematical convention owns a special name for a square matrix which is degenerate and thus has no inverse; it calls it a *singular* matrix.

[3]The notation $[A^{-1}]_{*j}$ means "the $j$th column of $A^{-1}$." Refer to § 11.1.3.

And what of a rectangular matrix? Is it degenerate? Well, no, not exactly, not necessarily. The definitions of the present particular section are meant for square matrices; they do not neatly apply to nonsquare ones. Refer to §§ 12.5.3 and 12.5.4. However, appending the right number of null rows or columns to a nonsquare matrix does turn it into a degenerate square, in which case the preceding argument applies. See also §§ 12.5.5, 13.4 and 13.6.

## 13.2 The exactly determined linear system

Section 11.1.1 has shown how the single matrix equation

$$A\mathbf{x} = \mathbf{b} \tag{13.3}$$

concisely represents an entire simultaneous system of linear scalar equations. If the system has $n$ scalar equations and $n$ scalar unknowns, then the matrix $A$ has square, $n \times n$ dimensionality. Furthermore, if the $n$ scalar equations are independent of one another, then the rows of $A$ are similarly independent, which gives $A$ full rank and makes it invertible. Under these conditions, one can solve (13.3) and the corresponding system of linear scalar equations by left-multiplying (13.3) by the $A^{-1}$ of (13.2) and (13.1) to reach the famous formula

$$\mathbf{x} = A^{-1}\mathbf{b}. \tag{13.4}$$

Inverting the square matrix $A$ of scalar coefficients, (13.4) concisely solves a simultaneous system of $n$ linear scalar equations in $n$ scalar unknowns. It is the classic motivational result of matrix theory.

It has taken the book two long chapters to reach (13.4). If one omits first to prepare the theoretical ground sufficiently to support more advanced matrix work, then one can indeed reach (13.4) with rather less effort than the book has done.[4] As the chapter's introduction has observed, however, we

[4]For motivational reasons, introductory, tutorial linear algebra textbooks like [30] and [42] rightly yet invariably invert the general square matrix of full rank much earlier, reaching (13.4) with less effort. The deferred price the student pays for the simpler-seeming approach of the tutorials is twofold. First, the student fails to develop the Gauss-Jordan decomposition properly, instead learning the less elegant but easier to grasp "row echelon form" of "Gaussian elimination" [30, Ch. 1][42, § 1.2]—which makes good matrix-arithmetic drill but leaves the student imperfectly prepared when the time comes to study kernels and eigensolutions or to read and write matrix-handling computer code. Second, in the long run the tutorials save no effort, because the student still must at some point develop the theory underlying matrix rank and supporting each of the several coincident properties of § 14.2. What the tutorials do is pedagogically necessary—it is how the

shall soon meet additional interesting applications of the matrix which in any case require the theoretical ground to have been prepared. Equation (13.4) is only the first fruit of the effort.

Where the inverse does not exist, where the square matrix $A$ is singular, the rows of the matrix are linearly dependent, meaning that the corresponding system actually contains fewer than $n$ useful scalar equations. Depending on the value of the driving vector $\mathbf{b}$, the superfluous equations either merely reproduce or flatly contradict information the other equations already supply. Either way, no unique solution to a linear system described by a singular square matrix is possible—though a good approximate solution is given by the *pseudoinverse* of § 13.6. In the language of § 12.5.5, the singular square matrix characterizes a system that is both underdetermined and overdetermined, thus degenerate.

## 13.3 The kernel

If a matrix $A$ has full column rank (§ 12.5.4), then the columns of $A$ are linearly independent and

$$A\mathbf{x} = 0 \tag{13.5}$$

is impossible if $I_n\mathbf{x} \neq 0$. If the matrix however lacks full column rank then (13.5) is possible even if $I_n\mathbf{x} \neq 0$. In either case, any $n$-element $\mathbf{x}$ (including $\mathbf{x} = 0$) that satisfies (13.5) belongs to the *kernel* of $A$.

Let $A$ be an $m \times n$ matrix of rank $r$. A second matrix,[5] $A^K$, minimally represents the kernel of $A$ if and only if

- $A^K$ has $n \times (n - r)$ dimensionality (which gives $A^K$ tall rectangular form unless $r = 0$),

writer first learned the matrix and probably how the reader first learned it, too—but it is appropriate to a tutorial, not to a study reference like this book.

In this book, where derivations prevail, the proper place to invert the general square matrix of full rank is here. Indeed, the inversion here goes smoothly, because Chs. 11 and 12 have laid under it a firm foundation upon which—and supplied it the right tools with which—to work.

[5]The conventional mathematical notation for the kernel of $A$ is $\ker\{A\}$, $\text{null}\{A\}$ or something nearly resembling one of the two—the notation seems to vary from editor to editor—which technically represent the kernel space itself, as opposed to the notation $A^K$ which represents a matrix whose columns address the kernel space. This book deëmphasizes the distinction and prefers the kernel matrix notation $A^K$.

If we were really precise, we might write not $A^K$ but $A^{K(n)}$ to match the $A^{-1(r)}$ of (11.49). The abbreviated notation $A^K$ is probably clear enough for most practical purposes, though, and surely more comprehensible to those who do not happen to have read this particular book.

- $A^K$ has full rank $n-r$ (that is, the columns of $A^K$ are linearly independent, which gives $A^K$ full column rank), and
- $A^K$ satisfies the equation

$$AA^K = 0. \tag{13.6}$$

The $n-r$ independent columns of the kernel matrix $A^K$ address the complete space $\mathbf{x} = A^K\mathbf{a}$ of vectors in the kernel, where the $(n-r)$-element vector $\mathbf{a}$ can have any value. In symbols,

$$A\mathbf{x} = A(A^K\mathbf{a}) = (AA^K)\mathbf{a} = 0.$$

The definition does not pretend that the kernel matrix $A^K$ is unique. Except when $A$ has full column rank the kernel matrix is not unique; there are infinitely many kernel matrices $A^K$ to choose from for a given matrix $A$. What is unique is not the kernel matrix but rather the space its columns address, and it is the latter space rather than $A^K$ as such that is technically the kernel (if you forget and call $A^K$ "a kernel," though, you'll be all right).

The *Gauss-Jordan kernel formula*[6]

$$A^K = S^{-1}K^{-1}H_rI_{n-r} = G_<^{-1}H_rI_{n-r} \tag{13.7}$$

gives a complete kernel $A^K$ of $A$, where $S^{-1}$, $K^{-1}$ and $G_<^{-1}$ are the factors their respective symbols indicate of the Gauss-Jordan decomposition's complementary form (12.3) and $H_r$ is the shift operator of § 11.9. Section 13.3.1 derives the formula, next.

### 13.3.1 The Gauss-Jordan kernel formula

To derive (13.7) is not easy. It begins from the statement of the linear system

$$A\mathbf{x} = \mathbf{b}, \text{ where } \mathbf{b} = 0 \text{ or } r = m, \text{ or both;} \tag{13.8}$$

and where $\mathbf{b}$ and $\mathbf{x}$ are respectively $m$- and $n$-element vectors and $A$ is an $m \times n$ matrix of rank $r$. This statement is broader than (13.5) requires but it serves § 13.4, too; so, for the moment, for generality's sake, we leave $\mathbf{b}$ unspecified but by the given proviso. Gauss-Jordan factoring $A$, by successive steps,

$$\begin{aligned} G_>I_rKS\mathbf{x} &= \mathbf{b}, \\ I_rKS\mathbf{x} &= G_>^{-1}\mathbf{b}, \\ I_r(K-I)S\mathbf{x} + I_rS\mathbf{x} &= G_>^{-1}\mathbf{b}. \end{aligned}$$

[6]The name *Gauss-Jordan kernel formula* is not standard as far as the writer is aware, but we would like a name for (13.7). This name seems as fitting as any.

Applying an identity from Table 12.2 on page 311,

$$I_r K (I_n - I_r) S\mathbf{x} + I_r S\mathbf{x} = G_>^{-1}\mathbf{b}.$$

Rearranging terms,

$$I_r S\mathbf{x} = G_>^{-1}\mathbf{b} - I_r K (I_n - I_r) S\mathbf{x}. \tag{13.9}$$

Equation (13.9) is interesting. It has $S\mathbf{x}$ on both sides, where $S\mathbf{x}$ is the vector $\mathbf{x}$ with elements reordered in some particular way. The equation has however on the left only $I_r S\mathbf{x}$, which is the first $r$ elements of $S\mathbf{x}$; and on the right only $(I_n - I_r)S\mathbf{x}$, which is the remaining $n - r$ elements.[7] No element of $S\mathbf{x}$ appears on both sides. Naturally this is no accident; we have (probably after some trial and error not recorded here) planned the steps leading to (13.9) to achieve precisely this effect. Equation (13.9) implies *that one can choose the last $n - r$ elements of $S\mathbf{x}$ freely, but that the choice then determines the first $r$ elements.*

The implication is significant. To express the implication more clearly we can rewrite (13.9) in the improved form

$$\begin{aligned} \mathbf{f} &= G_>^{-1}\mathbf{b} - I_r K H_r \mathbf{a}, \\ S\mathbf{x} &= \begin{bmatrix} \mathbf{f} \\ \mathbf{a} \end{bmatrix} = \mathbf{f} + H_r \mathbf{a}, \\ \mathbf{f} &\equiv I_r S\mathbf{x}, \\ \mathbf{a} &\equiv H_{-r}(I_n - I_r) S\mathbf{x}, \end{aligned} \tag{13.10}$$

where $\mathbf{a}$ represents the $n - r$ free elements of $S\mathbf{x}$ and $\mathbf{f}$ represents the $r$ dependent elements. This makes $\mathbf{f}$ and thereby also $\mathbf{x}$ functions of the free parameter $\mathbf{a}$ and the driving vector $\mathbf{b}$:

$$\begin{aligned} \mathbf{f}(\mathbf{a}, \mathbf{b}) &= G_>^{-1}\mathbf{b} - I_r K H_r \mathbf{a}, \\ S\mathbf{x}(\mathbf{a}, \mathbf{b}) &= \begin{bmatrix} \mathbf{f}(\mathbf{a}, \mathbf{b}) \\ \mathbf{a} \end{bmatrix} = \mathbf{f}(\mathbf{a}, \mathbf{b}) + H_r \mathbf{a}. \end{aligned} \tag{13.11}$$

If $\mathbf{b} = 0$ as (13.5) requires, then

$$\begin{aligned} \mathbf{f}(\mathbf{a}, 0) &= -I_r K H_r \mathbf{a}, \\ S\mathbf{x}(\mathbf{a}, 0) &= \begin{bmatrix} \mathbf{f}(\mathbf{a}, 0) \\ \mathbf{a} \end{bmatrix} = \mathbf{f}(\mathbf{a}, 0) + H_r \mathbf{a}. \end{aligned}$$

[7] Notice how we now associate the factor $(I_n - I_r)$ rightward as a row truncator, though it had first entered acting leftward as a column truncator. The flexibility to reassociate operators in such a way is one of many good reasons Chs. 11 and 12 have gone to such considerable trouble to develop the basic theory of the matrix.

Substituting the first line into the second,

$$S\mathbf{x}(\mathbf{a},0) = (I - I_r K)H_r\mathbf{a}. \tag{13.12}$$

In the event that $\mathbf{a} = \mathbf{e}_j$, where $1 \le j \le n-r$,

$$S\mathbf{x}(\mathbf{e}_j,0) = (I - I_r K)H_r\mathbf{e}_j.$$

For all the $\mathbf{e}_j$ at once,

$$S\mathbf{x}(I_{n-r},0) = (I - I_r K)H_r I_{n-r}.$$

But if all the $\mathbf{e}_j$ at once, the columns of $I_{n-r}$, exactly address the domain of $\mathbf{a}$, then the columns of $\mathbf{x}(I_{n-r},0)$ likewise exactly address the range of $\mathbf{x}(\mathbf{a},0)$. Equation (13.6) has already named this range $A^K$, by which[8]

$$SA^K = (I - I_r K)H_r I_{n-r}. \tag{13.13}$$

Left-multiplying by $S^{-1} = S^* = S^T$ produces the alternate kernel formula

$$A^K = S^{-1}(I - I_r K)H_r I_{n-r}. \tag{13.14}$$

The alternate kernel formula (13.14) is correct but not as simple as it could be. By the identity (11.76), eqn. (13.13) is

$$\begin{aligned} SA^K &= (I - I_r K)(I_n - I_r)H_r \\ &= [(I_n - I_r) - I_r K(I_n - I_r)]H_r \\ &= [(I_n - I_r) - (K - I)]H_r, \end{aligned} \tag{13.15}$$

[8]These are difficult steps. How does one justify replacing $\mathbf{a}$ by $\mathbf{e}_j$, then $\mathbf{e}_j$ by $I_{n-r}$, then $\mathbf{x}$ by $A^K$? One justifies them in that the columns of $I_{n-r}$ are the several $\mathbf{e}_j$, of which any $(n-r)$-element vector $\mathbf{a}$ can be constructed as the linear combination

$$\mathbf{a} = I_{n-r}\mathbf{a} = [\ \mathbf{e}_1 \quad \mathbf{e}_2 \quad \mathbf{e}_3 \quad \cdots \quad \mathbf{e}_{n-r}\ ]\mathbf{a} = \sum_{j=1}^{n-r} a_j\mathbf{e}_j$$

weighted by the elements of $\mathbf{a}$. Seen from one perspective, this seems trivial; from another perspective, baffling; until one grasps what is really going on here.

The idea is that if we can solve the problem for each elementary vector $\mathbf{e}_j$—that is, in aggregate, if we can solve the problem for the identity matrix $I_{n-r}$—then we shall implicitly have solved it for every $\mathbf{a}$ because $\mathbf{a}$ is a weighted combination of the $\mathbf{e}_j$ and the whole problem is linear. The solution

$$\mathbf{x} = A^K\mathbf{a}$$

for a given choice of $\mathbf{a}$ becomes a weighted combination of the solutions for each $\mathbf{e}_j$, with the elements of $\mathbf{a}$ again as the weights. And what are the solutions for each $\mathbf{e}_j$? Answer: the corresponding columns of $A^K$, which by definition are the independent values of $\mathbf{x}$ that cause $\mathbf{b} = 0$.

where we have used Table 12.2 again in the last step. How to proceed symbolically from (13.15) is not obvious, but if one sketches the matrices of (13.15) schematically with a pencil, and if one remembers that $K^{-1}$ is just $K$ with elements off the main diagonal negated, then it appears that

$$SA^K = K^{-1}H_r I_{n-r}. \tag{13.16}$$

The appearance is not entirely convincing,[9] but (13.16) though unproven still helps because it posits a hypothesis toward which to target the analysis.

Two variations on the identities of Table 12.2 also help. First, from the identity that

$$\frac{K + K^{-1}}{2} = I,$$

we have that

$$K - I = I - K^{-1}. \tag{13.17}$$

Second, right-multiplying by $I_r$ the identity that

$$I_r K^{-1}(I_n - I_r) = K^{-1} - I$$

and canceling terms, we have that

$$K^{-1}I_r = I_r \tag{13.18}$$

(which actually is pretty obvious if you think about it, since all of $K$'s interesting content lies by construction right of its $r$th column). Now we have enough to go on with. Substituting (13.17) and (13.18) into (13.15) yields

$$SA^K = [(I_n - K^{-1}I_r) - (I - K^{-1})]H_r.$$

Adding $0 = K^{-1}I_n H_r - K^{-1}I_n H_r$ and rearranging terms,

$$SA^K = K^{-1}(I_n - I_r)H_r + [K^{-1} - K^{-1}I_n - I + I_n]H_r.$$

Factoring,

$$SA^K = K^{-1}(I_n - I_r)H_r + [(K^{-1} - I)(I - I_n)]H_r.$$

According to Table 12.2, the quantity in square brackets is zero, so

$$SA^K = K^{-1}(I_n - I_r)H_r,$$

[9] Well, no, actually, the appearance pretty much is entirely convincing, but let us finish the proof symbolically nonetheless.

which, considering that the identity (11.76) has that $(I_n - I_r)H_r = H_r I_{n-r}$, proves (13.16). The final step is to left-multiply (13.16) by $S^{-1} = S^* = S^T$, reaching (13.7) that was to be derived.

One would like to feel sure that the columns of (13.7)'s $A^K$ actually addressed the whole kernel space of $A$ rather than only part. One would further like to feel sure that $A^K$ had no redundant columns; that is, that it had full rank. Moreover, the definition of $A^K$ in the section's introduction demands both of these features. In general such features would be hard to establish, but here the factors conveniently are Gauss-Jordan factors. Regarding the whole kernel space, $A^K$ addresses it because $A^K$ comes from all **a**. Regarding redundancy, $A^K$ lacks it because $SA^K$ lacks it, and $SA^K$ lacks it because according to (13.13) the last rows of $SA^K$ are $H_r I_{n-r}$. So, in fact, (13.7) has both features and does fit the definition.

### 13.3.2 Converting between kernel matrices

If $C$ is a reversible $(n-r)\times(n-r)$ operator by which we right-multiply (13.6), then the matrix

$$A'^K = A^K C \tag{13.19}$$

like $A^K$ evidently represents the kernel of $A$:

$$AA'^K = A(A^K C) = (AA^K)C = 0.$$

Indeed this makes sense: because the columns of $A^K C$ address the same space the columns of $A^K$ address, the two matrices necessarily represent the same underlying kernel. Moreover, *some* $C$ exists to convert $A^K$ into every alternate kernel matrix $A'^K$ of $A$. We know this because § 12.4 lets one replace the columns of $A^K$ with those of $A'^K$, reversibly, one column at a time, without altering the space addressed. (It might not let one replace the columns in sequence, but if out of sequence then a reversible permutation at the end corrects the order. Refer to §§ 12.5.1 and 12.5.2 for the pattern by which this is done.)

The orthonormalizing column operator $R^{-1}$ of (13.54) below incidentally tends to make a good choice for $C$.

### 13.3.3 The degree of freedom

A slightly vague but extraordinarily useful concept has emerged in this section, worth pausing briefly to appreciate. The concept is the concept of the *degree of freedom.*

A degree of freedom is a parameter one remains free to determine within some continuous domain. For example, Napoleon's artillerist[10] might have enjoyed as many as six degrees of freedom in firing a cannonball: two in where he chose to set up his cannon (one degree in north-south position, one in east-west); two in aim (azimuth and elevation); one in muzzle velocity (as governed by the quantity of gunpowder used to propel the ball); and one in time. A seventh potential degree of freedom, the height from which the artillerist fires, is of course restricted by the lay of the land: the artillerist can fire from a high place only if the place he has chosen to fire from happens to be up on a hill, for Napoleon had no flying cannon. Yet even among the six remaining degrees of freedom, the artillerist might find some impractical to exercise. The artillerist probably preloads the cannon always with a standard charge of gunpowder because, when he finds his target in the field, he cannot spare the time to unload the cannon and alter the charge: this costs one degree of freedom. Likewise, the artillerist must limber up the cannon and hitch it to a horse to shift it to better ground; for this too he cannot spare time in the heat of battle: this costs two degrees. And Napoleon might yell, "Fire!" canceling the time degree as well. Two degrees of freedom remain to the artillerist; but, since exactly two degrees are needed to hit some particular target on the battlefield, the two are enough.

Now consider what happens if the artillerist loses one of his last two remaining degrees of freedom. Maybe the cannon's carriage wheel is broken and the artillerist can no longer turn the cannon; that is, he can still choose firing elevation but no longer azimuth. In such a strait to hit some particular target on the battlefield, the artillerist needs somehow to recover another degree of freedom, for he needs two but has only one. If he disregards Napoleon's order, "Fire!" (maybe not a wise thing to do, but, anyway, ...) and waits for the target to traverse the cannon's fixed line of fire, then he can still hope to hit even with the broken carriage wheel; for could he choose neither azimuth nor the moment to fire, then he would almost surely miss.

Some apparent degrees of freedom are not real. For example, muzzle velocity gives the artillerist little control firing elevation does not also give. Other degrees of freedom are nonlinear in effect: a certain firing elevation gives maximum range; nearer targets can be hit by firing either higher or lower at the artillerist's discretion. On the other hand, too much gunpowder might break the cannon.

[10]The author, who has never fired an artillery piece (unless an arrow from a Boy Scout bow counts), invites any real artillerist among the readership to write in to improve the example.

All of this is hard to generalize in unambiguous mathematical terms, but the count of the degrees of freedom in a system is of high conceptual importance to the engineer nonetheless. Basically, the count captures the idea that to control $n$ output variables of some system takes at least $n$ independent input variables. The $n$ may possibly for various reasons still not suffice—it might be wise in some cases to allow $n + 1$ or $n + 2$—but in no event will fewer than $n$ do. Engineers of all kinds think in this way: an aeronautical engineer knows in advance that an airplane needs at least $n$ ailerons, rudders and other control surfaces for the pilot adequately to control the airplane; an electrical engineer knows in advance that a circuit needs at least $n$ potentiometers for the technician adequately to tune the circuit; and so on.

In geometry, a line brings a single degree of freedom. A plane brings two. A point brings none. If the line bends and turns like a mountain road, it still brings a single degree of freedom. And if the road reaches an intersection? Answer: still one degree. A degree of freedom has some continuous nature, not merely a discrete choice to turn left or right. On the other hand, a swimmer in a swimming pool enjoys three degrees of freedom (up-down, north-south, east-west) even though his domain in any of the three is limited to the small volume of the pool. The driver on the mountain road cannot claim a second degree of freedom at the mountain intersection (he can indeed claim a choice, but the choice being discrete lacks the proper character of a degree of freedom), but he might plausibly claim a second degree of freedom upon reaching the city, where the web or grid of streets is dense enough to approximate access to any point on the city's surface. Just how many streets it takes to turn the driver's "line" experience into a "plane" experience is a matter for the mathematician's discretion.

Reviewing (13.11), we find $n - r$ degrees of freedom in the general underdetermined linear system, represented by the $n - r$ free elements of $\mathbf{a}$. If the underdetermined system is not also overdetermined, if it is nondegenerate such that $r = m$, then it is guaranteed to have a family of solutions $\mathbf{x}$. This family is the topic of the next section.

## 13.4 The nonoverdetermined linear system

The exactly determined linear system of § 13.2 is common, but also common is the more general, nonoverdetermined linear system

$$A\mathbf{x} = \mathbf{b}, \tag{13.20}$$

in which $\mathbf{b}$ is a known, $m$-element vector; $\mathbf{x}$ is an unknown, $n$-element vector; and $A$ is a square or broad, $m \times n$ matrix of full row rank (§ 12.5.4)

$$r = m \leq n. \tag{13.21}$$

Except in the exactly determined edge case $r = m = n$ of § 13.2, the nonoverdetermined linear system has no unique solution but rather a family of solutions. This section delineates the family.

### 13.4.1 Particular and homogeneous solutions

The nonoverdetermined linear system (13.20) by definition admits more than one solution $\mathbf{x}$ for a given driving vector $\mathbf{b}$. Such a system is hard to solve all at once, though, so we prefer to split the system as

$$\begin{aligned} A\mathbf{x}_1 &= \mathbf{b}, \\ A(A^K\mathbf{a}) &= 0, \\ \mathbf{x} &= \mathbf{x}_1 + A^K\mathbf{a}, \end{aligned} \tag{13.22}$$

which, when the second line is added to the first and the third is substituted, makes the whole form (13.20). Splitting the system does not change it, but it does let us treat the system's first and second lines in (13.22) separately. In the split form, the symbol $\mathbf{x}_1$ represents any one $n$-element vector that happens to satisfy the form's first line—many are possible; the mathematician just picks one—and is called *a particular solution* of (13.20). The $(n - r)$-element vector $\mathbf{a}$ remains unspecified, whereupon $A^K\mathbf{a}$ represents the complete family of $n$-element vectors that satisfy the form's second line. The family of vectors expressible as $A^K\mathbf{a}$ is called *the homogeneous solution* of (13.20).

Notice the italicized articles *a* and *the.*

The Gauss-Jordan kernel formula (13.7) has given us $A^K$ and thereby the homogeneous solution, which renders the analysis of (13.20) already half done. To complete the analysis, it remains in § 13.4.2 to find a particular solution.

### 13.4.2 A particular solution

Any particular solution will do. Equation (13.11) has that

$$\begin{aligned} \mathbf{f}(\mathbf{a}, \mathbf{b}) &= G_>^{-1}\mathbf{b} - I_r K H_r \mathbf{a}, \\ (S)\left[\mathbf{x}_1(\mathbf{a}, \mathbf{b}) + A^K\mathbf{a}\right] &= \begin{bmatrix} \mathbf{f}(\mathbf{a}, \mathbf{b}) \\ \mathbf{a} \end{bmatrix} = \mathbf{f}(\mathbf{a}, \mathbf{b}) + H_r\mathbf{a}, \end{aligned}$$

where we have substituted the last line of (13.22) for $\mathbf{x}$. This holds for any $\mathbf{a}$ and $\mathbf{b}$. We are not free to choose the driving vector $\mathbf{b}$, but since we need only one particular solution, $\mathbf{a}$ can be anything we want. Why not

$$\mathbf{a} = 0?$$

Then

$$\begin{aligned} \mathbf{f}(0, \mathbf{b}) &= G_>^{-1}\mathbf{b}, \\ S\mathbf{x}_1(0, \mathbf{b}) &= \begin{bmatrix} \mathbf{f}(0, \mathbf{b}) \\ 0 \end{bmatrix} = \mathbf{f}(0, \mathbf{b}). \end{aligned}$$

That is,

$$\mathbf{x}_1 = S^{-1}G_>^{-1}\mathbf{b}. \tag{13.23}$$

### 13.4.3 The general solution

Assembling (13.7), (13.22) and (13.23) yields the general solution

$$\mathbf{x} = S^{-1}(G_>^{-1}\mathbf{b} + K^{-1}H_r I_{n-r}\mathbf{a}) \tag{13.24}$$

to the nonoverdetermined linear system (13.20).

In exact arithmetic (13.24) solves the nonoverdetermined linear system in theory exactly. Of course, practical calculations are usually done in limited precision, in which compounded rounding error in the last bit eventually disrupts (13.24) for matrices larger than some moderately large size. Avoiding unduly small pivots early in the Gauss-Jordan extends (13.24)'s reach to larger matrices, and for yet larger matrices a bewildering variety of more sophisticated techniques exists to mitigate the problem, which can be vexing because the problem arises even when the matrix $A$ is exactly known. Equation (13.24) is useful and correct, but one should at least be aware that it can in practice lose floating-point accuracy when the matrix it attacks grows too large. (It can also lose accuracy when the matrix's rows are almost dependent, but that is more the fault of the matrix than of the formula. See § 14.8, which addresses a related problem.)

## 13.5 The residual

Equations (13.2) and (13.4) solve the exactly determined linear system $A\mathbf{x} = \mathbf{b}$. Equation (13.24) broadens the solution to include the nonoverdetermined linear system. None of those equations however can handle the

overdetermined linear system, because for general $\mathbf{b}$ the overdetermined linear system

$$A\mathbf{x} \approx \mathbf{b} \tag{13.25}$$

has no exact solution. (See § 12.5.5 for the definitions of *underdetermined, overdetermined,* etc.)

One is tempted to declare the overdetermined system uninteresting because it has no solution and to leave the matter there, but this would be a serious mistake. In fact the overdetermined system is especially interesting, and the more so because it arises so frequently in applications. One seldom trusts a minimal set of data for important measurements, yet extra data imply an overdetermined system. We need to develop the mathematics to handle the overdetermined system properly.

The quantity[11,12]

$$\mathbf{r}(\mathbf{x}) \equiv \mathbf{b} - A\mathbf{x} \tag{13.26}$$

measures how nearly some candidate solution $\mathbf{x}$ solves the system (13.25). We call this quantity the *residual,* and the smaller, the better. More precisely, the smaller the nonnegative real scalar

$$[\mathbf{r}(\mathbf{x})]^*[\mathbf{r}(\mathbf{x})] = \sum_i |r_i(\mathbf{x})|^2 \tag{13.27}$$

is, called the *squared residual norm,* the more favorably we regard the candidate solution $\mathbf{x}$.

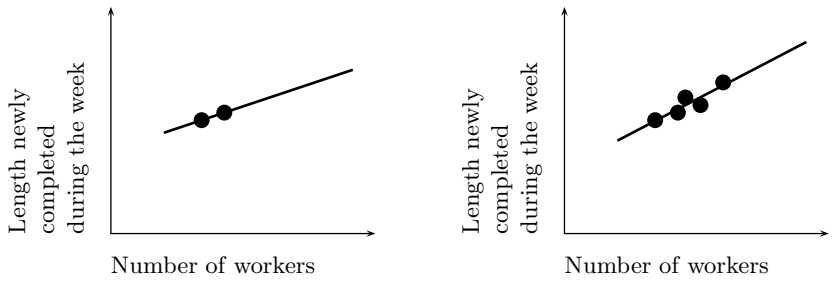
## 13.6 The Moore-Penrose pseudoinverse and the least-squares problem

A typical problem is to fit a straight line to some data. For example, suppose that we are building-construction contractors with a unionized work force, whose labor union can supply additional, fully trained labor on demand. Suppose further that we are contracted to build a long freeway and have been adding workers to the job in recent weeks to speed construction. On Saturday morning at the end of the second week, we gather and plot the production data on the left of Fig. 13.1. If $u_i$ and $b_i$ respectively represent the number of workers and the length of freeway completed during week $i$,

[11] Alas, the alphabet has only so many letters (see Appendix B). The $\mathbf{r}$ here is unrelated to matrix rank $r$.

[12] This is as [63] defines it. Some authors [47] however prefer to define $\mathbf{r}(\mathbf{x}) \equiv A\mathbf{x} - \mathbf{b}$, instead.

Figure 13.1: Fitting a line to measured data.




then we can fit a straight line $b = \sigma u + \gamma$ to the measured production data such that

$$\begin{bmatrix} u_1 & 1 \\ u_2 & 1 \end{bmatrix} \begin{bmatrix} \sigma \\ \gamma \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix},$$

inverting the matrix per §§ 13.1 and 13.2 to solve for $\mathbf{x} \equiv [\sigma \ \gamma]^T$, in the hope that the resulting line will predict future production accurately.

That is all mathematically irreproachable. By the fifth Saturday however we shall have gathered more production data, plotted on the figure's right, to which we should like to fit a better line to predict production more accurately. The added data present a problem. Statistically, the added data are welcome, but geometrically we need only two points to specify a line; what are we to do with the other three? The five points together overdetermine the linear system

$$\begin{bmatrix} u_1 & 1 \\ u_2 & 1 \\ u_3 & 1 \\ u_4 & 1 \\ u_5 & 1 \end{bmatrix} \begin{bmatrix} \sigma \\ \gamma \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \\ b_5 \end{bmatrix}.$$

There is no way to draw a single straight line $b = \sigma u + \gamma$ exactly through all five, for in placing the line we enjoy only two degrees of freedom.[13]

The proper approach is to draw among the data points a single straight line that misses the points as narrowly as possible. More precisely, the proper

[13] Section 13.3.3 characterized a line as enjoying only one degree of freedom. Why now two? The answer is that § 13.3.3 discussed travel along a line rather than placement of a line as here. Though both involve lines, they differ as driving an automobile differs from washing one. Do not let this confuse you.

approach chooses parameters $\sigma$ and $\gamma$ to minimize the squared residual norm $[\mathbf{r}(\mathbf{x})]^*[\mathbf{r}(\mathbf{x})]$ of § 13.5, given that

$$A = \begin{bmatrix} u_1 & 1 \\ u_2 & 1 \\ u_3 & 1 \\ u_4 & 1 \\ u_5 & 1 \\ \vdots & \end{bmatrix}, \quad \mathbf{x} = \begin{bmatrix} \sigma \\ \gamma \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \\ b_5 \\ \vdots \end{bmatrix}.$$

Such parameters constitute a *least-squares* solution.

The matrix $A$ in the example has two columns, data marching on the left, all ones on the right. This is a typical structure for $A$, but in general any matrix $A$ with any number of columns of any content might arise (because there were more than two relevant variables or because some data merited heavier weight than others, among many further reasons). Whatever matrix $A$ might arise from whatever source, this section attacks the difficult but important problem of approximating optimally a solution to the general, possibly unsolvable linear system (13.25), $A\mathbf{x} \approx \mathbf{b}$.

### 13.6.1 Least squares in the real domain

The least-squares problem is simplest when the matrix $A$ enjoys full column rank and no complex numbers are involved. In this case, we seek to minimize the squared residual norm

$$\begin{aligned} [\mathbf{r}(\mathbf{x})]^T[\mathbf{r}(\mathbf{x})] &= (\mathbf{b} - A\mathbf{x})^T(\mathbf{b} - A\mathbf{x}) \\ &= \mathbf{x}^T A^T A\mathbf{x} + \mathbf{b}^T\mathbf{b} - \left(\mathbf{x}^T A^T\mathbf{b} + \mathbf{b}^T A\mathbf{x}\right) \\ &= \mathbf{x}^T A^T A\mathbf{x} + \mathbf{b}^T\mathbf{b} - 2\mathbf{x}^T A^T\mathbf{b} \\ &= \mathbf{x}^T A^T \left(A\mathbf{x} - 2\mathbf{b}\right) + \mathbf{b}^T\mathbf{b}, \end{aligned}$$

in which the transpose is used interchangeably for the adjoint because all the numbers involved happen to be real. The norm is minimized where

$$\frac{d}{d\mathbf{x}}\left(\mathbf{r}^T\mathbf{r}\right) = 0$$

(in which $d/d\mathbf{x}$ is the Jacobian operator of § 11.10). A requirement that

$$\frac{d}{d\mathbf{x}}\left[\mathbf{x}^T A^T \left(A\mathbf{x} - 2\mathbf{b}\right) + \mathbf{b}^T\mathbf{b}\right] = 0$$

comes of combining the last two equations. Differentiating by the Jacobian product rule (11.79) yields the equation

$$\mathbf{x}^T A^T A + \left[A^T (A\mathbf{x} - 2\mathbf{b})\right]^T = 0;$$

or, after transposing the equation, rearranging terms and dividing by 2, the simplified equation

$$A^T A\mathbf{x} = A^T \mathbf{b}.$$

Assuming (as warranted by § 13.6.2, next) that the $n \times n$ square matrix $A^T A$ is invertible, the simplified equation implies the approximate but optimal least-squares solution

$$\mathbf{x} = \left(A^T A\right)^{-1} A^T \mathbf{b} \tag{13.28}$$

to the unsolvable linear system (13.25) in the restricted but quite typical case that $A$ and $\mathbf{b}$ are real and $A$ has full column rank.

Equation (13.28) plots the line on Fig. 13.1's right. As the reader can see, the line does not pass through all the points, for no line can; but it does pass pretty convincingly nearly among them. In fact it passes optimally nearly among them. No line can pass more nearly, in the squared-residual norm sense of (13.27).[14]

[14]Here is a nice example of the use of the mathematical adjective *optimal* in its adverbial form. "Optimal" means "best." Many problems in applied mathematics involve discovering the best of something. What constitutes the best however can be a matter of judgment, even of dispute. We will leave to the philosopher and the theologian the important question of what constitutes objective good, for applied mathematics is a poor guide to such mysteries. The role of applied mathematics is to construct suitable models to calculate quantities needed to achieve some definite good; its role is not, usually, to identify the good as good in the first place.

One generally establishes mathematical optimality by some suitable, nonnegative, real *cost function* or *metric,* and the less, the better. Strictly speaking, the mathematics cannot tell us which metric to use, but where no other consideration prevails the applied mathematician tends to choose the metric that best simplifies the mathematics at hand—and, really, that is about as good a way to choose a metric as any. The metric (13.27) is so chosen.

"But," comes the objection, "what if some more complicated metric is better?"

Well, if the other metric really, objectively is better, then one should probably use it. In general however the mathematical question is: what does one mean by "better?" Better by which metric? Each metric is better according to itself. This is where the mathematician's experience, taste and judgment come in.

In the present section's example, too much labor on the freeway job might actually slow construction rather than speed it. One could therefore seek to fit not a line but some downward-turning curve to the data. Mathematics offers many downward-turning curves. A circle, maybe? Not likely. An experienced mathematician would probably reject the circle on the aesthetic yet practical ground that the parabola $b = \alpha u^2 + \sigma u + \gamma$ lends

### 13.6.2 The invertibility of $A^*A$

Section 13.6.1 has assumed correctly but unwarrantedly that the product $A^TA$ were invertible for real $A$ of full column rank. For real $A$, it happens that $A^T = A^*$, so it only broadens the same assumption to suppose that the product $A^*A$ were invertible for complex $A$ of full column rank.[15] This subsection warrants the latter assumption, thereby incidentally also warranting the former.

Let $A$ be a complex, $m \times n$ matrix of full column rank $r = n \leq m$. Suppose falsely that $A^*A$ were not invertible but singular. Since the product $A^*A$ is a square, $n \times n$ matrix, this is to suppose (§ 13.1) that the product's rank $r' < n$ were less than full, implying (§ 12.5.4) that its columns (as its rows) depended on one another. This would mean that there existed a nonzero, $n$-element vector $\mathbf{u}$ for which

$$A^*A\mathbf{u} = 0, \quad I_n\mathbf{u} \neq 0.$$

Left-multiplying by $\mathbf{u}^*$ would give that

$$\mathbf{u}^*A^*A\mathbf{u} = 0, \quad I_n\mathbf{u} \neq 0,$$

or in other words that

$$(A\mathbf{u})^*(A\mathbf{u}) = \sum_{i=1}^{n} |[A\mathbf{u}]_i|^2 = 0, \quad I_n\mathbf{u} \neq 0.$$

But this could only be so if

$$A\mathbf{u} = 0, \quad I_n\mathbf{u} \neq 0,$$

impossible when the columns of $A$ are independent. The contradiction proves false the assumption which gave rise to it. The false assumption: that $A^*A$ were singular.

Thus, *the $n \times n$ product $A^*A$ is invertible for any tall or square, $m \times n$ matrix $A$ of full column rank $r = n \leq m$.*

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itself to easier analysis. Yet even fitting a mere straight line offers choices. One might fit the line to the points $(b_i, u_i)$ or $(\ln u_i, \ln b_i)$ rather than to the points $(u_i, b_i)$. The three resulting lines differ subtly. They predict production differently. The adjective "optimal" alone evidently does not always tell us all we need to know.

Section 6.3 offers a choice between averages that resembles in spirit this footnote's choice between metrics.

[15] Notice that if $A$ is tall, then $A^*A$ is a compact, $n \times n$ square, whereas $AA^*$ is a big, $m \times m$ square. It is the compact square that concerns this section. The big square is not very interesting and in any case is not invertible.

### 13.6.3 Positive definiteness

An $n \times n$ matrix $C$ is *positive definite* if and only if

$$\Im(\mathbf{u}^* C \mathbf{u}) = 0 \text{ and } \Re(\mathbf{u}^* C \mathbf{u}) > 0 \text{ for all } I_n \mathbf{u} \neq 0. \tag{13.29}$$

As in § 13.6.2, here also when a matrix $A$ has full column rank $r = n \leq m$ the product $\mathbf{u}^* A^* A \mathbf{u} = (A\mathbf{u})^*(A\mathbf{u})$ is real and positive for all nonzero, $n$-element vectors $\mathbf{u}$. Thus per (13.29) *the product $A^*A$ is positive definite for any matrix $A$ of full column rank.*

An $n \times n$ matrix $C$ is *nonnegative definite* if and only if

$$\Im(\mathbf{u}^* C \mathbf{u}) = 0 \text{ and } \Re(\mathbf{u}^* C \mathbf{u}) \geq 0 \text{ for all } \mathbf{u}. \tag{13.30}$$

By reasoning like the last paragraph's, *the product $A^*A$ is nonnegative definite for any matrix $A$ whatsoever.*

Such definitions might seem opaque, but their sense is that a positive definite operator never reverses the thing it operates on, that the product $A\mathbf{u}$ points more in the direction of $\mathbf{u}$ than of $-\mathbf{u}$. Section 13.8 explains further. A positive definite operator resembles a positive scalar in this sense.

### 13.6.4 The Moore-Penrose pseudoinverse

Not every $m \times n$ matrix $A$ enjoys full rank. According to (12.16), however, every $m \times n$ matrix $A$ of rank $r$ can be factored into a product[16]

$$A = BC$$

of an $m \times r$ tall or square matrix $B$ and an $r \times n$ broad or square matrix $C$, both of which factors themselves enjoy full rank $r$. (If $A$ happens to have full row or column rank, then one can just choose $B = I_m$ or $C = I_n$; but even if $A$ lacks full rank, the Gauss-Jordan decomposition of eqn. 12.2 finds at least the full-rank factorization $B = G_> I_r$, $C = I_r G_<$.) This being so, a conjecture seems warranted. Suppose that, inspired by (13.28), we manipulated (13.25) by the successive steps

$$\begin{aligned}
A\mathbf{x} &\approx \mathbf{b}, \\
BC\mathbf{x} &\approx \mathbf{b}, \\
(B^*B)^{-1}B^*BC\mathbf{x} &\approx (B^*B)^{-1}B^*\mathbf{b}, \\
C\mathbf{x} &\approx (B^*B)^{-1}B^*\mathbf{b}.
\end{aligned}$$

[16] This subsection uses the symbols $B$ and $\mathbf{b}$ for unrelated purposes, which is unfortunate but conventional. See footnote 11.

Then suppose that we changed

$$C^*\mathbf{u} \leftarrow \mathbf{x},$$

thus restricting $\mathbf{x}$ to the space addressed by the independent columns of $C^*$. Continuing,

$$\begin{aligned} CC^*\mathbf{u} &\approx (B^*B)^{-1}B^*\mathbf{b}, \\ \mathbf{u} &\approx (CC^*)^{-1}(B^*B)^{-1}B^*\mathbf{b}. \end{aligned}$$

Changing the variable back and (because we are conjecturing and can do as we like), altering the "$\approx$" sign to "$=$,"

$$\mathbf{x} = C^*(CC^*)^{-1}(B^*B)^{-1}B^*\mathbf{b}. \tag{13.31}$$

Equation (13.31) has a pleasingly symmetrical form, and we know from § 13.6.2 at least that the two matrices it tries to invert are invertible. So here is our conjecture:

- no $\mathbf{x}$ enjoys a smaller squared residual norm $\mathbf{r}^*\mathbf{r}$ than the $\mathbf{x}$ of (13.31) does; and
- among all $\mathbf{x}$ that enjoy the same, minimal squared residual norm, the $\mathbf{x}$ of (13.31) is strictly least in magnitude.

The conjecture is bold, but if you think about it in the right way it is not unwarranted under the circumstance. After all, (13.31) does resemble (13.28), the latter of which admittedly requires real $A$ of full column rank but does minimize the residual when its requirements are met; and, even if there were more than one $\mathbf{x}$ which minimized the residual, one of them might be smaller than the others: why not the $\mathbf{x}$ of (13.31)? One can but investigate.

The first point of the conjecture is symbolized

$$\mathbf{r}^*(\mathbf{x})\mathbf{r}(\mathbf{x}) \le \mathbf{r}^*(\mathbf{x}+\Delta\mathbf{x})\mathbf{r}(\mathbf{x}+\Delta\mathbf{x}),$$

where $\Delta\mathbf{x}$ represents the deviation, whether small, moderate or large, of some alternate $\mathbf{x}$ from the $\mathbf{x}$ of (13.31). According to (13.26), this is

$$[\mathbf{b}-A\mathbf{x}]^*[\mathbf{b}-A\mathbf{x}] \le [\mathbf{b}-(A)(\mathbf{x}+\Delta\mathbf{x})]^*[\mathbf{b}-(A)(\mathbf{x}+\Delta\mathbf{x})].$$

Reorganizing,

$$[\mathbf{b}-A\mathbf{x}]^*[\mathbf{b}-A\mathbf{x}] \le [(\mathbf{b}-A\mathbf{x})-A\,\Delta\mathbf{x}]^*[(\mathbf{b}-A\mathbf{x})-A\,\Delta\mathbf{x}].$$

Distributing factors and canceling like terms,

$$0 \leq -\Delta\mathbf{x}^*\, A^*(\mathbf{b} - A\mathbf{x}) - (\mathbf{b} - A\mathbf{x})^* A\, \Delta\mathbf{x} + \Delta\mathbf{x}^*\, A^* A\, \Delta\mathbf{x}.$$

But according to (13.31) and the full-rank factorization $A = BC$,

$$\begin{aligned} A^*(\mathbf{b} - A\mathbf{x}) &= A^*\mathbf{b} - A^*A\mathbf{x} \\ &= [C^*B^*][\mathbf{b}] - [C^*B^*][BC][C^*(CC^*)^{-1}(B^*B)^{-1}B^*\mathbf{b}] \\ &= C^*B^*\mathbf{b} - C^*(B^*B)(CC^*)(CC^*)^{-1}(B^*B)^{-1}B^*\mathbf{b} \\ &= C^*B^*\mathbf{b} - C^*B^*\mathbf{b} = 0, \end{aligned}$$

which reveals two of the inequality's remaining three terms to be zero, leaving an assertion that

$$0 \leq \Delta\mathbf{x}^*\, A^* A\, \Delta\mathbf{x}.$$

Each step in the present paragraph is reversible,[17] so the assertion in the last form is logically equivalent to the conjecture's first point, with which the paragraph began. Moreover, the assertion in the last form is correct because the product of any matrix and its adjoint according to § 13.6.3 is a nonnegative definite operator, thus establishing the conjecture's first point.

The conjecture's first point, now established, has it that no $\mathbf{x}+\Delta\mathbf{x}$ enjoys a smaller squared residual norm than the $\mathbf{x}$ of (13.31) does. It does not claim that no $\mathbf{x}+\Delta\mathbf{x}$ enjoys the same, minimal squared residual norm. The latter case is symbolized

$$\mathbf{r}^*(\mathbf{x})\mathbf{r}(\mathbf{x}) = \mathbf{r}^*(\mathbf{x}+\Delta\mathbf{x})\mathbf{r}(\mathbf{x}+\Delta\mathbf{x}),$$

or equivalently by the last paragraph's logic,

$$0 = \Delta\mathbf{x}^*\, A^* A\, \Delta\mathbf{x};$$

or in other words,

$$A\, \Delta\mathbf{x} = 0.$$

But $A = BC$, so this is to claim that

$$B(C\, \Delta\mathbf{x}) = 0,$$

which since $B$ has full column rank is possible only if

$$C\, \Delta\mathbf{x} = 0.$$

[17] The paragraph might inscrutably but logically instead have ordered the steps in reverse as in §§ 6.3.2 and 9.5. See Ch. 6's footnote 16.

Considering the product $\Delta\mathbf{x}^*\,\mathbf{x}$ in light of (13.31) and the last equation, we observe that

$$\begin{aligned}\Delta\mathbf{x}^*\,\mathbf{x} &= \Delta\mathbf{x}^*\,[C^*(CC^*)^{-1}(B^*B)^{-1}B^*\mathbf{b}] \\ &= [C\,\Delta\mathbf{x}]^*[(CC^*)^{-1}(B^*B)^{-1}B^*\mathbf{b}],\end{aligned}$$

which is to observe that

$$\Delta\mathbf{x}^*\,\mathbf{x} = 0$$

for any $\Delta\mathbf{x}$ for which $\mathbf{x}+\Delta\mathbf{x}$ achieves minimal squared residual norm.

Returning attention to the conjecture, its second point is symbolized

$$\mathbf{x}^*\mathbf{x} < (\mathbf{x}+\Delta\mathbf{x})^*(\mathbf{x}+\Delta\mathbf{x})$$

for any

$$\Delta\mathbf{x} \neq 0$$

for which $\mathbf{x}+\Delta\mathbf{x}$ achieves minimal squared residual norm (note that it's "$<$" this time, not "$\leq$" as in the conjecture's first point). Distributing factors and canceling like terms,

$$0 < \mathbf{x}^*\,\Delta\mathbf{x} + \Delta\mathbf{x}^*\,\mathbf{x} + \Delta\mathbf{x}^*\,\Delta\mathbf{x}.$$

But the last paragraph has found that $\Delta\mathbf{x}^*\,\mathbf{x} = 0$ for precisely such $\Delta\mathbf{x}$ as we are considering here, so the last inequality reduces to read

$$0 < \Delta\mathbf{x}^*\,\Delta\mathbf{x},$$

which naturally for $\Delta\mathbf{x} \neq 0$ is true. Since each step in the paragraph is reversible, reverse logic establishes the conjecture's second point.

With both its points established, the conjecture is true.

If $A = BC$ is a full-rank factorization, then the matrix[18]

$$A^\dagger \equiv C^*(CC^*)^{-1}(B^*B)^{-1}B^* \tag{13.32}$$

of (13.31) is called the *Moore-Penrose pseudoinverse* of $A$, more briefly the *pseudoinverse* of $A$. Whether underdetermined, exactly determined, overdetermined or even degenerate, every matrix has a Moore-Penrose pseudoinverse. Yielding the optimal approximation

$$\mathbf{x} = A^\dagger\mathbf{b}, \tag{13.33}$$

[18]Some books print $A^\dagger$ as $A^+$.

the Moore-Penrose solves the linear system (13.25) as well as the system can be solved—exactly if possible, with minimal squared residual norm if impossible. If $A$ is square and invertible, then the Moore-Penrose $A^\dagger = A^{-1}$ is just the inverse, and then of course (13.33) solves the system uniquely and exactly. Nothing can solve the system uniquely if $A$ has broad shape but the Moore-Penrose still solves the system exactly in that case as long as $A$ has full row rank, moreover minimizing the solution's squared magnitude $\mathbf{x}^*\mathbf{x}$ (which the solution of eqn. 13.23 fails to do). If $A$ lacks full row rank, then the Moore-Penrose solves the system as nearly as the system can be solved (as in Fig. 13.1) and as a side-benefit also minimizes $\mathbf{x}^*\mathbf{x}$. The Moore-Penrose is thus a general-purpose solver and approximator for linear systems. It is a significant discovery.[19]

## 13.7 The multivariate Newton-Raphson iteration

When we first met the Newton-Raphson iteration in § 4.8 we lacked the matrix notation and algebra to express and handle vector-valued functions adeptly. Now that we have the notation and algebra we can write down the multivariate Newton-Raphson iteration almost at once.

The iteration approximates the nonlinear vector function $\mathbf{f}(\mathbf{x})$ by its tangent

$$\tilde{\mathbf{f}}_k(\mathbf{x}) = \mathbf{f}(\mathbf{x}_k) + \left[\frac{d}{d\mathbf{x}}\mathbf{f}(\mathbf{x})\right]_{\mathbf{x}=\mathbf{x}_k} (\mathbf{x} - \mathbf{x}_k),$$

where $d\mathbf{f}/d\mathbf{x}$ is the Jacobian derivative of § 11.10. It then approximates the root $\mathbf{x}_{k+1}$ as the point at which $\tilde{\mathbf{f}}_k(\mathbf{x}_{k+1}) = 0$:

$$\tilde{\mathbf{f}}_k(\mathbf{x}_{k+1}) = 0 = \mathbf{f}(\mathbf{x}_k) + \left[\frac{d}{d\mathbf{x}}\mathbf{f}(\mathbf{x})\right]_{\mathbf{x}=\mathbf{x}_k} (\mathbf{x}_{k+1} - \mathbf{x}_k).$$

Solving for $\mathbf{x}_{k+1}$ (approximately if necessary), we have that

$$\mathbf{x}_{k+1} = \mathbf{x} - \left[\frac{d}{d\mathbf{x}}\mathbf{f}(\mathbf{x})\right]^\dagger \mathbf{f}(\mathbf{x})\Bigg|_{\mathbf{x}=\mathbf{x}_k}, \tag{13.34}$$

where $[\cdot]^\dagger$ is the Moore-Penrose pseudoinverse of § 13.6—which is just the ordinary inverse $[\cdot]^{-1}$ of § 13.1 if $\mathbf{f}$ and $\mathbf{x}$ happen each to have the same number of elements. Refer to § 4.8 and Fig. 4.5.[20]

[19] [6, § 3.3][50, "Moore-Penrose generalized inverse"]
[20] [53]

Despite the Moore-Penrose notation of (13.34), the Newton-Raphson iteration is not normally meant to be applied at a value of $\mathbf{x}$ for which the Jacobian is degenerate. The iteration intends rather in light of (13.32) that

$$\left[\frac{d}{d\mathbf{x}}\mathbf{f}(\mathbf{x})\right]^{\dagger} = \begin{cases} [d\mathbf{f}/d\mathbf{x}]^{*}\left([d\mathbf{f}/d\mathbf{x}]\,[d\mathbf{f}/d\mathbf{x}]^{*}\right)^{-1} & \text{if } r = m \leq n, \\ [d\mathbf{f}/d\mathbf{x}]^{-1} & \text{if } r = m = n, \\ \left([d\mathbf{f}/d\mathbf{x}]^{*}\,[d\mathbf{f}/d\mathbf{x}]\right)^{-1}[d\mathbf{f}/d\mathbf{x}]^{*} & \text{if } r = n \leq m, \end{cases} \tag{13.35}$$

where $B = I_m$ in the first case and $C = I_n$ in the last. It does not intend to use the full (13.32). If both $r < m$ and $r < n$—which is to say, if the Jacobian is degenerate—then (13.35) fails, as though the curve of Fig. 4.5 ran horizontally at the test point—when one quits, restarting the iteration from another point.

## 13.8 The dot product

The *dot product* of two vectors, also called the *inner product,*[21] is the product of the two vectors to the extent to which they run in the same direction. It is written

$$\mathbf{a} \cdot \mathbf{b}.$$

In general,

$$\mathbf{a} \cdot \mathbf{b} = (a_1\mathbf{e}_1 + a_2\mathbf{e}_2 + \cdots + a_n\mathbf{e}_n) \cdot (b_1\mathbf{e}_1 + b_2\mathbf{e}_2 + \cdots + b_n\mathbf{e}_n).$$

But if the dot product is to mean anything, it must be that

$$\mathbf{e}_i \cdot \mathbf{e}_j = \delta_{ij}. \tag{13.36}$$

Therefore,

$$\mathbf{a} \cdot \mathbf{b} = a_1b_1 + a_2b_2 + \cdots + a_nb_n;$$

or, more concisely,

$$\mathbf{a} \cdot \mathbf{b} \equiv \mathbf{a}^T\mathbf{b} = \sum_{j=-\infty}^{\infty} a_jb_j. \tag{13.37}$$

[21] The term *inner product* is often used to indicate a broader class of products than the one defined here, especially in some of the older literature. Where used, the notation usually resembles $\langle \mathbf{a}, \mathbf{b} \rangle$ or $(\mathbf{b}, \mathbf{a})$, both of which mean $\mathbf{a}^* \cdot \mathbf{b}$ (or, more broadly, some similar product), except that which of $\mathbf{a}$ and $\mathbf{b}$ is conjugated depends on the author. Most recently, at least in the author's country, the usage $\langle \mathbf{a}, \mathbf{b} \rangle \equiv \mathbf{a}^* \cdot \mathbf{b}$ seems to be emerging as standard where the dot is not used [6, § 3.1][21, Ch. 4]. This book prefers the dot.

The dot notation does not worry whether its arguments are column or row vectors, incidentally:

$$\mathbf{a} \cdot \mathbf{b} = \mathbf{a} \cdot \mathbf{b}^T = \mathbf{a}^T \cdot \mathbf{b} = \mathbf{a}^T \cdot \mathbf{b}^T = \mathbf{a}^T \mathbf{b}.$$

That is, if either vector is wrongly oriented, the notation implicitly reorients it before using it. (The more orderly notation $\mathbf{a}^T\mathbf{b}$ by contrast assumes that both are proper column vectors.)

Where vectors may have complex elements, usually one is not interested in $\mathbf{a} \cdot \mathbf{b}$ so much as in

$$\mathbf{a}^* \cdot \mathbf{b} \equiv \mathbf{a}^*\mathbf{b} = \sum_{j=-\infty}^{\infty} a_j^* b_j. \tag{13.38}$$

The reason is that

$$\Re(\mathbf{a}^* \cdot \mathbf{b}) = \Re(\mathbf{a}) \cdot \Re(\mathbf{b}) + \Im(\mathbf{a}) \cdot \Im(\mathbf{b}),$$

with the product of the imaginary parts added not subtracted, thus honoring the right Argand sense of "the product of the two vectors to the extent to which they run in the same direction."

By the Pythagorean theorem, the dot product

$$|\mathbf{a}|^2 = \mathbf{a}^* \cdot \mathbf{a} \tag{13.39}$$

gives the square of a vector's magnitude, always real, never negative. The unit vector in $\mathbf{a}$'s direction then is

$$\hat{\mathbf{a}} \equiv \frac{\mathbf{a}}{|\mathbf{a}|} = \frac{\mathbf{a}}{\sqrt{\mathbf{a}^* \cdot \mathbf{a}}}, \tag{13.40}$$

from which

$$|\hat{\mathbf{a}}|^2 = \hat{\mathbf{a}}^* \cdot \hat{\mathbf{a}} = 1. \tag{13.41}$$

When two vectors do not run in the same direction at all, such that

$$\mathbf{a}^* \cdot \mathbf{b} = 0, \tag{13.42}$$

the two vectors are said to lie *orthogonal* to one another. Geometrically this puts them at right angles. For other angles $\theta$ between two vectors,

$$\hat{\mathbf{a}}^* \cdot \hat{\mathbf{b}} = \cos\theta, \tag{13.43}$$

which formally defines the angle $\theta$ even when $\mathbf{a}$ and $\mathbf{b}$ have more than three elements each.

## 13.9 The complex vector triangle inequalities

The triangle inequalities (2.44) and (3.21) lead one to hypothesize generally that

$$|\mathbf{a}| - |\mathbf{b}| \leq |\mathbf{a} + \mathbf{b}| \leq |\mathbf{a}| + |\mathbf{b}| \tag{13.44}$$

for any complex, $n$-dimensional vectors $\mathbf{a}$ and $\mathbf{b}$.

The proof of the sum hypothesis that $|\mathbf{a} + \mathbf{b}| \leq |\mathbf{a}| + |\mathbf{b}|$ is by contradiction. We suppose falsely that

$$|\mathbf{a} + \mathbf{b}| > |\mathbf{a}| + |\mathbf{b}| .$$

Squaring and using (13.39),

$$(\mathbf{a} + \mathbf{b})^* \cdot (\mathbf{a} + \mathbf{b}) > \mathbf{a}^* \cdot \mathbf{a} + 2\,|\mathbf{a}|\,|\mathbf{b}| + \mathbf{b}^* \cdot \mathbf{b}.$$

Distributing factors and canceling like terms,

$$\mathbf{a}^* \cdot \mathbf{b} + \mathbf{b}^* \cdot \mathbf{a} > 2\,|\mathbf{a}|\,|\mathbf{b}| .$$

Splitting $\mathbf{a}$ and $\mathbf{b}$ each into real and imaginary parts on the inequality's left side and then halving both sides,

$$\Re(\mathbf{a}) \cdot \Re(\mathbf{b}) + \Im(\mathbf{a}) \cdot \Im(\mathbf{b}) > |\mathbf{a}|\,|\mathbf{b}| .$$

Defining the new, $2n$-dimensional *real* vectors

$$\mathbf{f} \equiv \begin{bmatrix} \Re(\mathbf{a}_1) \\ \Im(\mathbf{a}_1) \\ \Re(\mathbf{a}_2) \\ \Im(\mathbf{a}_2) \\ \vdots \\ \Re(\mathbf{a}_n) \\ \Im(\mathbf{a}_n) \end{bmatrix}, \quad \mathbf{g} \equiv \begin{bmatrix} \Re(\mathbf{b}_1) \\ \Im(\mathbf{b}_1) \\ \Re(\mathbf{b}_2) \\ \Im(\mathbf{b}_2) \\ \vdots \\ \Re(\mathbf{b}_n) \\ \Im(\mathbf{b}_n) \end{bmatrix},$$

we make the inequality to be

$$\mathbf{f} \cdot \mathbf{g} > |\mathbf{f}|\,|\mathbf{g}| ,$$

in which we observe that the left side must be positive because the right side is nonnegative. (This naturally is impossible for any case in which $\mathbf{f} = 0$ or $\mathbf{g} = 0$, among others, but wishing to establish impossibility for all cases we pretend not to notice and continue reasoning as follows.) Squaring again,

$$(\mathbf{f} \cdot \mathbf{g})^2 > (\mathbf{f} \cdot \mathbf{f})(\mathbf{g} \cdot \mathbf{g});$$

or, in other words,

$$\sum_{i,j} f_i g_i f_j g_j > \sum_{i,j} f_i^2 g_j^2.$$

Reordering factors,

$$\sum_{i,j} [(f_i g_j)(g_i f_j)] > \sum_{i,j} (f_i g_j)^2.$$

Subtracting $\sum_i (f_i g_i)^2$ from each side,

$$\sum_{i \neq j} [(f_i g_j)(g_i f_j)] > \sum_{i \neq j} (f_i g_j)^2,$$

which we can cleverly rewrite in the form

$$\sum_{i<j} [2(f_i g_j)(g_i f_j)] > \sum_{i<j} [(f_i g_j)^2 + (g_i f_j)^2],$$

where $\sum_{i<j} = \sum_{i=1}^{2n-1} \sum_{j=i+1}^{2n}$. Transferring all terms to the inequality's right side,

$$0 > \sum_{i<j} [(f_i g_j)^2 + 2(f_i g_j)(g_i f_j) + (g_i f_j)^2].$$

This is

$$0 > \sum_{i<j} [f_i g_j + g_i f_j]^2,$$

which, since we have constructed the vectors $\mathbf{f}$ and $\mathbf{g}$ to have real elements only, is impossible in all cases. The contradiction proves false the assumption that gave rise to it, thus establishing the sum hypothesis of (13.44).

The difference hypothesis that $|\mathbf{a}| - |\mathbf{b}| \leq |\mathbf{a} + \mathbf{b}|$ is established by defining a vector $\mathbf{c}$ such that

$$\mathbf{a} + \mathbf{b} + \mathbf{c} = 0,$$

whereupon according to the sum hypothesis (which we have already established),

$$|\mathbf{a} + \mathbf{c}| \leq |\mathbf{a}| + |\mathbf{c}|,$$
$$|\mathbf{b} + \mathbf{c}| \leq |\mathbf{b}| + |\mathbf{c}|.$$

That is,

$$|-\mathbf{b}| \leq |\mathbf{a}| + |-\mathbf{a} - \mathbf{b}|,$$
$$|-\mathbf{a}| \leq |\mathbf{b}| + |-\mathbf{a} - \mathbf{b}|,$$

which is the difference hypothesis in disguise. This completes the proof of (13.44).

As in § 3.10, here too we can extend the sum inequality to the even more general form

$$\left|\sum_k \mathbf{a}_k\right| \leq \sum_k |\mathbf{a}_k| \,. \tag{13.45}$$

## 13.10 The orthogonal complement

The $m \times (m-r)$ kernel (§ 13.3)[22]

$$A^\perp \equiv A^{*K} \tag{13.46}$$

is an interesting matrix. By definition of the kernel, the columns of $A^{*K}$ are the independent vectors $\mathbf{u}_j$ for which $A^*\mathbf{u}_j = 0$, which—inasmuch as the rows of $A^*$ are the adjoints of the *columns* of $A$—is possible only when each $\mathbf{u}_j$ lies orthogonal to every column of $A$. This says that the columns of $A^\perp \equiv A^{*K}$ address the complete space of vectors that lie orthogonal to $A$'s columns, such that

$$A^{\perp *}A = 0 = A^*A^\perp. \tag{13.47}$$

The matrix $A^\perp$ is called the *orthogonal complement*[23] or *perpendicular matrix* to $A$.

Among other uses, the orthogonal complement $A^\perp$ supplies the columns $A$ lacks to reach full row rank. Properties include that

$$\begin{aligned} A^{*K} &= A^\perp, \\ A^{*\perp} &= A^K. \end{aligned} \tag{13.48}$$

## 13.11 Gram-Schmidt orthonormalization

If a vector $\mathbf{x} = A^K\mathbf{a}$ belongs to a kernel space $A^K$ (§ 13.3), then so equally does any $\alpha\mathbf{x}$. If the vectors $\mathbf{x}_1 = A^K\mathbf{a}_1$ and $\mathbf{x}_2 = A^K\mathbf{a}_2$ both belong, then so does $\alpha_1\mathbf{x}_1 + \alpha_2\mathbf{x}_2$. If I claim $A^K = [3\ 4\ 5; -1\ 1\ 0]^T$ to represent a kernel, then you are not mistaken arbitrarily to rescale each column of my $A^K$ by a separate nonzero factor, instead for instance representing the same kernel

[22] The symbol $A^\perp$ [30][6][42] can be pronounced "A perp," short for "A perpendicular," since by (13.47) $A^\perp$ is in some sense perpendicular to $A$.

If we were really precise, we might write not $A^\perp$ but $A^{\perp(m)}$. Refer to footnote 5.

[23] [30, § 3.VI.3]

as $A^K = [6\ 8\ 0\text{xA};\ \frac{1}{7}\ -\frac{1}{7}\ 0]^T$. Kernel vectors have no inherent scale. Style generally asks the applied mathematician to remove the false appearance of scale by using (13.40) *to normalize* the columns of a kernel matrix to unit magnitude before reporting them. The same goes for the eigenvectors of Ch. 14 to come.

Where a kernel matrix $A^K$ has two or more columns (or a repeated eigenvalue has two or more eigenvectors), style generally asks the applied mathematician not only to normalize but also *to orthogonalize* the columns before reporting them. One orthogonalizes a vector $\mathbf{b}$ with respect to a vector $\mathbf{a}$ by subtracting from $\mathbf{b}$ a multiple of $\mathbf{a}$ such that

$$\begin{aligned}\mathbf{a}^* \cdot \mathbf{b}_\perp &= 0,\\ \mathbf{b}_\perp &\equiv \mathbf{b} - \beta\mathbf{a},\end{aligned}$$

where the symbol $\mathbf{b}_\perp$ represents the orthogonalized vector. Substituting the second of these equations into the first and solving for $\beta$ yields

$$\beta = \frac{\mathbf{a}^* \cdot \mathbf{b}}{\mathbf{a}^* \cdot \mathbf{a}}.$$

Hence,

$$\begin{aligned}\mathbf{a}^* \cdot \mathbf{b}_\perp &= 0,\\ \mathbf{b}_\perp &\equiv \mathbf{b} - \frac{\mathbf{a}^* \cdot \mathbf{b}}{\mathbf{a}^* \cdot \mathbf{a}}\mathbf{a}.\end{aligned} \tag{13.49}$$

But according to (13.40), $\mathbf{a} = \hat{\mathbf{a}}\sqrt{\mathbf{a}^* \cdot \mathbf{a}}$; and according to (13.41), $\hat{\mathbf{a}}^* \cdot \hat{\mathbf{a}} = 1$; so,

$$\mathbf{b}_\perp = \mathbf{b} - \hat{\mathbf{a}}(\hat{\mathbf{a}}^* \cdot \mathbf{b}); \tag{13.50}$$

or, in matrix notation,

$$\mathbf{b}_\perp = \mathbf{b} - \hat{\mathbf{a}}(\hat{\mathbf{a}}^*)(\mathbf{b}).$$

This is arguably better written

$$\mathbf{b}_\perp = \left[I - (\hat{\mathbf{a}})(\hat{\mathbf{a}}^*)\right]\mathbf{b} \tag{13.51}$$

(observe that it's $[\hat{\mathbf{a}}][\hat{\mathbf{a}}^*]$, a matrix, rather than the scalar $[\hat{\mathbf{a}}^*][\hat{\mathbf{a}}]$).

One *orthonormalizes* a set of vectors by orthogonalizing them with respect to one another, then by normalizing each of them to unit magnitude. The procedure to orthonormalize several vectors

$$\{\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \ldots, \mathbf{x}_n\}$$

therefore is as follows. First, normalize $\mathbf{x}_1$ by (13.40); call the result $\hat{\mathbf{x}}_{1\perp}$. Second, orthogonalize $\mathbf{x}_2$ with respect to $\hat{\mathbf{x}}_{1\perp}$ by (13.50) or (13.51), then normalize it; call the result $\hat{\mathbf{x}}_{2\perp}$. Third, orthogonalize $\mathbf{x}_3$ with respect to $\hat{\mathbf{x}}_{1\perp}$ then to $\hat{\mathbf{x}}_{2\perp}$, then normalize it; call the result $\hat{\mathbf{x}}_{3\perp}$. Proceed in this manner through the several $\mathbf{x}_j$. Symbolically,

$$\begin{aligned}\hat{\mathbf{x}}_{j\perp} &= \frac{\mathbf{x}_{j\perp}}{\sqrt{\mathbf{x}_{j\perp}^* \mathbf{x}_{j\perp}}}, \\ \mathbf{x}_{j\perp} &\equiv \left[\prod_{i=1}^{j-1} (I - \hat{\mathbf{x}}_{i\perp}\hat{\mathbf{x}}_{i\perp}^*)\right] \mathbf{x}_j.\end{aligned} \tag{13.52}$$

By the vector replacement principle of § 12.4 in light of (13.49), the resulting orthonormal set of vectors

$$\{\hat{\mathbf{x}}_{1\perp}, \hat{\mathbf{x}}_{2\perp}, \hat{\mathbf{x}}_{3\perp}, \ldots, \hat{\mathbf{x}}_{n\perp}\}$$

addresses the same space as did the original set.

Orthonormalization naturally works equally for any linearly independent set of vectors, not only for kernel vectors or eigenvectors. By the technique, one can conveniently replace a set of independent vectors by an equivalent, neater, orthonormal set which addresses precisely the same space.

### 13.11.1 Efficient implementation

To turn an equation like the latter line of (13.52) into an efficient numerical algorithm sometimes demands some extra thought, in perspective of whatever it happens to be that one is trying to accomplish. If all one wants is some vectors orthonormalized, then the equation as written is neat but is overkill because the product $\hat{\mathbf{x}}_{i\perp}\hat{\mathbf{x}}_{i\perp}^*$ is a matrix, whereas the product $\hat{\mathbf{x}}_{i\perp}^*\mathbf{x}_j$ implied by (13.50) is just a scalar. Fortunately, one need not apply the latter line of (13.52) exactly as written. One can instead introduce intermediate vectors $\mathbf{x}_{ji}$, representing the $\prod$ multiplication in the admittedly messier form

$$\begin{aligned}\mathbf{x}_{j1} &\equiv \mathbf{x}_j, \\ \mathbf{x}_{j(i+1)} &\equiv \mathbf{x}_{ji} - (\hat{\mathbf{x}}_{i\perp}^* \cdot \mathbf{x}_{ji})\,\hat{\mathbf{x}}_{i\perp}, \\ \mathbf{x}_{j\perp} &= \mathbf{x}_{jj}.\end{aligned} \tag{13.53}$$

Besides obviating the matrix $I - \hat{\mathbf{x}}_{i\perp}\hat{\mathbf{x}}_{i\perp}^*$ and the associated matrix multiplication, the messier form (13.53) has the significant additional practical virtue

that it lets one forget each intermediate vector $\mathbf{x}_{ji}$ immediately after using it. (A well-written orthonormalizing computer program reserves memory for one intermediate vector only, which memory it repeatedly overwrites—and, actually, probably does not even reserve that much, working rather in the memory space it has already reserved for $\hat{\mathbf{x}}_{j\perp}$.)[24]

Other equations one algorithmizes can likewise benefit from thoughtful rendering.

### 13.11.2 The Gram-Schmidt decomposition

The orthonormalization technique this section has developed is named the *Gram-Schmidt process.* One can turn it into the *Gram-Schmidt decomposition*

$$\begin{aligned} A &= QR = QUDS, \\ R &\equiv UDS, \end{aligned} \tag{13.54}$$

also called the *orthonormalizing* or *QR decomposition,* by an algorithm that somewhat resembles the Gauss-Jordan algorithm of § 12.3.3; except that (12.4) here becomes

$$A = \tilde{Q}\tilde{U}\tilde{D}\tilde{S} \tag{13.55}$$

and initially $\tilde{Q} \leftarrow A$. By elementary column operations based on (13.52) and (13.53), the algorithm gradually transforms $\tilde{Q}$ into a dimension-limited, $m \times r$ matrix $Q$ of orthonormal columns, distributing the inverse elementaries to $\tilde{U}$, $\tilde{D}$ and $\tilde{S}$ according to Table 12.1—where the latter three working matrices ultimately become the extended-operational factors $U$, $D$ and $S$ of (13.54).

Borrowing the language of computer science we observe that the indices $i$ and $j$ of (13.52) and (13.53) imply a two-level nested loop, one level looping over $j$ and the other over $i$. The equations suggest *$j$-major nesting,* with the loop over $j$ at the outer level and the loop over $i$ at the inner, such that the several $(i, j)$ index pairs occur in the sequence (reading left to right then top to bottom)

$$\begin{array}{cccc} (1,2) & & & \\ (1,3) & (2,3) & & \\ (1,4) & (2,4) & (3,4) & \\ \cdots & \cdots & \cdots & \ddots \end{array}$$

[24] [66, "Gram-Schmidt process," 04:48, 11 Aug. 2007]

In reality, however, (13.53)'s middle line requires only that no $\hat{\mathbf{x}}_{i\perp}$ be used before it is fully calculated; otherwise that line does not care which $(i, j)$ pair follows which. The $i$-major nesting

$$\begin{array}{cccc} (1,2) & (1,3) & (1,4) & \cdots \\ & (2,3) & (2,4) & \cdots \\ & & (3,4) & \cdots \\ & & & \ddots \end{array}$$

bringing the very same index pairs in a different sequence, is just as valid. We choose $i$-major nesting on the subtle ground that it affords better information to the choice of column index $p$ during the algorithm's step 3.

The algorithm, in detail:

1. Begin by initializing

$$\tilde{U} \leftarrow I,\ \tilde{D} \leftarrow I,\ \tilde{S} \leftarrow I,$$
$$\tilde{Q} \leftarrow A,$$
$$i \leftarrow 1.$$

2. (Besides arriving at this point from step 1 above, the algorithm also reënters here from step 9 below.) Observe that $\tilde{U}$ enjoys the major partial unit triangular form $L^{\{i-1\}T}$ (§ 11.8.5), that $\tilde{D}$ is a general scaling operator (§ 11.7.2) with $\tilde{d}_{jj} = 1$ for all $j \geq i$, that $\tilde{S}$ is permutor (§ 11.7.1), and that the first through $(i-1)$th columns of $\tilde{Q}$ consist of mutually orthonormal unit vectors.

3. Choose a column $p \geq i$ of $\tilde{Q}$ containing at least one nonzero element. (The simplest choice is perhaps $p = i$ as long as the $i$th column does not happen to be null, but one might instead prefer to choose the column of greatest magnitude, or to choose randomly, among other heuristics.) If $\tilde{Q}$ is null in and rightward of its $i$th column such that no column $p \geq i$ remains available to choose, then skip directly to step 10.

4. Observing that (13.55) can be expanded to read

$$\begin{aligned} A &= \Big(\tilde{Q}T_{[i\leftrightarrow p]}\Big)\Big(T_{[i\leftrightarrow p]}\tilde{U}T_{[i\leftrightarrow p]}\Big)\Big(T_{[i\leftrightarrow p]}\tilde{D}T_{[i\leftrightarrow p]}\Big)\Big(T_{[i\leftrightarrow p]}\tilde{S}\Big) \\ &= \Big(\tilde{Q}T_{[i\leftrightarrow p]}\Big)\Big(T_{[i\leftrightarrow p]}\tilde{U}T_{[i\leftrightarrow p]}\Big)\tilde{D}\Big(T_{[i\leftrightarrow p]}\tilde{S}\Big), \end{aligned}$$

where the latter line has applied a rule from Table 12.1, interchange the chosen $p$th column to the $i$th position by

$$\begin{aligned}\tilde{Q} &\leftarrow \tilde{Q}T_{[i\leftrightarrow p]},\\ \tilde{U} &\leftarrow T_{[i\leftrightarrow p]}\tilde{U}T_{[i\leftrightarrow p]},\\ \tilde{S} &\leftarrow T_{[i\leftrightarrow p]}\tilde{S}.\end{aligned}$$

5. Observing that (13.55) can be expanded to read

$$A = \left(\tilde{Q}T_{(1/\alpha)[i]}\right)\left(T_{\alpha[i]}\tilde{U}T_{(1/\alpha)[i]}\right)\left(T_{\alpha[i]}\tilde{D}\right)\tilde{S},$$

normalize the $i$th column of $\tilde{Q}$ by

$$\begin{aligned}\tilde{Q} &\leftarrow \tilde{Q}T_{(1/\alpha)[i]},\\ \tilde{U} &\leftarrow T_{\alpha[i]}\tilde{U}T_{(1/\alpha)[i]},\\ \tilde{D} &\leftarrow T_{\alpha[i]}\tilde{D},\end{aligned}$$

where

$$\alpha = \sqrt{\left[\tilde{Q}\right]^{*}_{*i} \cdot \left[\tilde{Q}\right]_{*i}}.$$

6. Initialize

$$j \leftarrow i + 1.$$

7. (Besides arriving at this point from step 6 above, the algorithm also reënters here from step 8 below.) If $j > n$ then skip directly to step 9. Otherwise, observing that (13.55) can be expanded to read

$$A = \left(\tilde{Q}T_{-\beta[ij]}\right)\left(T_{\beta[ij]}\tilde{U}\right)\tilde{D}\tilde{S},$$

orthogonalize the $j$th column of $\tilde{Q}$ per (13.53) with respect to the $i$th column by

$$\begin{aligned}\tilde{Q} &\leftarrow \tilde{Q}T_{-\beta[ij]},\\ \tilde{U} &\leftarrow T_{\beta[ij]}\tilde{U},\end{aligned}$$

where

$$\beta = \left[\tilde{Q}\right]^{*}_{*i} \cdot \left[\tilde{Q}\right]_{*j}.$$

8. Increment
$$j \leftarrow j + 1$$
and return to step 7.

9. Increment
$$i \leftarrow i + 1$$
and return to step 2.

10. Let
$$Q \equiv \tilde{Q},\ U \equiv \tilde{U},\ D \equiv \tilde{D},\ S \equiv \tilde{S},$$
$$r = i - 1.$$
End.

Though the Gram-Schmidt algorithm broadly resembles the Gauss-Jordan, at least two significant differences stand out: (i) the Gram-Schmidt is one-sided because it operates only on the columns of $\tilde{Q}$, never on the rows; (ii) since $Q$ is itself dimension-limited, the Gram-Schmidt decomposition (13.54) needs and has no explicit factor $I_r$.

As in § 12.5.7, here also one sometimes prefers that $S = I$. The algorithm optionally supports this preference if the $m \times n$ matrix $A$ has full column rank $r = n$, when null columns cannot arise, if one always chooses $p = i$ during the algorithm's step 3. Such optional discipline maintains $S = I$ when desired.

Whether $S = I$ or not, the matrix $Q = QI_r$ has only $r$ columns, so one can write (13.54) as
$$A = (QI_r)(R).$$
Reassociating factors, this is
$$A = (Q)(I_r R), \tag{13.56}$$
which per (12.16) is a proper full-rank factorization with which one can compute the pseudoinverse $A^\dagger$ of $A$ (see eqn. 13.32, above; but see also eqn. 13.65, below).

If the Gram-Schmidt decomposition (13.54) looks useful, it is even more useful than it looks. The most interesting of its several factors is the $m \times r$ orthonormalized matrix $Q$, whose orthonormal columns address the same space the columns of $A$ themselves address. If $Q$ reaches the maximum possible rank $r = m$, achieving square, $m \times m$ shape, then it becomes a *unitary matrix*—the subject of § 13.12. Before treating the unitary matrix, however, let us pause to extract a kernel from the Gram-Schmidt decomposition in § 13.11.3, next.

### 13.11.3 The Gram-Schmidt kernel formula

Like the Gauss-Jordan decomposition in (13.7), the Gram-Schmidt decomposition too brings a kernel formula. To develop and apply it, one decomposes an $m \times n$ matrix

$$A = QR \tag{13.57}$$

per the Gram-Schmidt (13.54) and its algorithm in § 13.11.2. Observing that the $r$ independent columns of the $m \times r$ matrix $Q$ address the same space the columns of $A$ address, one then constructs the $m \times (r+m)$ matrix

$$A' \equiv Q + I_m H_{-r} = \begin{bmatrix} Q & I_m \end{bmatrix} \tag{13.58}$$

and decomposes it too,

$$A' = Q'R', \tag{13.59}$$

again by Gram-Schmidt—with the differences that, this time, one chooses $p = 1, 2, 3, \ldots, r$ during the first $r$ instances of the algorithm's step 3, and that one skips the unnecessary step 7 for all $j \leq r$; on the ground that the earlier Gram-Schmidt application of (13.57) has already orthonormalized first $r$ columns of $A'$, which columns, after all, are just $Q$. The resulting $m \times m$, full-rank square matrix

$$Q' = Q + A^\perp H_{-r} = \begin{bmatrix} Q & A^\perp \end{bmatrix} \tag{13.60}$$

consists of

- $r$ columns on the left that address the same space the columns of $A$ address and
- $m-r$ columns on the right that give a complete orthogonal complement (§ 13.10) $A^\perp$ of $A$.

Each column has unit magnitude and conveniently lies orthogonal to every other column, left and right.

Equation (13.60) is probably the more useful form, but the *Gram-Schmidt kernel formula* as such,

$$A^{*K} = A^\perp = Q' H_r I_{m-r}, \tag{13.61}$$

extracts the rightward columns that express the kernel, not of $A$, but of $A^*$. To compute the kernel of a matrix $B$ by Gram-Schmidt one sets $A = B^*$ and applies (13.57) through (13.61). Refer to (13.48).

In either the form (13.60) or the form (13.61), the Gram-Schmidt kernel formula does everything the Gauss-Jordan kernel formula (13.7) does and in at least one sense does it better; for, if one wants a Gauss-Jordan kernel orthonormalized, then one must orthonormalize it as an extra step, whereas the Gram-Schmidt kernel comes already orthonormalized.

Being square, the $m \times m$ matrix $Q'$ is a unitary matrix, as the last paragraph of § 13.11.2 has alluded. The unitary matrix is the subject of § 13.12 that follows.

## 13.12 The unitary matrix

When the orthonormalized matrix $Q$ of the Gram-Schmidt decomposition (13.54) is square, having the maximum possible rank $r = m$, it brings one property so interesting that the property merits a section of its own. The property is that

$$Q^*Q = I_m = QQ^*. \tag{13.62}$$

The reason that $Q^*Q = I_m$ is that $Q$'s columns are orthonormal, and that the very definition of orthonormality demands that the dot product $[Q]^*_{*i} \cdot [Q]_{*j}$ of orthonormal columns be zero unless $i = j$, when the dot product of a unit vector with itself is unity. That $I_m = QQ^*$ is unexpected, however, until one realizes[25] that the equation $Q^*Q = I_m$ characterizes $Q^*$ to be the rank-$m$ inverse of $Q$, and that § 13.1 lets any rank-$m$ inverse (orthonormal or otherwise) attack just as well from the right as from the left. Thus,

$$Q^{-1} = Q^*, \tag{13.63}$$

a very useful property. A matrix $Q$ that satisfies (13.62), whether derived from the Gram-Schmidt or from elsewhere, is called a *unitary matrix.* (Note that the permutor of § 11.7.1 enjoys the property of eqn. 13.63 precisely because it is unitary.)

One immediate consequence of (13.62) is that *a square matrix with either orthonormal columns or orthonormal rows is unitary and has both.*

*The product of two or more unitary matrices is itself unitary* if the matrices are of the same dimensionality. To prove it, consider the product

$$Q = Q_a Q_b \tag{13.64}$$

of $m \times m$ unitary matrices $Q_a$ and $Q_b$. Let the symbols $\mathbf{q}_j$, $\mathbf{q}_{aj}$ and $\mathbf{q}_{bj}$ respectively represent the $j$th columns of $Q$, $Q_a$ and $Q_b$ and let the symbol $q_{bij}$ represent the $i$th element of $\mathbf{q}_{bj}$. By the columnwise interpretation

[25] [21, § 4.4]

(§ 11.1.3) of matrix multiplication,

$$\mathbf{q}_j = \sum_i q_{bij}\mathbf{q}_{ai}.$$

The adjoint dot product of any two of $Q$'s columns then is

$$\mathbf{q}^*_{j'} \cdot \mathbf{q}_j = \sum_{i,i'} q^*_{bi'j'} q_{bij} \mathbf{q}^*_{ai'} \cdot \mathbf{q}_{ai}.$$

But $\mathbf{q}^*_{ai'} \cdot \mathbf{q}_{ai} = \delta_{i'i}$ because $Q_a$ is unitary,[26] so

$$\mathbf{q}^*_{j'} \cdot \mathbf{q}_j = \sum_i q^*_{bij'} q_{bij} = \mathbf{q}^*_{bj'} \cdot \mathbf{q}_{bj} = \delta_{j'j},$$

which says neither more nor less than that the columns of $Q$ are orthonormal, which is to say that $Q$ is unitary, as was to be demonstrated.

*Unitary operations preserve length.* That is, operating on an $m$-element vector by an $m \times m$ unitary matrix does not alter the vector's magnitude. To prove it, consider the system

$$Q\mathbf{x} = \mathbf{b}.$$

Multiplying the system by its own adjoint yields

$$\mathbf{x}^* Q^* Q \mathbf{x} = \mathbf{b}^* \mathbf{b}.$$

But according to (13.62), $Q^*Q = I_m$; so,

$$\mathbf{x}^* \mathbf{x} = \mathbf{b}^* \mathbf{b},$$

as was to be demonstrated.

Equation (13.63) lets one use the Gram-Schmidt decomposition (13.54) to invert a square matrix as

$$A^{-1} = R^{-1}Q^* = S^* D^{-1} U^{-1} Q^*. \tag{13.65}$$

Unitary extended operators are certainly possible, for if $Q$ is an $m \times m$ dimension-limited matrix, then the extended operator

$$Q_\infty = Q + (I - I_m),$$

[26] This is true only for $1 \le i \le m$, but you knew that already.

which is just $Q$ with ones running out the main diagonal from its active region, itself meets the unitary criterion (13.62) for $m = \infty$.

Unitary matrices are so easy to handle that they can sometimes justify significant effort to convert a model to work in terms of them if possible. We shall meet the unitary matrix again in §§ 14.10 and 14.12.

The chapter as a whole has demonstrated at least in theory (and usually in practice) techniques to solve any linear system characterized by a matrix of finite dimensionality, whatever the matrix's rank or shape. It has explained how to orthonormalize a set of vectors and has derived from the explanation the useful Gram-Schmidt decomposition. As the chapter's introduction had promised, the matrix has shown its worth here; for without the matrix's notation, arithmetic and algebra most of the chapter's findings would have lain beyond practical reach. And even so, the single most interesting agent of matrix arithmetic remains yet to be treated. This last is the eigenvalue, and it is the subject of Ch. 14, next.

# Chapter 14

# The eigenvalue

The *eigenvalue* is a scalar by which a square matrix scales a vector without otherwise changing it, such that

$$A\mathbf{v} = \lambda\mathbf{v}.$$

This chapter analyzes the eigenvalue and the associated *eigenvector* it scales.

Before treating the eigenvalue proper, the chapter gathers from across Chs. 11 through 14 several properties all invertible square matrices share, assembling them in § 14.2 for reference. One of these regards the *determinant,* which opens the chapter.

## 14.1 The determinant

Through Chs. 11, 12 and 13 the theory of the matrix has developed slowly but pretty straightforwardly. Here comes the first unexpected turn.

It begins with an arbitrary-seeming definition. The *determinant* of an $n \times n$ square matrix $A$ is the sum of $n!$ terms, each term the product of $n$ elements, no two elements from the same row or column, terms of positive parity adding to and terms of negative parity subtracting from the sum—a term's parity (§ 11.6) being the parity of the permutor (§ 11.7.1) marking the positions of the term's elements.

Unless you already know about determinants, the definition alone might seem hard to parse, so try this. The inverse of the general $2 \times 2$ square matrix

$$A_2 = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix},$$

by the Gauss-Jordan method or any other convenient technique, is found to be

$$A_2^{-1} = \frac{\begin{bmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{bmatrix}}{a_{11}a_{22} - a_{12}a_{21}}.$$

The quantity[1]

$$\det A_2 = a_{11}a_{22} - a_{12}a_{21}$$

in the denominator is defined to be the *determinant* of $A_2$. Each of the determinant's terms includes one element from each column of the matrix and one from each row, with parity giving the term its $\pm$ sign. The determinant of the general $3 \times 3$ square matrix by the same rule is

$$\begin{aligned} \det A_3 &= (a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32}) \\ &\quad - (a_{13}a_{22}a_{31} + a_{12}a_{21}a_{33} + a_{11}a_{23}a_{32}); \end{aligned}$$

and indeed if we tediously invert such a matrix symbolically, we do find that quantity in the denominator there.

The parity rule merits a more careful description. The parity of a term like $a_{12}a_{23}a_{31}$ is positive because the parity of the permutor, or interchange quasielementary (§ 11.7.1),

$$P = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$$

marking the positions of the term's elements is positive. The parity of a term like $a_{13}a_{22}a_{31}$ is negative for the same reason. The determinant comprehends all possible such terms, $n!$ in number, half of positive parity and half of negative. (How do we know that exactly half are of positive and half, negative? Answer: by pairing the terms. For every term like $a_{12}a_{23}a_{31}$ whose marking permutor is $P$, there is a corresponding $a_{13}a_{22}a_{31}$ whose marking permutor is $T_{[1\leftrightarrow 2]}P$, necessarily of opposite parity. The sole exception to the rule is the $1 \times 1$ square matrix, which has no second term to pair.)

[1]The determinant $\det A$ used to be written $|A|$, an appropriately terse notation for which the author confesses some nostalgia. The older notation $|A|$ however unluckily suggests "the magnitude of $A$," which though not quite the wrong idea is not quite the right idea, either. The magnitude $|z|$ of a scalar or $|\mathbf{u}|$ of a vector is a real-valued, nonnegative, nonanalytic function of the elements of the quantity in question, whereas the determinant $\det A$ is a complex-valued, analytic function. The book follows convention by denoting the determinant as $\det A$ for this reason among others.

Normally the context implies a determinant's rank $n$, but the nonstandard notation

$$\det{}^{(n)} A$$

is available especially to call the rank out, stating explicitly that the determinant has exactly $n!$ terms. (See also §§ 11.3.5 and 11.5 and eqn. 11.49.[2])

It is admitted[3] that we have not, as yet, actually shown the determinant to be a generally useful quantity; we have merely motivated and defined it. Historically the determinant probably emerged not from abstract considerations but for the mundane reason that the quantity it represents occurred frequently in practice (as in the $A_2^{-1}$ example above). Nothing however logically prevents one from simply defining some quantity which, at first, one merely suspects will later prove useful. So we do here.[4]

### 14.1.1 Basic properties

The determinant $\det A$ enjoys several useful basic properties.

- If

$$c_{i*} = \begin{cases} a_{i''*} & \text{when } i = i', \\ a_{i'*} & \text{when } i = i'', \\ a_{i*} & \text{otherwise,} \end{cases}$$

  or if

$$c_{*j} = \begin{cases} a_{*j''} & \text{when } j = j', \\ a_{*j'} & \text{when } j = j'', \\ a_{*j} & \text{otherwise,} \end{cases}$$

  where $i'' \neq i'$ and $j'' \neq j'$, then

$$\det C = -\det A. \tag{14.1}$$

  Interchanging rows or columns negates the determinant.

- If

$$c_{i*} = \begin{cases} \alpha a_{i*} & \text{when } i = i', \\ a_{i*} & \text{otherwise,} \end{cases}$$

[2] And further Ch. 13's footnotes 5 and 22.
[3] [21, § 1.2]
[4] [21, Ch. 1]

or if

$$c_{*j} = \begin{cases} \alpha a_{*j} & \text{when } j = j', \\ a_{*j} & \text{otherwise,} \end{cases}$$

then

$$\det C = \alpha \det A. \tag{14.2}$$

Scaling a single row or column of a matrix scales the matrix's determinant by the same factor. (Equation 14.2 tracks the linear scaling property of § 7.3.3 and of eqn. 11.2.)

- If

$$c_{i*} = \begin{cases} a_{i*} + b_{i*} & \text{when } i = i', \\ a_{i*} = b_{i*} & \text{otherwise,} \end{cases}$$

or if

$$c_{*j} = \begin{cases} a_{*j} + b_{*j} & \text{when } j = j', \\ a_{*j} = b_{*j} & \text{otherwise,} \end{cases}$$

then

$$\det C = \det A + \det B. \tag{14.3}$$

If one row or column of a matrix $C$ is the sum of the corresponding rows or columns of two other matrices $A$ and $B$, while the three matrices remain otherwise identical, then the determinant of the one matrix is the sum of the determinants of the other two. (Equation 14.3 tracks the linear superposition property of § 7.3.3 and of eqn. 11.2.)

- If

$$c_{i'*} = 0,$$

or if

$$c_{*j'} = 0,$$

then

$$\det C = 0. \tag{14.4}$$

A matrix with a null row or column also has a null determinant.

- If

$$c_{i''*} = \gamma c_{i'*},$$

or if

$$c_{*j''} = \gamma c_{*j'},$$

where $i'' \neq i'$ and $j'' \neq j'$, then

$$\det C = 0. \tag{14.5}$$

The determinant is zero if one row or column of the matrix is a multiple of another.

- The determinant of the adjoint is just the determinant's conjugate, and the determinant of the transpose is just the determinant itself:

$$\begin{aligned} \det C^* &= (\det C)^*; \\ \det C^T &= \det C. \end{aligned} \tag{14.6}$$

These basic properties are all fairly easy to see if the definition of the determinant is clearly understood. Equations (14.2), (14.3) and (14.4) come because each of the $n!$ terms in the determinant's expansion has exactly one element from row $i'$ or column $j'$. Equation (14.1) comes because a row or column interchange reverses parity. Equation (14.6) comes because according to § 11.7.1, the permutors $P$ and $P^*$ always have the same parity, and because the adjoint operation individually conjugates each element of $C$. Finally, (14.5) comes because, in this case, every term in the determinant's expansion finds an equal term of opposite parity to offset it. Or, more formally, (14.5) comes because the following procedure does not alter the matrix: (i) scale row $i''$ or column $j''$ by $1/\gamma$; (ii) scale row $i'$ or column $j'$ by $\gamma$; (iii) interchange rows $i' \leftrightarrow i''$ or columns $j' \leftrightarrow j''$. Not altering the matrix, the procedure does not alter the determinant either; and indeed according to (14.2), step (ii)'s effect on the determinant cancels that of step (i). However, according to (14.1), step (iii) negates the determinant. Hence the net effect of the procedure is to negate the determinant—to negate the very determinant the procedure is not permitted to alter. The apparent contradiction can be reconciled only if the determinant is zero to begin with.

From the foregoing properties the following further property can be deduced.

- If

$$c_{i*} = \begin{cases} a_{i*} + \alpha a_{i'*} & \text{when } i = i'', \\ a_{i*} & \text{otherwise,} \end{cases}$$

or if

$$c_{*j} = \begin{cases} a_{*j} + \alpha a_{*j'} & \text{when } j = j'', \\ a_{*j} & \text{otherwise,} \end{cases}$$

where $i'' \neq i'$ and $j'' \neq j'$, then

$$\det C = \det A. \tag{14.7}$$

Adding to a row or column of a matrix a multiple of another row or column does not change the matrix's determinant.

To derive (14.7) for rows (the column proof is similar), one defines a matrix $B$ such that

$$b_{i*} \equiv \begin{cases} \alpha a_{i'*} & \text{when } i = i'', \\ a_{i*} & \text{otherwise.} \end{cases}$$

From this definition, $b_{i''*} = \alpha a_{i'*}$ whereas $b_{i'*} = a_{i'*}$, so

$$b_{i''*} = \alpha b_{i'*},$$

which by (14.5) guarantees that

$$\det B = 0.$$

On the other hand, the three matrices $A$, $B$ and $C$ differ only in the $(i'')$th row, where $[C]_{i''*} = [A]_{i''*} + [B]_{i''*}$; so, according to (14.3),

$$\det C = \det A + \det B.$$

Equation (14.7) results from combining the last two equations.

### 14.1.2 The determinant and the elementary operator

Section 14.1.1 has it that interchanging, scaling or adding rows or columns of a matrix respectively negates, scales or does not alter the matrix's determinant. But the three operations named are precisely the operations of the three elementaries of § 11.4. Therefore,

$$\begin{aligned} \det T_{[i\leftrightarrow j]}A &= -\det A = \det AT_{[i\leftrightarrow j]}, \\ \det T_{\alpha[i]}A &= \alpha \det A = \det AT_{\alpha[j]}, \\ \det T_{\alpha[ij]}A &= \det A = \det AT_{\alpha[ij]}, \\ &1 \leq (i,j) \leq n,\ i \neq j, \end{aligned} \tag{14.8}$$

for any $n \times n$ square matrix $A$. Obviously also,

$$\begin{aligned} \det IA &= \det A = \det AI, \\ \det I_n A &= \det A = \det AI_n, \\ \det I &= 1 = \det I_n. \end{aligned} \tag{14.9}$$

If $A$ is taken to represent an arbitrary product of identity matrices ($I_n$ and/or $I$) and elementary operators, then a significant consequence of (14.8) and (14.9), applied recursively, is that the determinant of a product is the product of the determinants, at least where identity matrices and elementary operators are concerned. In symbols,[5]

$$\begin{aligned} \det\left(\prod_k M_k\right) &= \prod_k \det M_k, \\ M_k &\in \left\{I_n, I, T_{[i\leftrightarrow j]}, T_{\alpha[i]}, T_{\alpha[ij]}\right\}, \\ 1 \le (i,j) &\le n. \end{aligned} \tag{14.10}$$

This matters because, as the Gauss-Jordan decomposition of § 12.3 has shown, one can build up any square matrix of full rank by applying elementary operators to $I_n$. Section 14.1.4 will put the rule (14.10) to good use.

### 14.1.3 The determinant of a singular matrix

Equation (14.8) gives elementary operators the power to alter a matrix's determinant almost arbitrarily—almost arbitrarily, but not quite. What an $n \times n$ elementary operator[6] cannot do is to change an $n \times n$ matrix's determinant to or from zero. Once zero, a determinant remains zero under the action of elementary operators. Once nonzero, always nonzero. Elementary operators being reversible have no power to breach this barrier.

Another thing $n \times n$ elementaries cannot do according to § 12.5.3 is to change an $n \times n$ matrix's rank. Nevertheless, such elementaries can reduce any $n \times n$ matrix reversibly to $I_r$, where $r \le n$ is the matrix's rank, by the Gauss-Jordan algorithm of § 12.3. Equation (14.4) has that the $n \times n$ determinant of $I_r$ is zero if $r < n$, so it follows that the $n \times n$ determinant of every rank-$r$ matrix is similarly zero if $r < n$; and complementarily that the $n \times n$ determinant of a rank-$n$ matrix is never zero. Singular matrices always have zero determinants; full-rank square matrices never do. One can evidently tell the singularity or invertibility of a square matrix from its determinant alone.

[5] Notation like "$\in$", first met in § 2.3, can be too fancy for applied mathematics, but it does help here. The notation $M_k \in \{\ldots\}$ restricts $M_k$ to be any of the things between the braces. As it happens though, in this case, (14.11) below is going to erase the restriction.

[6] That is, an elementary operator which honors an $n \times n$ active region. See § 11.3.2.

### 14.1.4 The determinant of a matrix product

Sections 14.1.2 and 14.1.3 suggest the useful rule that

$$\det AB = \det A \det B. \tag{14.11}$$

To prove the rule, we consider three distinct cases.

The first case is that $A$ is singular. In this case, $B$ acts as a column operator on $A$, whereas according to § 12.5.2 no operator has the power to promote $A$ in rank. Hence the product $AB$ is no higher in rank than $A$, which says that $AB$ is no less singular than $A$, which implies that $AB$ like $A$ has a null determinant. Evidently (14.11) holds in the first case.

The second case is that $B$ is singular. The proof here resembles that of the first case.

The third case is that neither matrix is singular. Here, we use Gauss-Jordan to decompose both matrices into sequences of elementary operators and rank-$n$ identity matrices, for which

$$\begin{aligned}
\det AB &= \det \{[A]\,[B]\} \\
&= \det \left\{ \left[ \left( \prod T \right) I_n \left( \coprod T \right) \right] \left[ \left( \prod T \right) I_n \left( \coprod T \right) \right] \right\} \\
&= \left( \prod \det T \right) \det I_n \left( \coprod \det T \right) \left( \prod \det T \right) \det I_n \left( \coprod \det T \right) \\
&= \det \left[ \left( \prod T \right) I_n \left( \coprod T \right) \right] \det \left[ \left( \prod T \right) I_n \left( \coprod T \right) \right] \\
&= \det A \det B,
\end{aligned}$$

which is a schematic way of pointing out in light of (14.10) merely that since $A$ and $B$ are products of identity matrices and elementaries, the determinant of the product is the product of the determinants.

So it is that (14.11) holds in all three cases, as was to be demonstrated. *The determinant of a matrix product is the product of the matrix determinants.*

### 14.1.5 Determinants of inverse and unitary matrices

From (14.11) it follows that

$$\det A^{-1} = \frac{1}{\det A} \tag{14.12}$$

because $A^{-1}A = I_n$ and $\det I_n = 1$.

From (14.6) it follows that if $Q$ is a unitary matrix (§ 13.12), then

$$\begin{aligned} \det Q^* \det Q &= 1, \\ |\det Q| &= 1. \end{aligned} \tag{14.13}$$

This reason is that $|\det Q|^2 = (\det Q)^*(\det Q) = \det Q^* \det Q = \det Q^* Q = \det Q^{-1} Q = \det I_n = 1$.

### 14.1.6 Inverting the square matrix by determinant

The Gauss-Jordan algorithm comfortably inverts concrete matrices of moderate size, but swamps one in nearly interminable algebra when *symbolically* inverting general matrices larger than the $A_2$ at the section's head. Slogging through the algebra to invert $A_3$ symbolically nevertheless (the reader need not actually do this unless he desires a long exercise), one quite incidentally discovers a clever way to factor the determinant:

$$\begin{aligned} C^T A &= (\det A) I_n = A C^T; \\ c_{ij} &\equiv \det R_{ij}; \\ [R_{ij}]_{i'j'} &\equiv \begin{cases} 1 & \text{if } i' = i \text{ and } j' = j, \\ 0 & \text{if } i' = i \text{ or } j' = j \text{ but not both}, \\ a_{i'j'} & \text{otherwise.} \end{cases} \end{aligned} \tag{14.14}$$

Pictorially,

$$R_{ij} = \begin{bmatrix} & \vdots & \vdots & \vdots & \vdots & \vdots & \\ \cdots & * & * & 0 & * & * & \cdots \\ \cdots & * & * & 0 & * & * & \cdots \\ \cdots & 0 & 0 & 1 & 0 & 0 & \cdots \\ \cdots & * & * & 0 & * & * & \cdots \\ \cdots & * & * & 0 & * & * & \cdots \\ & \vdots & \vdots & \vdots & \vdots & \vdots & \end{bmatrix},$$

same as $A$ except in the $i$th row and $j$th column. The matrix $C$, called the *cofactor* of $A$, then consists of the determinants of the various $R_{ij}$.

Another way to write (14.14) is

$$[C^T A]_{ij} = (\det A)\delta_{ij} = [A C^T]_{ij}, \tag{14.15}$$

which comprises two cases. In the case that $i = j$,

$$[A C^T]_{ij} = [A C^T]_{ii} = \sum_\ell a_{i\ell} c_{i\ell} = \sum_\ell a_{i\ell} \det R_{i\ell} = \det A = (\det A)\delta_{ij},$$

wherein the equation $\sum_\ell a_{i\ell} \det R_{i\ell} = \det A$ states that $\det A$, being a determinant, consists of several terms, each term including one factor from each row of $A$, where $a_{i\ell}$ provides the $i$th row and $R_{i\ell}$ provides the other rows.[7] In the case that $i \neq j$,

$$[AC^T]_{ij} = \sum_\ell a_{i\ell} c_{j\ell} = \sum_\ell a_{i\ell} \det R_{j\ell} = 0 = (\det A)(0) = (\det A)\delta_{ij},$$

wherein $\sum_\ell a_{i\ell} \det R_{j\ell}$ is the determinant, not of $A$ itself, but rather of $A$ with the $j$th row replaced by a copy of the $i$th, which according to (14.5) evaluates to zero. Similar equations can be written for $[C^T A]_{ij}$ in both cases. The two cases together prove (14.15), hence also (14.14).

Dividing (14.14) by $\det A$, we have that[8]

$$\begin{aligned} A^{-1}A &= I_n = AA^{-1}, \\ A^{-1} &= \frac{C^T}{\det A}. \end{aligned} \tag{14.16}$$

Equation (14.16) inverts a matrix by determinant. In practice, it inverts small matrices nicely, through about $4 \times 4$ dimensionality (the $A_2^{-1}$ equation at the head of the section is just eqn. 14.16 for $n = 2$). It inverts $5 \times 5$ and even $6 \times 6$ matrices reasonably, too—especially with the help of a computer to do the arithmetic. Though (14.16) still holds in theory for yet larger matrices, and though symbolically it expresses the inverse of an abstract, $n \times n$ matrix concisely whose entries remain unspecified, for concrete matrices much bigger than $4 \times 4$ to $6 \times 6$ or so its several determinants begin to grow too great and too many for practical calculation. The Gauss-Jordan technique (or even the Gram-Schmidt technique) is preferred to invert concrete matrices above a certain size for this reason.[9]

## 14.2 Coincident properties

Chs. 11, 12 and 13, plus this chapter up to the present point, have discovered several coincident properties of the invertible $n \times n$ square matrix. One does

[7] This is a bit subtle, but if you actually write out $A_3$ and its cofactor $C_3$ symbolically, trying (14.15) on them, then you will soon see what is meant.

[8] Cramer's rule [21, § 1.6], of which the reader may have heard, results from applying (14.16) to (13.4). However, Cramer's rule is really nothing more than (14.16) in a less pleasing form, so this book does not treat Cramer's rule as such.

[9] For very large matrices, even the Gauss-Jordan grows impractical, due to compound floating-point rounding error and the maybe large but nonetheless limited quantity of available computer memory. Iterative techniques ([chapter not yet written]) serve to invert such matrices approximately.

not feel the full impact of the coincidence when these properties are left scattered across the long chapters; so, let us gather and summarize the properties here. A square, $n \times n$ matrix evidently has either all of the following properties or none of them, never some but not others.

- The matrix is invertible (§ 13.1).
- Its rows are linearly independent (§§ 12.1 and 12.3.4).
- Its columns are linearly independent (§ 12.5.4).
- Its columns address the same space the columns of $I_n$ address, and its rows address the same space the rows of $I_n$ address (§ 12.5.7).
- The Gauss-Jordan algorithm reduces it to $I_n$ (§ 12.3.3). (In this, per § 12.5.3, the choice of pivots does not matter.)
- Decomposing it, the Gram-Schmidt algorithm achieves a fully square, unitary, $n \times n$ factor $Q$ (§ 13.11.2).
- It has full rank $r = n$ (§ 12.5.4).
- The linear system $A\mathbf{x} = \mathbf{b}$ it represents has a unique $n$-element solution $\mathbf{x}$, given any specific $n$-element driving vector $\mathbf{b}$ (§ 13.2).
- The determinant $\det A \neq 0$ (§ 14.1.3).
- None of its eigenvalues is zero (§ 14.3, below).

The square matrix which has one of these properties, has all of them. The square matrix which lacks one, lacks all. Assuming exact arithmetic, a square matrix is either invertible, with all that that implies, or singular; never both. The distinction between invertible and singular matrices is theoretically as absolute as (and indeed is analogous to) the distinction between nonzero and zero scalars.

Whether the distinction is always useful is another matter. Usually the distinction is indeed useful, but a matrix can be *almost* singular just as a scalar can be almost zero. Such a matrix is known, among other ways, by its unexpectedly small determinant. Now it is true: in exact arithmetic, a nonzero determinant, no matter how small, implies a theoretically invertible matrix. Practical matrices however often have entries whose values are imprecisely known; and even when they don't, the computers which invert them tend to do arithmetic imprecisely in floating-point. Matrices which live

on the hazy frontier between invertibility and singularity resemble the infinitesimals of § 4.1.1. They are called *ill-conditioned* matrices. Section 14.8 develops the topic.

## 14.3 The eigenvalue itself

We stand ready at last to approach the final major agent of matrix arithmetic, the *eigenvalue.* Suppose a square, $n \times n$ matrix $A$, a nonzero $n$-element vector

$$\mathbf{v} = I_n \mathbf{v} \neq 0, \tag{14.17}$$

and a scalar $\lambda$, together such that

$$A\mathbf{v} = \lambda \mathbf{v}, \tag{14.18}$$

or in other words such that $A\mathbf{v} = \lambda I_n \mathbf{v}$. If so, then

$$[A - \lambda I_n]\mathbf{v} = 0. \tag{14.19}$$

Since $I_n \mathbf{v}$ is nonzero, the last equation is true if and only if the matrix $[A - \lambda I_n]$ is singular—which in light of § 14.1.3 is to demand that

$$\det(A - \lambda I_n) = 0. \tag{14.20}$$

The left side of (14.20) is an $n$th-order polynomial in $\lambda$, the *characteristic polynomial,* whose $n$ roots are the *eigenvalues*[10] of the matrix $A$.

What is an eigenvalue, really? An eigenvalue is a scalar a matrix resembles under certain conditions. When a matrix happens to operate on the right *eigenvector* $\mathbf{v}$, it is all the same whether one applies the entire matrix or just the eigenvalue to the vector. The matrix scales the eigenvector by the eigenvalue without otherwise altering the vector, changing the vector's

[10] An example:

$$\begin{aligned}
A &= \begin{bmatrix} 2 & 0 \\ 3 & -1 \end{bmatrix}, \\
\det(A - \lambda I_n) &= \det \begin{bmatrix} 2-\lambda & 0 \\ 3 & -1-\lambda \end{bmatrix} \\
&= (2-\lambda)(-1-\lambda) - (0)(3) \\
&= \lambda^2 - \lambda - 2 = 0, \\
\lambda &= -1 \text{ or } 2.
\end{aligned}$$

magnitude but not its direction. The eigenvalue alone takes the place of the whole, hulking matrix. This is what (14.18) means. Of course it works only when $\mathbf{v}$ happens to be the right *eigenvector,* which § 14.4 discusses.

Observe incidentally that the characteristic polynomial of an $n \times n$ matrix always enjoys full order $n$, regardless of the matrix's rank. The reason lies in the determinant $\det(A - \lambda I_n)$, which comprises exactly $n!$ determinant-terms (we say "determinant-terms" rather than "terms" here only to avoid confusing the determinant's terms with the characteristic polynomial's), only one of which, $(a_{11} - \lambda)(a_{22} - \lambda) \cdots (a_{nn} - \lambda)$, gathers elements straight down the main diagonal of the matrix $[A - \lambda I_n]$. When multiplied out, this main-diagonal determinant-term evidently contributes a $(-\lambda)^n$ to the characteristic polynomial, whereas none of the other determinant-terms finds enough factors of $\lambda$ to reach order $n$. (If unsure, take your pencil and just calculate the characteristic polynomials of the $3 \times 3$ matrices $I_3$ and 0. You will soon see what is meant.)

On the other hand, nothing prevents $\lambda = 0$. When $\lambda = 0$, (14.20) makes $\det A = 0$, which as we have said is the sign of a singular matrix. Zero eigenvalues and singular matrices always travel together. *Singular matrices each have at least one zero eigenvalue; nonsingular matrices never do.*

The eigenvalues of a matrix's inverse are the inverses of the matrix's eigenvalues. That is,

$$\lambda_j' \lambda_j = 1 \quad \text{for all } 1 \le j \le n \text{ if } A'A = I_n = AA'. \tag{14.21}$$

The reason behind (14.21) comes from answering the question: if $A\mathbf{v}_j$ scales $\mathbf{v}_j$ by the factor $\lambda_j$, then what does $A'A\mathbf{v}_j = I\mathbf{v}_j$ do to $\mathbf{v}_j$?

Naturally one must solve (14.20)'s $n$th-order polynomial to locate the actual eigenvalues. One solves it by the same techniques by which one solves any polynomial: the quadratic formula (2.2); the cubic and quartic methods of Ch. 10; the Newton-Raphson iteration (4.31). On the other hand, the determinant (14.20) can be impractical to expand for a large matrix; here iterative techniques help: see [chapter not yet written].[11]

## 14.4 The eigenvector

It is an odd fact that (14.19) and (14.20) reveal the eigenvalues $\lambda$ of a square matrix $A$ while obscuring the associated *eigenvectors* $\mathbf{v}$. Once one has calculated an eigenvalue, though, one can feed it back to calculate the

[11] The inexpensive [21] also covers the topic competently and readably.

associated eigenvector. According to (14.19), the eigenvectors are the $n$-element vectors for which

$$[A - \lambda I_n]\mathbf{v} = 0,$$

which is to say that the eigenvectors are the vectors of the kernel space of the degenerate matrix $[A - \lambda I_n]$—which one can calculate (among other ways) by the Gauss-Jordan kernel formula (13.7) or the Gram-Schmidt kernel formula (13.61).

An eigenvalue and its associated eigenvector, taken together, are sometimes called an *eigensolution.*

## 14.5 Eigensolution facts

Many useful or interesting mathematical facts concern the eigensolution, among them the following.

- *If the eigensolutions of $A$ are $(\lambda_j, \mathbf{v}_j)$, then the eigensolutions of $A + \alpha I_n$ are $(\lambda_j + \alpha, \mathbf{v}_j)$.* The eigenvalues move over by $\alpha I_n$ while the eigenvectors remain fixed. This is seen by adding $\alpha\mathbf{v}_j$ to both sides of the definition $A\mathbf{v}_j = \lambda\mathbf{v}_j$.

- *$A$ matrix and its inverse share the same eigenvectors with inverted eigenvalues.* Refer to (14.21) and its explanation in § 14.3.

- *Eigenvectors corresponding to distinct eigenvalues are always linearly independent of one another.* To prove this fact, consider several independent eigenvectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_{k-1}$ respectively with distinct eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_{k-1}$, and further consider another eigenvector $\mathbf{v}_k$ which might or might not be independent but which too has a distinct eigenvalue $\lambda_k$. Were $\mathbf{v}_k$ dependent, which is to say, did nontrivial coefficients $c_j$ exist such that

$$\mathbf{v}_k = \sum_{j=1}^{k-1} c_j \mathbf{v}_j,$$

  then left-multiplying the equation by $A - \lambda_k$ would yield

$$0 = \sum_{j=1}^{k-1} (\lambda_j - \lambda_k) c_j \mathbf{v}_j,$$

impossible since the $k-1$ eigenvectors are independent. Thus $\mathbf{v}_k$ too is independent, whereupon by induction from a start case of $k = 1$ we conclude that there exists no dependent eigenvector with a distinct eigenvalue.

- *If an $n \times n$ square matrix $A$ has $n$ independent eigenvectors* (which is always so if the matrix has $n$ distinct eigenvalues and often so even otherwise), *then any $n$-element vector can be expressed as a unique linear combination of the eigenvectors.* This is a simple consequence of the fact that the $n \times n$ matrix $V$ whose columns are the several eigenvectors $\mathbf{v}_j$ has full rank $r = n$. Unfortunately some matrices with repeated eigenvalues also have repeated eigenvectors—as for example, curiously,[12] $[1\ 0; 1\ 1]^T$, whose double eigenvalue $\lambda = 1$ has the single eigenvector $[1\ 0]^T$. Section 14.10.2 speaks of matrices of the last kind.

- *An $n \times n$ square matrix whose eigenvectors are linearly independent of one another cannot share all eigensolutions with any other $n \times n$ square matrix.* This fact proceeds from the last point, that every $n$-element vector $\mathbf{x}$ is a unique linear combination of independent eigenvectors. Neither of the two proposed matrices $A_1$ and $A_2$ could scale any of the eigenvector components of $\mathbf{x}$ differently than the other matrix did, so $A_1\mathbf{x} - A_2\mathbf{x} = (A_1 - A_2)\mathbf{x} = 0$ for all $\mathbf{x}$, which in turn is possible only if $A_1 = A_2$.

- *A positive definite matrix has only real, positive eigenvalues. A nonnegative definite matrix has only real, nonnegative eigenvalues.* Were it not so, then $\mathbf{v}^*A\mathbf{v} = \lambda\mathbf{v}^*\mathbf{v}$ (in which $\mathbf{v}^*\mathbf{v}$ naturally is a positive real scalar) would violate the criterion for positive or nonnegative definiteness. See § 13.6.3.

- *Every $n \times n$ square matrix has at least one eigensolution* if $n > 0$, because according to the fundamental theorem of algebra (6.1) the matrix's characteristic polynomial (14.20) has at least one root, an eigenvalue, which by definition would be no eigenvalue if it had no eigenvector to scale, and for which (14.19) necessarily admits at least one nonzero solution $\mathbf{v}$ because its matrix $A - \lambda I_n$ is degenerate.

[12] [29]

## 14.6 Diagonalization

Any $n \times n$ matrix with $n$ independent eigenvectors (which class per § 14.5 includes, but is not limited to, every $n \times n$ matrix with $n$ distinct eigenvalues) can be *diagonalized* as

$$A = V \Lambda V^{-1}, \tag{14.22}$$

where

$$\Lambda = \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 & 0 \\ 0 & \lambda_2 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & \lambda_{n-1} & 0 \\ 0 & 0 & \cdots & 0 & \lambda_n \end{bmatrix}$$

is an otherwise empty $n \times n$ matrix with the eigenvalues of $A$ set along its main diagonal and

$$V = \begin{bmatrix} \mathbf{v}_1 & \mathbf{v}_2 & \cdots & \mathbf{v}_{n-1} & \mathbf{v}_n \end{bmatrix}$$

is an $n \times n$ matrix whose columns are the eigenvectors of $A$. This is so because the identity $A\mathbf{v}_j = \mathbf{v}_j \lambda_j$ holds for all $1 \leq j \leq n$; or, expressed more concisely, because the identity

$$AV = V\Lambda$$

holds.[13] The matrix $V$ is invertible because its columns the eigenvectors are independent, from which (14.22) follows. Equation (14.22) is called the *eigenvalue decomposition,* the *diagonal decomposition* or the *diagonalization* of the square matrix $A$.

One might object that we had shown only how to compose some matrix $V \Lambda V^{-1}$ with the correct eigenvalues and independent eigenvectors, but had failed to show that the matrix was actually $A$. However, we need not show this, because § 14.5 has already demonstrated that two matrices with the same eigenvalues and independent eigenvectors are in fact the same matrix, whereby the product $V \Lambda V^{-1}$ can be nothing other than $A$.

An $n \times n$ matrix with $n$ independent eigenvectors (which class, again, includes every $n \times n$ matrix with $n$ distinct eigenvalues and also includes many matrices with fewer) is called a *diagonalizable* matrix. Besides factoring a diagonalizable matrix by (14.22), one can apply the same formula to compose a diagonalizable matrix with desired eigensolutions.

[13] If this seems confusing, then consider that the $j$th column of the product $AV$ is $A\mathbf{v}_j$, whereas the $j$th column of $\Lambda$ having just the one element acts to scale $V$'s $j$th column only.

The diagonal matrix $\text{diag}\{\mathbf{x}\}$ of (11.55) is trivially diagonalizable as $\text{diag}\{\mathbf{x}\} = I_n \,\text{diag}\{\mathbf{x}\} I_n$.

It is a curious and useful fact that

$$A^2 = (V\Lambda V^{-1})(V\Lambda V^{-1}) = V\Lambda^2 V^{-1}$$

and by extension that

$$A^k = V\Lambda^k V^{-1} \tag{14.23}$$

for any diagonalizable matrix $A$. The diagonal matrix $\Lambda^k$ is nothing more than the diagonal matrix $\Lambda$ with each element individually raised to the $k$th power, such that

$$\left[\Lambda^k\right]_{ij} = \delta_{ij}\lambda_j^k.$$

Changing $z \leftarrow k$ implies the generalization[14]

$$\begin{aligned} A^z &= V\Lambda^z V^{-1}, \\ \left[\Lambda^z\right]_{ij} &= \delta_{ij}\lambda_j^z, \end{aligned} \tag{14.24}$$

good for any diagonalizable $A$ and complex $z$.

Nondiagonalizable matrices are troublesome and interesting. The nondiagonalizable matrix vaguely resembles the singular matrix in that both represent edge cases and can be hard to handle numerically; but the resemblance ends there, and a matrix can be either without being the other. The $n \times n$ null matrix for example is singular but still diagonalizable. What a nondiagonalizable matrix is in essence is a matrix with a repeated eigensolution: the same eigenvalue with the same eigenvector, twice or more. More formally, a nondiagonalizable matrix is a matrix with an $n$-fold eigenvalue whose corresponding eigenvector space fewer than $n$ eigenvectors fully characterize. Section 14.10.2 will have more to say about the nondiagonalizable matrix.

## 14.7 Remarks on the eigenvalue

Eigenvalues and their associated eigenvectors stand among the principal reasons one goes to the considerable trouble to develop matrix theory as we have done in recent chapters. The idea that a matrix resembles a humble scalar in the right circumstance is powerful. Among the reasons for this

[14] It may not be clear however according to (5.13) which branch of $\lambda_j^z$ one should choose at each index $j$, especially if $A$ has negative or complex eigenvalues.

is that a matrix can represent an iterative process, operating repeatedly on a vector $\mathbf{v}$ to change it first to $A\mathbf{v}$, then to $A^2\mathbf{v}$, $A^3\mathbf{v}$ and so on. The *dominant eigenvalue* of $A$, largest in magnitude, tends then to transform $\mathbf{v}$ into the associated eigenvector, gradually but relatively eliminating all other components of $\mathbf{v}$. Should the dominant eigenvalue have greater than unit magnitude, it destabilizes the iteration; thus one can sometimes judge the stability of a physical process indirectly by examining the eigenvalues of the matrix which describes it. Then there is the edge case of the nondiagonalizable matrix, which matrix surprisingly covers only part of its domain with eigenvectors. All this is fairly deep mathematics. It brings an appreciation of the matrix for reasons which were anything but apparent from the outset of Ch. 11.

Remarks continue in §§ 14.10.2 and 14.13.

## 14.8 Matrix condition

The largest in magnitude of the several eigenvalues of a diagonalizable operator $A$, denoted here $\lambda_{\max}$, tends to dominate the iteration $A^k\mathbf{x}$. Section 14.7 has named $\lambda_{\max}$ the *dominant eigenvalue* for this reason.

One sometimes finds it convenient to normalize a dominant eigenvalue by defining a new operator $A' \equiv A/|\lambda_{\max}|$, whose own dominant eigenvalue $\lambda_{\max}/|\lambda_{\max}|$ has unit magnitude. In terms of the new operator, the iteration becomes $A^k\mathbf{x} = |\lambda_{\max}|^k A'^k\mathbf{x}$, leaving one free to carry the magnifying effect $|\lambda_{\max}|^k$ separately if one prefers to do so. However, the scale factor $1/|\lambda_{\max}|$ scales all eigenvalues equally; thus, if $A$'s eigenvalue of *smallest* magnitude is denoted $\lambda_{\min}$, then the corresponding eigenvalue of $A'$ is $\lambda_{\min}/|\lambda_{\max}|$. If zero, then both matrices according to § 14.3 are singular; if nearly zero, then both matrices are ill conditioned.

Such considerations lead us to define the *condition* of a diagonalizable matrix quantitatively as[15]

$$\kappa \equiv \left|\frac{\lambda_{\max}}{\lambda_{\min}}\right|, \tag{14.25}$$

by which

$$\kappa \geq 1 \tag{14.26}$$

is always a real number of no less than unit magnitude. For best invertibility, $\kappa = 1$ would be ideal (it would mean that all eigenvalues had the same magnitude), though in practice quite a broad range of $\kappa$ is usually acceptable.

[15] [62]

Could we always work in exact arithmetic, the value of $\kappa$ might not interest us much as long as it stayed finite; but in computer floating point, or where the elements of $A$ are known only within some tolerance, infinite $\kappa$ tends to emerge imprecisely rather as large $\kappa \gg 1$. An *ill-conditioned* matrix by definition[16] is a matrix of large $\kappa \gg 1$. The applied mathematician handles such a matrix with due skepticism.

Matrix condition so defined turns out to have another useful application. Suppose that a diagonalizable matrix $A$ is precisely known but that the corresponding driving vector $\mathbf{b}$ is not. If

$$A(\mathbf{x} + \delta\mathbf{x}) = \mathbf{b} + \delta\mathbf{b},$$

where $\delta\mathbf{b}$ is the error in $\mathbf{b}$ and $\delta\mathbf{x}$ is the resultant error in $\mathbf{x}$, then one should like to bound the ratio $|\delta\mathbf{x}| \,/\, |\mathbf{x}|$ to ascertain the reliability of $\mathbf{x}$ as a solution. Transferring $A$ to the equation's right side,

$$\mathbf{x} + \delta\mathbf{x} = A^{-1}(\mathbf{b} + \delta\mathbf{b}).$$

Subtracting $\mathbf{x} = A^{-1}\mathbf{b}$ and taking the magnitude,

$$|\delta\mathbf{x}| = \left|A^{-1}\,\delta\mathbf{b}\right|.$$

Dividing by $|\mathbf{x}| = \left|A^{-1}\mathbf{b}\right|$,

$$\frac{|\delta\mathbf{x}|}{|\mathbf{x}|} = \frac{\left|A^{-1}\,\delta\mathbf{b}\right|}{|A^{-1}\mathbf{b}|}.$$

The quantity $\left|A^{-1}\,\delta\mathbf{b}\right|$ cannot exceed $\left|\lambda_{\min}^{-1}\,\delta\mathbf{b}\right|$. The quantity $\left|A^{-1}\mathbf{b}\right|$ cannot fall short of $\left|\lambda_{\max}^{-1}\mathbf{b}\right|$. Thus,

$$\frac{|\delta\mathbf{x}|}{|\mathbf{x}|} \leq \frac{\left|\lambda_{\min}^{-1}\,\delta\mathbf{b}\right|}{\left|\lambda_{\max}^{-1}\mathbf{b}\right|} = \left|\frac{\lambda_{\max}}{\lambda_{\min}}\right| \frac{|\delta\mathbf{b}|}{|\mathbf{b}|}.$$

That is,

$$\frac{|\delta\mathbf{x}|}{|\mathbf{x}|} \leq \kappa \frac{|\delta\mathbf{b}|}{|\mathbf{b}|}. \tag{14.27}$$

Condition, incidentally, might technically be said to apply to scalars as well as to matrices, but ill condition remains a property of matrices alone. According to (14.25), the condition of every nonzero scalar is happily $\kappa = 1$.

[16] There is of course no definite boundary, no particular edge value of $\kappa$, less than which a matrix is well conditioned, at and beyond which it turns ill-conditioned; but you knew that already. If I tried to claim that a matrix with a fine $\kappa = 3$ were ill conditioned, for instance, or that one with a wretched $\kappa = 2^{0\text{x}18}$ were well conditioned, then you might not credit me—but the mathematics nevertheless can only give the number; it remains to the mathematician to interpret it.

## 14.9 The similarity transformation

Any collection of vectors assembled into a matrix can serve as a *basis* by which other vectors can be expressed. For example, if the columns of

$$B = \begin{bmatrix} 1 & -1 \\ 0 & 2 \\ 0 & 1 \end{bmatrix}$$

are regarded as a basis, then the vector

$$B\begin{bmatrix} 5 \\ 1 \end{bmatrix} = 5\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + 1\begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 4 \\ 2 \\ 1 \end{bmatrix}$$

is $(5, 1)$ in the basis $B$: five times the first basis vector plus once the second. The basis provides the units from which other vectors can be built.

Particularly interesting is the $n \times n$, invertible *complete basis* $B$, in which the $n$ basis vectors are independent and address the same full space the columns of $I_n$ address. If

$$\mathbf{x} = B\mathbf{u}$$

then $\mathbf{u}$ represents $\mathbf{x}$ in the basis $B$. Left-multiplication by $B$ evidently converts out of the basis. Left-multiplication by $B^{-1}$,

$$\mathbf{u} = B^{-1}\mathbf{x},$$

then does the reverse, converting into the basis. One can therefore convert any operator $A$ to work within a complete basis $B$ by the successive steps

$$\begin{aligned} A\mathbf{x} &= \mathbf{b}, \\ AB\mathbf{u} &= \mathbf{b}, \\ [B^{-1}AB]\mathbf{u} &= B^{-1}\mathbf{b}, \end{aligned}$$

by which the operator $B^{-1}AB$ is seen to be the operator $A$, only transformed to work within the basis[17,18] $B$.

[17] The reader may need to ponder the basis concept a while to grasp it, but the concept is simple once grasped and little purpose would be served by dwelling on it here. Basically, the idea is that one can build the same vector from alternate building blocks, not only from the standard building blocks $\mathbf{e}_1$, $\mathbf{e}_2$, $\mathbf{e}_3$, etc.—except that the right word for the relevant "building block" is *basis vector*. The books [30] and [42] introduce the basis more gently; one might consult one of those if needed.

[18] The professional matrix literature sometimes distinguishes by typeface between the matrix $B$ and the basis $\mathsf{B}$ its columns represent. Such semantical distinctions seem a little too fine for applied use, though. This book just uses $B$.

The conversion from $A$ into $B^{-1}AB$ is called a *similarity transformation.* If $B$ happens to be unitary (§ 13.12), then the conversion is also called a *unitary transformation.* The matrix $B^{-1}AB$ the transformation produces is said to be *similar* (or, if $B$ is unitary, *unitarily similar*) to the matrix $A$. We have already met the similarity transformation in §§ 11.5 and 12.2. Now we have the theory to appreciate it properly.

Probably the most important property of the similarity transformation is that it alters no eigenvalues. That is, if

$$A\mathbf{x} = \lambda\mathbf{x},$$

then, by successive steps,

$$\begin{aligned} B^{-1}A(BB^{-1})\mathbf{x} &= \lambda B^{-1}\mathbf{x}, \\ [B^{-1}AB]\mathbf{u} &= \lambda\mathbf{u}. \end{aligned} \tag{14.28}$$

*The eigenvalues of $A$ and the similar $B^{-1}AB$ are the same* for any square, $n \times n$ matrix $A$ and any invertible, square, $n \times n$ matrix $B$.

## 14.10 The Schur decomposition

The *Schur decomposition* of an arbitrary, $n \times n$ square matrix $A$ is

$$A = QU_SQ^*, \tag{14.29}$$

where $Q$ is an $n \times n$ unitary matrix whose inverse, as for any unitary matrix (§ 13.12), is $Q^{-1} = Q^*$; and where $U_S$ is a general upper triangular matrix which can have any values (even zeros) along its main diagonal. The Schur decomposition is slightly obscure, is somewhat tedious to derive and is of limited use in itself, but serves a theoretical purpose.[19] We derive it here for this reason.

[19] The alternative is to develop the interesting but difficult *Jordan canonical form,* which for brevity's sake this chapter prefers to omit.

### 14.10.1 Derivation

Suppose that[20] (for some reason, which will shortly grow clear) we have a matrix $B$ of the form

$$B = \begin{bmatrix} \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \\ \cdots & * & * & * & * & * & * & * & \cdots \\ \cdots & 0 & * & * & * & * & * & * & \cdots \\ \cdots & 0 & 0 & * & * & * & * & * & \cdots \\ \cdots & 0 & 0 & 0 & * & * & * & * & \cdots \\ \cdots & 0 & 0 & 0 & 0 & * & * & * & \cdots \\ \cdots & 0 & 0 & 0 & 0 & * & * & * & \cdots \\ \cdots & 0 & 0 & 0 & 0 & * & * & * & \cdots \\ & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}, \tag{14.30}$$

where the $i$th row and $i$th column are depicted at center. Suppose further that we wish to transform $B$ not only similarly but unitarily into

$$C \equiv W^* B W = \begin{bmatrix} \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \\ \cdots & * & * & * & * & * & * & * & \cdots \\ \cdots & 0 & * & * & * & * & * & * & \cdots \\ \cdots & 0 & 0 & * & * & * & * & * & \cdots \\ \cdots & 0 & 0 & 0 & * & * & * & * & \cdots \\ \cdots & 0 & 0 & 0 & 0 & * & * & * & \cdots \\ \cdots & 0 & 0 & 0 & 0 & 0 & * & * & \cdots \\ \cdots & 0 & 0 & 0 & 0 & 0 & * & * & \cdots \\ & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}, \tag{14.31}$$

where $W$ is an $n \times n$ unitary matrix, and where we do not mind if any or all of the $*$ elements change in going from $B$ to $C$ but we require zeros in the indicated spots. Let $B_o$ and $C_o$ represent the $(n-i) \times (n-i)$ submatrices in the lower right corners respectively of $B$ and $C$, such that

$$\begin{aligned} B_o &\equiv I_{n-i} H_{-i} B H_i I_{n-i}, \\ C_o &\equiv I_{n-i} H_{-i} C H_i I_{n-i}, \end{aligned} \tag{14.32}$$

[20] This subsection assigns various capital Roman letters to represent the several matrices and submatrices it manipulates. Its choice of letters except in (14.29) is not standard and carries no meaning elsewhere. The writer had to choose some letters and these are ones he chose.

This footnote mentions the fact because good mathematical style avoid assigning letters that already bear a conventional meaning in a related context (for example, this book avoids writing $A\mathbf{x} = \mathbf{b}$ as $T\mathbf{e} = \mathbf{i}$, not because the latter is wrong but because it would be extremely confusing). The Roman alphabet provides only twenty-six capitals, though, of which this subsection uses too many to be allowed to reserve any. See Appendix B.

where $H_k$ is the shift operator of § 11.9. Pictorially,

$$B_o = \begin{bmatrix} * & * & * & \cdots \\ * & * & * & \cdots \\ * & * & * & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix}, \quad C_o = \begin{bmatrix} * & * & * & \cdots \\ 0 & * & * & \cdots \\ 0 & * & * & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix}.$$

Equation (14.31) seeks an $n \times n$ unitary matrix $W$ to transform the matrix $B$ into a new matrix $C \equiv W^*BW$ such that $C$ fits the form (14.31) stipulates. The question remains as to whether a unitary $W$ exists that satisfies the form and whether for general $B$ we can discover a way to calculate it. To narrow the search, because we need not find every $W$ that satisfies the form but only one such $W$, let us look first for a $W$ that fits the restricted template

$$W = I_i + H_i W_o H_{-i} = \begin{bmatrix} \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \\ \cdots & 1 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\ \cdots & 0 & 1 & 0 & 0 & 0 & 0 & 0 & \cdots \\ \cdots & 0 & 0 & 1 & 0 & 0 & 0 & 0 & \cdots \\ \cdots & 0 & 0 & 0 & 1 & 0 & 0 & 0 & \cdots \\ \cdots & 0 & 0 & 0 & 0 & * & * & * & \cdots \\ \cdots & 0 & 0 & 0 & 0 & * & * & * & \cdots \\ \cdots & 0 & 0 & 0 & 0 & * & * & * & \cdots \\ & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}, \tag{14.33}$$

which contains a smaller, $(n-i) \times (n-i)$ unitary submatrix $W_o$ in its lower right corner and resembles $I_n$ elsewhere. Beginning from (14.31), we have by successive, reversible steps that

$$\begin{aligned} C &= W^*BW \\ &= (I_i + H_i W_o^* H_{-i})(B)(I_i + H_i W_o H_{-i}) \\ &= I_i B I_i + I_i B H_i W_o H_{-i} + H_i W_o^* H_{-i} B I_i \\ &\qquad + H_i W_o^* H_{-i} B H_i W_o H_{-i}. \end{aligned}$$

The unitary submatrix $W_o$ has only $n-i$ columns and $n-i$ rows, so $I_{n-i}W_o = W_o = W_o I_{n-i}$. Thus,

$$\begin{aligned} C &= I_i B I_i + I_i B H_i W_o I_{n-i} H_{-i} + H_i I_{n-i} W_o^* H_{-i} B I_i \\ &\qquad + H_i I_{n-i} W_o^* I_{n-i} H_{-i} B H_i I_{n-i} W_o I_{n-i} H_{-i} \\ &= I_i[B] I_i + I_i[B H_i W_o H_{-i}](I_n - I_i) + (I_n - I_i)[H_i W_o^* H_{-i} B] I_i \\ &\qquad + (I_n - I_i)[H_i W_o^* B_o W_o H_{-i}](I_n - I_i), \end{aligned}$$

where the last step has used (14.32) and the identity (11.76). The four terms on the equation's right, each term with rows and columns neatly truncated, represent the four quarters of $C \equiv W^*BW$—upper left, upper right, lower left and lower right, respectively. The lower left term is null because

$$\begin{aligned}(I_n - I_i)[H_i W_o^* H_{-i} B]I_i &= (I_n - I_i)[H_i W_o^* I_{n-i} H_{-i} B I_i]I_i \\ &= (I_n - I_i)[H_i W_o^* H_{-i}][(I_n - I_i)BI_i]I_i \\ &= (I_n - I_i)[H_i W_o^* H_{-i}][0]I_i = 0,\end{aligned}$$

leaving

$$\begin{aligned}C &= I_i[B]I_i + I_i[BH_i W_o H_{-i}](I_n - I_i) \\ &\quad + (I_n - I_i)[H_i W_o^* B_o W_o H_{-i}](I_n - I_i).\end{aligned}$$

But the upper left term makes the upper left areas of $B$ and $C$ the same, and the upper right term does not bother us because we have not restricted the content of $C$'s upper right area. Apparently any $(n-i)\times(n-i)$ unitary submatrix $W_o$ whatsoever obeys (14.31) in the lower left, upper left and upper right.

That leaves the lower right. Left- and right-multiplying (14.31) by the truncator $(I_n - I_i)$ to focus solely on the lower right area, we have the reduced requirement that

$$(I_n - I_i)C(I_n - I_i) = (I_n - I_i)W^*BW(I_n - I_i). \tag{14.34}$$

Further left-multiplying by $H_{-i}$, right-multiplying by $H_i$, and applying the identity (11.76) yields

$$I_{n-i}H_{-i}CH_iI_{n-i} = I_{n-i}H_{-i}W^*BWH_iI_{n-i};$$

or, substituting from (14.32),

$$C_o = I_{n-i}H_{-i}W^*BWH_iI_{n-i}.$$

Expanding $W$ per (14.33),

$$C_o = I_{n-i}H_{-i}(I_i + H_i W_o^* H_{-i})B(I_i + H_i W_o H_{-i})H_i I_{n-i};$$

or, since $I_{n-i}H_{-i}I_i = 0 = I_i H_i I_{n-i}$,

$$\begin{aligned}C_o &= I_{n-i}H_{-i}(H_i W_o^* H_{-i})B(H_i W_o H_{-i})H_i I_{n-i} \\ &= I_{n-i}W_o^* H_{-i}BH_i W_o I_{n-i} \\ &= W_o^* I_{n-i}H_{-i}BH_i I_{n-i}W_o.\end{aligned}$$

Per (14.32), this is

$$C_o = W_o^* B_o W_o. \tag{14.35}$$

The steps from (14.34) to (14.35) are reversible, so the latter is as good a way to state the reduced requirement as the former is. To achieve a unitary transformation of the form (14.31), therefore, it suffices to satisfy (14.35).

The increasingly well-stocked armory of matrix theory we now have to draw from makes satisfying (14.35) possible as follows. Observe per § 14.5 that every square matrix has at least one eigensolution. Let $(\lambda_o, \mathbf{v}_o)$ represent an eigensolution of $B_o$—*any* eigensolution of $B_o$—with $\mathbf{v}_o$ normalized to unit magnitude. Form the broad, $(n-i) \times (n-i+1)$ matrix

$$F \equiv \begin{bmatrix} \mathbf{v}_o & \mathbf{e}_1 & \mathbf{e}_2 & \mathbf{e}_3 & \cdots & \mathbf{e}_{n-i} \end{bmatrix}.$$

Decompose $F$ by the Gram-Schmidt technique of § 13.11.2, choosing $p = 1$ during the first instance of the algorithm's step 3 (though choosing any permissible $p$ thereafter), to obtain

$$F = Q_F R_F.$$

Noting that the Gram-Schmidt algorithm orthogonalizes only rightward, observe that the first column of the $(n-i) \times (n-i)$ unitary matrix $Q_F$ remains simply the first column of $F$, which is the unit eigenvector $\mathbf{v}_o$:

$$[Q_F]_{*1} = Q_F \mathbf{e}_1 = \mathbf{v}_o.$$

Transform $B_o$ unitarily by $Q_F$ to define the new matrix

$$G \equiv Q_F^* B_o Q_F,$$

then transfer factors to reach the equation

$$Q_F G Q_F^* = B_o.$$

Right-multiplying by $Q_F \mathbf{e}_1 = \mathbf{v}_o$ and noting that $B_o \mathbf{v}_o = \lambda_o \mathbf{v}_o$, observe that

$$Q_F G \mathbf{e}_1 = \lambda_o \mathbf{v}_o.$$

Left-multiplying by $Q_F^*$,

$$G \mathbf{e}_1 = \lambda_o Q_F^* \mathbf{v}_o.$$

Noting that the Gram-Schmidt process has rendered orthogonal to $\mathbf{v}_o$ all columns of $Q_F$ but the first, which is $\mathbf{v}_o$, observe that

$$G \mathbf{e}_1 = \lambda_o Q_F^* \mathbf{v}_o = \lambda_o \mathbf{e}_1 = \begin{bmatrix} \lambda_o \\ 0 \\ 0 \\ \vdots \end{bmatrix},$$

which means that

$$G = \begin{bmatrix} \lambda_o & * & * & \cdots \\ 0 & * & * & \cdots \\ 0 & * & * & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix},$$

which fits the very form (14.32) the submatrix $C_o$ is required to have. Conclude therefore that

$$\begin{aligned} W_o &= Q_F, \\ C_o &= G, \end{aligned} \tag{14.36}$$

where $Q_F$ and $G$ are as this paragraph develops, together constitute a valid choice for $W_o$ and $C_o$, satisfying the reduced requirement (14.35) and thus also the original requirement (14.31).

Equation (14.36) completes a failsafe technique to transform unitarily any square matrix $B$ of the form (14.30) into a square matrix $C$ of the form (14.31). Naturally the technique can be applied recursively as

$$B|_{i=i'} = C|_{i=i'-1}, \quad 1 \le i' \le n, \tag{14.37}$$

because the form (14.30) of $B$ at $i = i'$ is nothing other than the form (14.31) of $C$ at $i = i' - 1$. Therefore, if we let

$$B|_{i=0} = A, \tag{14.38}$$

then it follows by induction that

$$B|_{i=n} = U_S, \tag{14.39}$$

where per (14.30) the matrix $U_S$ has the general upper triangular form the Schur decomposition (14.29) requires. Moreover, because the product of unitary matrices according to (13.64) is itself a unitary matrix, we have that

$$Q = \prod_{i'=0}^{n-1} \left( W|_{i=i'} \right), \tag{14.40}$$

which along with (14.39) accomplishes the Schur decomposition.

### 14.10.2 The nondiagonalizable matrix

The characteristic equation (14.20) of the general upper triangular matrix $U_S$ is

$$\det(U_S - \lambda I_n) = 0.$$

Unlike most determinants, this determinant brings only the one term

$$\det(U_S - \lambda I_n) = \prod_{i=1}^{n} (u_{Sii} - \lambda) = 0$$

whose factors run straight down the main diagonal, where the determinant's $n! - 1$ other terms are all zero because each of them includes at least one zero factor from below the main diagonal.[21] Hence no element above the main diagonal of $U_S$ even influences the eigenvalues, which apparently are

$$\lambda_i = u_{Sii}, \tag{14.41}$$

the main-diagonal elements.

According to (14.28), similarity transformations preserve eigenvalues. The Schur decomposition (14.29) is in fact a similarity transformation; and, as we have seen, every matrix $A$ has a Schur decomposition. If therefore

$$A = QU_SQ^*,$$

then *the eigenvalues of $A$ are just the values along the main diagonal of $U_S$.*[22]

One might think that the Schur decomposition offered an easy way to calculate eigenvalues, but it is less easy than it first appears because one must

[21] The determinant's definition in § 14.1 makes the following two propositions equivalent: (i) that a determinant's term which includes one or more factors above the main diagonal also includes one or more factors below; (ii) that the only permutor that marks no position below the main diagonal is the one which also marks no position above. In either form, the proposition's truth might seem less than obvious until viewed from the proper angle.

Consider a permutor $P$. If $P$ marked no position below the main diagonal, then it would necessarily have $p_{nn} = 1$, else the permutor's bottom row would be empty which is not allowed. In the next-to-bottom row, $p_{(n-1)(n-1)} = 1$, because the $n$th column is already occupied. In the next row up, $p_{(n-2)(n-2)} = 1$; and so on, thus affirming the proposition.

[22] An unusually careful reader might worry that $A$ and $U_S$ had the same eigenvalues with different multiplicities. It would be surprising if it actually were so; but, still, one would like to give a sounder reason than the participle "surprising." Consider however that

$$\begin{aligned} A - \lambda I_n &= QU_SQ^* - \lambda I_n = Q[U_S - Q^*(\lambda I_n)Q]Q^* \\ &= Q[U_S - \lambda(Q^*I_nQ)]Q^* = Q[U_S - \lambda I_n]Q^*. \end{aligned}$$

According to (14.11) and (14.13), this equation's determinant is

$$\det[A - \lambda I_n] = \det\{Q[U_S - \lambda I_n]Q^*\} = \det Q \det[U_S - \lambda I_n] \det Q^* = \det[U_S - \lambda I_n],$$

which says that $A$ and $U_S$ have not only the same eigenvalues but also the same characteristic polynomials, thus further the same eigenvalue multiplicities.

calculate eigenvalues to reach the Schur decomposition in the first place. Whatever practical merit the Schur decomposition might have or lack, however, it brings at least the theoretical benefit of (14.41): every square matrix without exception has a Schur decomposition, whose triangular factor $U_S$ openly lists all eigenvalues along its main diagonal.

This theoretical benefit pays when some of the $n$ eigenvalues of an $n \times n$ square matrix $A$ repeat. By the Schur decomposition, one can construct a second square matrix $A'$, as near as desired to $A$ but having $n$ distinct eigenvalues, simply by perturbing the main diagonal of $U_S$ to[23]

$$\begin{aligned} U'_S &\equiv U_S + \epsilon \operatorname{diag}\{\mathbf{u}\}, \\ u_{i'} &\neq u_i \quad \text{if } \lambda_{i'} = \lambda_i, \end{aligned} \tag{14.42}$$

where $|\epsilon| \ll 1$ and where $\mathbf{u}$ is an arbitrary vector that meets the criterion given. Though infinitesimally near $A$, the modified matrix $A' = QU'_SQ^*$ unlike $A$ has $n$ (maybe infinitesimally) distinct eigenvalues. With sufficient toil, one can analyze such perturbed eigenvalues and their associated eigenvectors similarly as § 9.6.2 has analyzed perturbed poles.

Equation (14.42) brings us to the nondiagonalizable matrix of the subsection's title. Section 14.6 and its diagonalization formula (14.22) diagonalize any matrix with distinct eigenvalues and even any matrix with repeated eigenvalues but distinct eigenvectors, but fail where eigenvectors repeat. Equation (14.42) separates eigenvalues, thus also eigenvectors—for according to § 14.5 eigenvectors of distinct eigenvalues never depend on one another—permitting a nonunique but still sometimes usable form of diagonalization in the limit $\epsilon \to 0$ even when the matrix in question is strictly nondiagonalizable.

The finding that every matrix is arbitrarily nearly diagonalizable illuminates a question the chapter has evaded up to the present point. The question: does a $p$-fold root in the characteristic polynomial (14.20) necessarily imply a $p$-fold eigenvalue in the corresponding matrix? The existence of the nondiagonalizable matrix casts a shadow of doubt until one realizes that every nondiagonalizable matrix is arbitrarily nearly diagonalizable—and, better, is arbitrarily nearly diagonalizable with distinct eigenvalues. If you claim that a matrix has a triple eigenvalue and someone disputes the claim, then you can show him a nearly identical matrix with three infinitesimally distinct eigenvalues. That is the essence of the idea. We will leave the answer in that form.

[23] Equation (11.55) defines the $\operatorname{diag}\{\cdot\}$ notation.

Generalizing the nondiagonalizability concept leads one eventually to the ideas of the *generalized eigenvector*[24] (which solves the higher-order linear system $[A - \lambda I]^k \mathbf{v} = 0$) and the *Jordan canonical form,*[25] which together roughly track the sophisticated conventional pole-separation technique of § 9.6.5. Then there is a kind of sloppy Schur form called a Hessenberg form which allows content in $U_S$ along one or more subdiagonals just beneath the main diagonal. One could profitably propose and prove any number of useful theorems concerning the nondiagonalizable matrix and its generalized eigenvectors, or concerning the eigenvalue problem[26] more broadly, in more and less rigorous ways, but for the time being we will let the matter rest there.

## 14.11 The Hermitian matrix

An $m \times m$ square matrix $A$ that is its own adjoint,

$$A^* = A, \tag{14.43}$$

is called a *Hermitian* or *self-adjoint* matrix. Properties of the Hermitian matrix include that

- its eigenvalues are real,
- its eigenvectors corresponding to distinct eigenvalues lie orthogonal to one another, and
- it is unitarily diagonalizable (§§ 13.12 and 14.6) such that

$$A = V \Lambda V^*. \tag{14.44}$$

That the eigenvalues are real is proved by letting $(\lambda, \mathbf{v})$ represent an eigensolution of $A$ and constructing the product $\mathbf{v}^* A \mathbf{v}$, for which

$$\lambda^* \mathbf{v}^* \mathbf{v} = (A\mathbf{v})^* \mathbf{v} = \mathbf{v}^* A \mathbf{v} = \mathbf{v}^* (A\mathbf{v}) = \lambda \mathbf{v}^* \mathbf{v}.$$

That is,

$$\lambda^* = \lambda,$$

which naturally is possible only if $\lambda$ is real.

[24] [23, Ch. 7]
[25] [21, Ch. 5]
[26] [68]

That eigenvectors corresponding to distinct eigenvalues lie orthogonal to one another is proved[27] by letting $(\lambda_1, \mathbf{v}_1)$ and $(\lambda_2, \mathbf{v}_2)$ represent eigensolutions of $A$ and constructing the product $\mathbf{v}_2^* A \mathbf{v}_1$, for which

$$\lambda_2^* \mathbf{v}_2^* \mathbf{v}_1 = (A\mathbf{v}_2)^* \mathbf{v}_1 = \mathbf{v}_2^* A \mathbf{v}_1 = \mathbf{v}_2^* (A\mathbf{v}_1) = \lambda_1 \mathbf{v}_2^* \mathbf{v}_1.$$

That is,

$$\lambda_2^* = \lambda_1 \quad \text{or} \quad \mathbf{v}_2^* \mathbf{v}_1 = 0.$$

But according to the last paragraph all eigenvalues are real; the eigenvalues $\lambda_1$ and $\lambda_2$ are no exceptions. Hence,

$$\lambda_2 = \lambda_1 \quad \text{or} \quad \mathbf{v}_2^* \mathbf{v}_1 = 0.$$

To prove the last hypothesis of the three needs first some definitions as follows. Given an $m \times m$ matrix $A$, let the $s$ columns of the $m \times s$ matrix $V_o$ represent the $s$ independent eigenvectors of $A$ such that (i) each column has unit magnitude and (ii) columns whose eigenvectors share the same eigenvalue lie orthogonal to one another. Let the $s \times s$ diagonal matrix $\Lambda_o$ carry the eigenvalues on its main diagonal such that

$$AV_o = V_o \Lambda_o,$$

where the distinction between the matrix $\Lambda_o$ and the full eigenvalue matrix $\Lambda$ of (14.22) is that the latter always includes a $p$-fold eigenvalue $p$ times, whereas the former includes a $p$-fold eigenvalue only as many times as the eigenvalue enjoys independent eigenvectors. Let the $m-s$ columns of the $m \times (m-s)$ matrix $V_o^\perp$ represent the complete orthogonal complement (§ 13.10) to $V_o$—perpendicular to all eigenvectors, each column of unit magnitude—such that

$$V_o^{\perp *} V_o = 0 \quad \text{and} \quad V_o^{\perp *} V_o^\perp = I_{m-s}.$$

Recall from § 14.5 that $s \neq 0$ but $0 < s \leq m$ because every square matrix has at least one eigensolution. Recall from § 14.6 that $s = m$ if and only if $A$ is diagonalizable.[28]

[27] [42, § 8.1]

[28] A concrete example: the invertible but nondiagonalizable matrix

$$A = \begin{bmatrix} -1 & 0 & 0 & 0 \\ -6 & 5 & \frac{5}{2} & -\frac{5}{2} \\ 0 & 0 & 5 & 0 \\ 0 & 0 & 0 & 5 \end{bmatrix}$$

has a single eigenvalue at $\lambda = -1$ and a triple eigenvalue at $\lambda = 5$, the latter of whose

With these definitions in hand, we can now prove by contradiction that all Hermitian matrices are diagonalizable, falsely supposing a nondiagonalizable Hermitian matrix $A$, whose $V_o^\perp$ (since $A$ is supposed to be nondiagonalizable, implying that $s < m$) would have at least one column. For such a matrix $A$, $s \times (m-s)$ and $(m-s) \times (m-s)$ auxiliary matrices $F$ and $G$ necessarily would exist such that

$$AV_o^\perp = V_oF + V_o^\perp G,$$

not due to any unusual property of the product $AV_o^\perp$ but for the mundane reason that the columns of $V_o$ and $V_o^\perp$ together by definition addressed the space of *all* $m$-element vectors—including the columns of $AV_o^\perp$. Left-multiplying by $V_o^*$, we would have by successive steps that

$$\begin{aligned}
V_o^*AV_o^\perp &= V_o^*V_oF + V_o^*V_o^\perp G, \\
(AV_o)^*V_o^\perp &= I_sF + V_o^*V_o^\perp G, \\
(V_o\Lambda_o)^*V_o^\perp &= F + V_o^*V_o^\perp G, \\
\Lambda_o^*V_o^*V_o^\perp &= F + V_o^*V_o^\perp G, \\
\Lambda_o^*(0) &= F + (0)G, \\
0 &= F,
\end{aligned}$$

where we had relied on the assumption that $A$ were Hermitian and thus that, as proved above, its distinctly eigenvalued eigenvectors lay orthogonal to one another; in consequence of which $A^* = A$ and $V_o^*V_o = I_s$.

The finding that $F = 0$ reduces the $AV_o^\perp$ equation above to read

$$AV_o^\perp = V_o^\perp G.$$

In the reduced equation the matrix $G$ would have at least one eigensolution, not due to any unusual property of $G$ but because according to § 14.5 every

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eigenvector space is fully characterized by two eigenvectors rather than three such that

$$V_o = \begin{bmatrix} \frac{1}{\sqrt{2}} & 0 & 0 \\ \frac{1}{\sqrt{2}} & 1 & 0 \\ 0 & 0 & \frac{1}{\sqrt{2}} \\ 0 & 0 & \frac{1}{\sqrt{2}} \end{bmatrix}, \quad \Lambda_o = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 5 \end{bmatrix}, \quad V_o^\perp = \begin{bmatrix} 0 \\ 0 \\ \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \end{bmatrix}.$$

The orthogonal complement $V_o^\perp$ supplies the missing vector, not an eigenvector but perpendicular to them all.

In the example, $m = 4$ and $s = 3$.

All vectors in the example are reported with unit magnitude. The two $\lambda = 5$ eigenvectors are reported in mutually orthogonal form, but notice that eigenvectors corresponding to distinct eigenvalues need not be orthogonal when $A$ is not Hermitian.

square matrix, $1 \times 1$ or larger, has at least one eigensolution. Let $(\mu, \mathbf{w})$ represent an eigensolution of $G$. Right-multiplying by the $(m-s)$-element vector $\mathbf{w} \neq 0$, we would have by successive steps that

$$\begin{aligned} AV_o^\perp \mathbf{w} &= V_o^\perp G\mathbf{w}, \\ A(V_o^\perp \mathbf{w}) &= \mu(V_o^\perp \mathbf{w}). \end{aligned}$$

The last equation claims that $(\mu, V_o^\perp \mathbf{w})$ were an eigensolution of $A$, when we had supposed that all of $A$'s eigenvectors lay in the space addressed by the columns of $V_o$, thus by construction did not lie in the space addressed by the columns of $V_o^\perp$. The contradiction proves false the assumption that gave rise to it. The assumption: that a nondiagonalizable Hermitian $A$ existed. We conclude that all Hermitian matrices are diagonalizable—and conclude further that they are *unitarily* diagonalizable on the ground that their eigenvectors lie orthogonal to one another—as was to be demonstrated.

Having proven that all Hermitian matrices are diagonalizable and have real eigenvalues and orthogonal eigenvectors, one wonders whether the converse holds: are all diagonalizable matrices with real eigenvalues and orthogonal eigenvectors Hermitian? To show that they are, one can construct the matrix described by the diagonalization formula (14.22),

$$A = V\Lambda V^*,$$

where $V^{-1} = V^*$ because this $V$ is unitary (§ 13.12). The equation's adjoint is

$$A^* = V\Lambda^* V^*.$$

But all the eigenvalues here are real, which means that $\Lambda^* = \Lambda$ and the right sides of the two equations are the same. That is, $A^* = A$ as was to be demonstrated. *All diagonalizable matrices with real eigenvalues and orthogonal eigenvectors are Hermitian.*

This section brings properties that greatly simplify many kinds of matrix analysis. The properties demand a Hermitian matrix, which might seem a severe and unfortunate restriction—except that one can left-multiply any exactly determined linear system $C\mathbf{x} = \mathbf{d}$ by $C^*$ to get the equivalent Hermitian system

$$[C^*C]\mathbf{x} = [C^*\mathbf{d}], \tag{14.45}$$

in which $A = C^*C$ and $\mathbf{b} = C^*\mathbf{d}$, for which the properties obtain.[29]

[29] The device (14.45) worsens a matrix's condition and may be undesirable for this reason, but it works in theory at least.

## 14.12 The singular-value decomposition

Occasionally an elegant idea awaits discovery, overlooked, almost in plain sight. If the unlikely thought occurred to you to take the square root of a matrix, then the following idea is one you might discover.[30]

Consider the $n \times n$ product $A^*A$ of a tall or square, $m \times n$ matrix $A$ of full column rank

$$r = n \leq m$$

and its adjoint $A^*$. The product $A^*A$ is invertible according to § 13.6.2; is positive definite according to § 13.6.3; and, since $(A^*A)^* = A^*A$, is clearly Hermitian according to § 14.11; thus is unitarily diagonalizable according to (14.44) as

$$A^*A = V\Lambda V^*. \tag{14.46}$$

Here, the $n \times n$ matrices $\Lambda$ and $V$ represent respectively the eigenvalues and eigenvectors not of $A$ but of the product $A^*A$. Though nothing requires the product's eigenvectors to be real, because the product is positive definite § 14.5 does require all of its eigenvalues to be real and moreover positive—which means among other things that the eigenvalue matrix $\Lambda$ has full rank. That the eigenvalues, the diagonal elements of $\Lambda$, are real and positive is a useful fact; for just as a real, positive scalar has a real, positive square root, so equally has $\Lambda$ a real, positive square root under these conditions. Let the symbol $\Sigma = \sqrt{\Lambda}$ represent the $n \times n$ real, positive square root of the eigenvalue matrix $\Lambda$ such that

$$\begin{aligned} \Lambda &= \Sigma^*\Sigma, \\ \Sigma^* = \Sigma &= \begin{bmatrix} +\sqrt{\lambda_1} & 0 & \cdots & 0 & 0 \\ 0 & +\sqrt{\lambda_2} & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & +\sqrt{\lambda_{n-1}} & 0 \\ 0 & 0 & \cdots & 0 & +\sqrt{\lambda_n} \end{bmatrix}, \end{aligned} \tag{14.47}$$

where the *singular values* of $A$ populate $\Sigma$'s diagonal. Applying (14.47) to (14.46) then yields

$$\begin{aligned} A^*A &= V\Sigma^*\Sigma V^*, \\ V^*A^*AV &= \Sigma^*\Sigma. \end{aligned} \tag{14.48}$$

[30] [66, "Singular value decomposition," 14:29, 18 Oct. 2007]

Now consider the $m \times m$ matrix $U$ such that

$$\begin{aligned} AV\Sigma^{-1} &= UI_n, \\ AV &= U\Sigma, \\ A &= U\Sigma V^*. \end{aligned} \tag{14.49}$$

Substituting (14.49)'s second line into (14.48)'s second line gives the equation

$$\Sigma^* U^* U \Sigma = \Sigma^* \Sigma;$$

but $\Sigma\Sigma^{-1} = I_n$, so left- and right-multiplying respectively by $\Sigma^{-*}$ and $\Sigma^{-1}$ leaves that

$$I_n U^* U I_n = I_n,$$

which says neither more nor less than that the first $n$ columns of $U$ are orthonormal. Equation (14.49) does not constrain the last $m - n$ columns of $U$, leaving us free to make them anything we want. Why not use Gram-Schmidt to make them orthonormal, too, thus making $U$ a unitary matrix? If we do this, then the surprisingly simple (14.49) constitutes the *singular-value decomposition* of $A$.

If $A$ happens to have broad shape then we can decompose $A^*$, instead, so this case poses no special trouble. Apparently every full-rank matrix has a singular-value decomposition.

But what of the matrix of less than full rank $r < n$? In this case the product $A^*A$ is singular and has only $s < n$ nonzero eigenvalues (it may be that $s = r$, but this is irrelevant to the proof at hand). However, if the $s$ nonzero eigenvalues are arranged first in $\Lambda$, then (14.49) becomes

$$\begin{aligned} AV\Sigma^{-1} &= UI_s, \\ AV &= U\Sigma, \\ A &= U\Sigma V^*. \end{aligned} \tag{14.50}$$

The product $A^*A$ is nonnegative definite in this case and $\Sigma\Sigma^{-1} = I_s$, but the reasoning is otherwise the same as before. Apparently every matrix of less than full rank has a singular-value decomposition, too.

If $A$ happens to be an invertible square matrix, then the singular-value decomposition evidently inverts it as

$$A^{-1} = V\Sigma^{-1}U^*. \tag{14.51}$$

## 14.13 General remarks on the matrix

Chapters 11 through 14 have derived the uncomfortably bulky but—incredibly—approximately minimal knot of theory one needs to grasp the matrix properly and to use it with moderate versatility. As far as the writer knows, no one has yet discovered a satisfactory way to untangle the knot. The choice to learn the basic theory of the matrix is almost an all-or-nothing choice; and how many scientists and engineers would rightly choose the "nothing" if the matrix did not serve so very many applications as it does? Since it does serve so very many, the "all" it must be.[31] Applied mathematics brings nothing else quite like it.

These several matrix chapters have not covered every topic they might. The topics they omit fall roughly into two classes. One is the class of more advanced and more specialized matrix theory, about which we will have more to say in a moment. The other is the class of basic matrix theory these chapters do not happen to use. The essential agents of matrix analysis—multiplicative associativity, rank, inversion, pseudoinversion, the kernel, the orthogonal complement, orthonormalization, the eigenvalue, diagonalization and so on—are the same in practically all books on the subject, but the way the agents are developed differs. This book has chosen a way that needs some tools like truncators other books omit, but does not need other tools like projectors other books[32] include. What has given these chapters their hefty bulk is not so much the immediate development of the essential agents as the preparatory development of theoretical tools used to construct the essential agents, yet most of the tools are of limited interest in themselves; it is the agents that matter. Tools like the projector not used here tend to be omitted here or deferred to later chapters, not because they are altogether useless but because they are not used *here* and because the present chapters are already too long. The reader who understands the Moore-Penrose pseudoinverse and/or the Gram-Schmidt process reasonably well can after all pretty easily figure out how to construct a projector without explicit instructions thereto, should the need arise.[33]

[31] Of course, one might avoid true understanding and instead work by memorized rules. That is not always a bad plan, really; but if that were *your* plan then it seems spectacularly unlikely that you would be reading a footnote buried beneath the further regions of the hinterland of Chapter 14 in such a book as this.

[32] Such as [30, § 3.VI.3], a lengthy but well-knit tutorial this writer recommends.

[33] Well, since we have brought it up (though only as an example of tools these chapters have avoided bringing up), briefly: a projector is a matrix that flattens an arbitrary vector $\mathbf{b}$ into its nearest shadow $\tilde{\mathbf{b}}$ within some restricted subspace. If the columns of $A$ represent the subspace, then $\mathbf{x}$ represents $\tilde{\mathbf{b}}$ in the subspace basis iff $A\mathbf{x} = \tilde{\mathbf{b}}$, which is to

Paradoxically and thankfully, more advanced and more specialized matrix theory though often harder tends to come in smaller, more manageable increments: the Cholesky decomposition, for instance; or the conjugate-gradient algorithm. The theory develops endlessly. From the present pause one could proceed directly to such topics. However, since this is the *first* proper pause these several matrix chapters have afforded, since the book is *Derivations of Applied Mathematics* rather than *Derivations of Applied Matrices,* maybe we ought to take advantage to change the subject.

say that $A\mathbf{x} \approx \mathbf{b}$, whereupon $\mathbf{x} = A^\dagger \mathbf{b}$. That is, per (13.32),

$$\tilde{\mathbf{b}} = A\mathbf{x} = AA^\dagger \mathbf{b} = [BC][C^*(CC^*)^{-1}(B^*B)^{-1}B^*]\mathbf{b} = B(B^*B)^{-1}B^*\mathbf{b},$$

in which the matrix $B(B^*B)^{-1}B^*$ is the projector. Thence it is readily shown that the deviation $\mathbf{b} - \tilde{\mathbf{b}}$ lies orthogonal to the shadow $\tilde{\mathbf{b}}$. More broadly defined, any matrix $M$ for which $M^2 = M$ is a projector. One can approach the projector in other ways, but there are two ways at least.

# Chapter 15

# Vector analysis

Leaving the matrix, this chapter and the next turn to a curiously underappreciated agent of applied mathematics, the three-dimensional geometrical vector, first met in §§ 3.3, 3.4 and 3.9. Seen from one perspective, the three-dimensional geometrical vector is the $n = 3$ special case of the general, $n$-dimensional vector of Chs. 11 through 14. Because its three elements represent the three dimensions of the physical world, however, the three-dimensional geometrical vector merits closer attention and special treatment.[1]

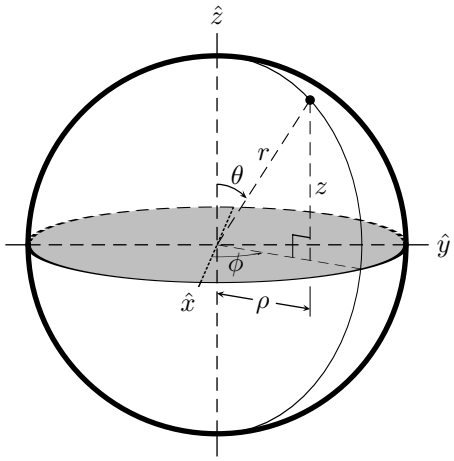
It also merits a shorter name. Where the geometrical context is clear—as it is in this chapter and the next—we will call the three-dimensional geometrical vector just a *vector*. A name like "matrix vector" or "$n$-dimensional vector" can disambiguate the vector of Chs. 11 through 14 where necessary but, since the three-dimensional geometrical vector is in fact a vector, it usually is not necessary to disambiguate. The lone word *vector* serves.

In the present chapter's context and according to § 3.3, a vector consists of an amplitude of some kind plus a direction. Per § 3.9, three scalars called *coordinates* suffice together to specify the amplitude and direction and thus the vector, the three being $(x, y, x)$ in the rectangular coordinate system, $(\rho; \phi, z)$ in the cylindrical coordinate system, or $(r; \theta; \phi)$ in the spherical spherical coordinate system—as Fig. 15.1 illustrates and Table 3.4 on page 67 interrelates—among other, more exotic possibilities (§ 15.7).

The vector brings an elegant notation. This chapter and Ch. 16 detail

[1] [12, Ch. 2]

Figure 15.1: A point on a sphere, in spherical $(r; \theta; \phi)$ and cylindrical $(\rho; \phi, z)$ coordinates. (The axis labels bear circumflexes in this figure only to disambiguate the $\hat{z}$ axis from the cylindrical coordinate $z$. See also Fig. 15.5.)



it. Without the notation, one would write an expression like

$$\frac{(z-z') - [\partial z'/\partial x]_{x=x',y=y'}\,(x-x') - [\partial z'/\partial y]_{x=x',y=y'}\,(y-y')}{\sqrt{[1+(\partial z'/\partial x)^2+(\partial z'/\partial y)^2]_{x=x',y=y'}\,[(x-x')^2+(y-y')^2+(z-z')^2]}}$$

for the aspect coefficient relative to a local surface normal (and if the sentence's words do not make sense to you yet, don't worry; just look the symbols over and appreciate the expression's bulk). The same coefficient in standard vector notation is

$$\hat{\mathbf{n}} \cdot \Delta\hat{\mathbf{r}}.$$

Besides being more evocative (once one has learned to read it) and much more compact, the standard vector notation brings the major advantage of freeing a model's geometry from reliance on any particular coordinate system. Reorienting axes (§ 15.1) for example knots the former expression like spaghetti but does not disturb the latter expression at all.

Two-dimensional geometrical vectors arise in practical modeling about as often as three-dimensional geometrical vectors do. Fortunately, the two-dimensional case needs little special treatment, for it is just the three-dimensional with $z = 0$ or $\theta = 2\pi/4$ (see however § 15.6).

Here at the outset, a word on complex numbers seems in order. Unlike most of the rest of the book this chapter and the next will work chiefly in real numbers, or at least in real coordinates. Notwithstanding, complex coordinates are possible. Indeed, in the rectangular coordinate system complex coordinates are perfectly appropriate and are straightforward enough to handle. The cylindrical and spherical systems however, which these chapters also treat, were not conceived with complex coordinates in mind; and, although it might with some theoretical subtlety be possible to treat complex radii, azimuths and elevations consistently as three-dimensional coordinates, these chapters will not try to do so.[2] (This is not to say that you cannot have a complex vector like, say, $\hat{\boldsymbol{\rho}}[3+j2] - \hat{\boldsymbol{\phi}}[1/4]$ in a nonrectangular basis. You can have such a vector, it is fine, and these chapters will not avoid it. What these chapters will avoid are complex nonrectangular *coordinates* like $[3 + j2; -1/4, 0]$.)

Vector addition will already be familiar to the reader from Ch. 3 or (quite likely) from earlier work outside this book. This chapter therefore begins with the reorientation of axes in § 15.1 and vector multiplication in § 15.2.

[2]The author would be interested to learn if there existed an uncontrived scientific or engineering application that actually used complex, nonrectangular coordinates.

## 15.1 Reorientation

Matrix notation expresses the rotation of axes (3.5) as

$$\begin{bmatrix} \hat{\mathbf{x}}' \\ \hat{\mathbf{y}}' \\ \hat{\mathbf{z}}' \end{bmatrix} = \begin{bmatrix} \cos\phi & \sin\phi & 0 \\ -\sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \hat{\mathbf{x}} \\ \hat{\mathbf{y}} \\ \hat{\mathbf{z}} \end{bmatrix}.$$

In three dimensions however one can do more than just to rotate the $x$ and $y$ axes about the $z$. One can reorient the three axes generally as follows.

### 15.1.1 The Tait-Bryan rotations

With a *yaw* and a *pitch* to point the $x$ axis in the desired direction plus a *roll* to position the $y$ and $z$ axes as desired about the new $x$ axis,[3] one can reorient the three axes generally:

$$\begin{bmatrix} \hat{\mathbf{x}}' \\ \hat{\mathbf{y}}' \\ \hat{\mathbf{z}}' \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\psi & \sin\psi \\ 0 & -\sin\psi & \cos\psi \end{bmatrix} \begin{bmatrix} \cos\theta & 0 & -\sin\theta \\ 0 & 1 & 0 \\ \sin\theta & 0 & \cos\theta \end{bmatrix} \begin{bmatrix} \cos\phi & \sin\phi & 0 \\ -\sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \hat{\mathbf{x}} \\ \hat{\mathbf{y}} \\ \hat{\mathbf{z}} \end{bmatrix}; \tag{15.1}$$

or, inverting per (3.6),

$$\begin{bmatrix} \hat{\mathbf{x}} \\ \hat{\mathbf{y}} \\ \hat{\mathbf{z}} \end{bmatrix} = \begin{bmatrix} \cos\phi & -\sin\phi & 0 \\ \sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\psi & -\sin\psi \\ 0 & \sin\psi & \cos\psi \end{bmatrix} \begin{bmatrix} \hat{\mathbf{x}}' \\ \hat{\mathbf{y}}' \\ \hat{\mathbf{z}}' \end{bmatrix}. \tag{15.2}$$

These are called the *Tait-Bryan rotations,* or alternately the *Cardano rotations.*[4,5]

[3] The English maritime verbs *to yaw, to pitch* and *to roll* describe the rotational motion of a vessel at sea. For a vessel to yaw is for her to rotate about her vertical axis, so her bow (her forwardmost part) yaws from side to side. For a vessel to pitch is for her to rotate about her "beam axis," so her bow pitches up and down. For a vessel to roll is for her to rotate about her "fore-aft axis" such that she rocks or lists (leans) without changing the direction she points [66, "Glossary of nautical terms," 23:00, 20 May 2008]. In the Tait-Bryan rotations as explained in this book, to yaw is to rotate about the $z$ axis, to pitch about the $y$, and to roll about the $x$ [36]. In the Euler rotations as explained in this book later in the present section, however, the axes are assigned to the vessel differently such that to yaw is to rotate about the $x$ axis, to pitch about the $y$, and to roll about the $z$. This implies that the Tait-Bryan vessel points $x$-ward whereas the Euler vessel points $z$-ward. The reason to shift perspective so is to maintain the semantics of the symbols $\theta$ and $\phi$ (though not $\psi$) according to Fig. 15.1.

If this footnote seems confusing, then read (15.1) and (15.7) which are correct.

[4] The literature seems to agree on no standard order among the three Tait-Bryan rotations; and, though the rotational angles are usually named $\phi$, $\theta$ and $\psi$, which angle gets which name admittedly depends on the author.

[5] [11]

Notice in (15.1) and (15.2) that the transpose (though curiously not the adjoint) of each $3 \times 3$ Tait-Bryan factor is also its inverse.

In concept, the Tait-Bryan equations (15.1) and (15.2) say nearly all one needs to say about reorienting axes in three dimensions; but, still, the equations can confuse the uninitiated. Consider a vector

$$\mathbf{v} = \hat{\mathbf{x}}x + \hat{\mathbf{y}}y + \hat{\mathbf{z}}z. \tag{15.3}$$

It is not the vector one reorients but rather the axes used to describe the vector. Envisioning the axes as in Fig. 15.1 with the $z$ axis upward, one first yaws the $x$ axis through an angle $\phi$ toward the $y$ then pitches it downward through an angle $\theta$ away from the $z$. Finally, one rolls the $y$ and $z$ axes through an angle $\psi$ about the new $x$, all the while maintaining the three axes rigidly at right angles to one another. These three Tait-Bryan rotations can orient axes any way. Yet, even once one has clearly visualized the Tait-Bryan sequence, the prospect of applying (15.2) (which inversely represents the sequence) to (15.3) can still seem daunting until one rewrites the latter equation in the form

$$\mathbf{v} = \begin{bmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}, \tag{15.4}$$

after which the application is straightforward. There results

$$\mathbf{v}' = \hat{\mathbf{x}}'x' + \hat{\mathbf{y}}'y' + \hat{\mathbf{z}}'z',$$

where

$$\begin{bmatrix} x' \\ y' \\ z' \end{bmatrix} \equiv \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\psi & \sin\psi \\ 0 & -\sin\psi & \cos\psi \end{bmatrix} \begin{bmatrix} \cos\theta & 0 & -\sin\theta \\ 0 & 1 & 0 \\ \sin\theta & 0 & \cos\theta \end{bmatrix} \begin{bmatrix} \cos\phi & \sin\phi & 0 \\ -\sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}, \tag{15.5}$$

and where Table 3.4 converts to cylindrical or spherical coordinates if and as desired. Since (15.5) resembles (15.1), it comes as no surprise that its inverse,

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} \cos\phi & -\sin\phi & 0 \\ \sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\psi & -\sin\psi \\ 0 & \sin\psi & \cos\psi \end{bmatrix} \begin{bmatrix} x' \\ y' \\ z' \end{bmatrix}, \tag{15.6}$$

resembles (15.2).

### 15.1.2 The Euler rotations

A useful alternative to the Tait-Bryan rotations are the *Euler rotations,* which view the problem of reorientation from the perspective of the $z$ axis rather than of the $x$. The Euler rotations consist of a roll and a pitch followed by another roll, without any explicit yaw:[6]

$$\begin{bmatrix} \hat{\mathbf{x}}' \\ \hat{\mathbf{y}}' \\ \hat{\mathbf{z}}' \end{bmatrix} = \begin{bmatrix} \cos\psi & \sin\psi & 0 \\ -\sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cos\theta & 0 & -\sin\theta \\ 0 & 1 & 0 \\ \sin\theta & 0 & \cos\theta \end{bmatrix} \begin{bmatrix} \cos\phi & \sin\phi & 0 \\ -\sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \hat{\mathbf{x}} \\ \hat{\mathbf{y}} \\ \hat{\mathbf{z}} \end{bmatrix}; \tag{15.7}$$

and inversely

$$\begin{bmatrix} \hat{\mathbf{x}} \\ \hat{\mathbf{y}} \\ \hat{\mathbf{z}} \end{bmatrix} = \begin{bmatrix} \cos\phi & -\sin\phi & 0 \\ \sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{bmatrix} \begin{bmatrix} \cos\psi & -\sin\psi & 0 \\ \sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \hat{\mathbf{x}}' \\ \hat{\mathbf{y}}' \\ \hat{\mathbf{z}}' \end{bmatrix}. \tag{15.8}$$

Whereas the Tait-Bryan point the $x$ axis first, the Euler tactic is to point first the $z$.

So, that's it. One can reorient three axes arbitrarily by rotating them in pairs about the $z$, $y$ and $x$ or the $z$, $y$ and $z$ axes in sequence—or, generalizing, in pairs about any of the three axes so long as the axis of the middle rotation differs from the axes (Tait-Bryan) or axis (Euler) of the first and last. A firmer grasp of the reorientation of axes in three dimensions comes with practice, but those are the essentials of it.

## 15.2 Multiplication

One can multiply a vector in any of three ways. The first, scalar multiplication, is trivial: if a vector $\mathbf{v}$ is as defined by (15.3), then

$$\psi\mathbf{v} = \hat{\mathbf{x}}\psi x + \hat{\mathbf{y}}\psi y + \hat{\mathbf{z}}\psi z. \tag{15.9}$$

Such scalar multiplication evidently scales a vector's length without diverting its direction. The other two forms of vector multiplication involve multiplying a vector by another vector and are the subjects of the two subsections that follow.

[6] As for the Tait-Bryan, for the Euler also the literature agrees on no standard sequence. What one author calls a pitch, another might call a yaw, and some prefer to roll twice about the $x$ axis rather than the $z$. What makes a reorientation an Euler rather than a Tait-Bryan is that the Euler rolls twice.

### 15.2.1 The dot product

We first met the dot product in § 13.8. It works similarly for the geometrical vectors of this chapter as for the matrix vectors of Ch. 13:

$$\mathbf{v}_1 \cdot \mathbf{v}_2 = x_1 x_2 + y_1 y_2 + z_1 z_2, \tag{15.10}$$

which, if the vectors $\mathbf{v}_1$ and $\mathbf{v}_2$ are real, is the product of the two vectors to the extent to which they run in the same direction. It is the product to the extent to which the vectors run in the same direction because one can reorient axes to point $\hat{\mathbf{x}}'$ in the direction of $\mathbf{v}_1$, whereupon $\mathbf{v}_1 \cdot \mathbf{v}_2 = x_1' x_2'$ since $y_1'$ and $z_1'$ have vanished.

Naturally, to be valid, the dot product must not vary under a reorientation of axes; and indeed if we write (15.10) in matrix notation,

$$\mathbf{v}_1 \cdot \mathbf{v}_2 = \begin{bmatrix} x_1 & y_1 & z_1 \end{bmatrix} \begin{bmatrix} x_2 \\ y_2 \\ z_2 \end{bmatrix}, \tag{15.11}$$

and then expand each of the two factors on the right according to (15.6), we see that the dot product does not in fact vary. As in (13.43) of § 13.8, here too the relationship

$$\begin{aligned} \mathbf{v}_1^* \cdot \mathbf{v}_2 &= v_1^* v_2 \cos\theta, \\ \hat{\mathbf{v}}_1^* \cdot \hat{\mathbf{v}}_2 &= \cos\theta, \end{aligned} \tag{15.12}$$

gives the angle $\theta$ between two vectors according Fig. 3.1's cosine if the vectors are real, by definition hereby if complex. Consequently, the two vectors are mutually orthogonal—that is, the vectors run at right angles $\theta = 2\pi/4$ to one another—if and only if

$$\mathbf{v}_1^* \cdot \mathbf{v}_2 = 0.$$

That the dot product is commutative,

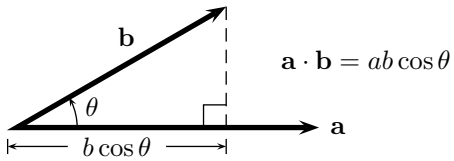
$$\mathbf{v}_2 \cdot \mathbf{v}_1 = \mathbf{v}_1 \cdot \mathbf{v}_2, \tag{15.13}$$

is obvious from (15.10). Fig. 15.2 illustrates the dot product.

### 15.2.2 The cross product

The dot product of two vectors according to § 15.2.1 is a scalar. One can also multiply two vectors to obtain a vector, however, and it is often useful to do so. As the dot product is the product of two vectors to the extent to

Figure 15.2: The dot product.




which they run in the same direction, the *cross product* is the product of two vectors to the extent to which they run in different directions. Unlike the dot product the cross product is a vector, defined in rectangular coordinates as

$$\begin{aligned}\mathbf{v}_1 \times \mathbf{v}_2 &= \begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \end{vmatrix} \\ &\equiv \hat{\mathbf{x}}(y_1 z_2 - z_1 y_2) + \hat{\mathbf{y}}(z_1 x_2 - x_1 z_2) + \hat{\mathbf{z}}(x_1 y_2 - y_1 x_2),\end{aligned} \tag{15.14}$$

where the $|\cdot|$ notation is a mnemonic (actually a pleasant old determinant notation § 14.1 could have but did not happen to use) whose semantics are as shown.

As the dot product, the cross product too is invariant under reorientation. One could demonstrate this fact by multiplying out (15.2) and (15.6) then substituting the results into (15.14): a lengthy, unpleasant exercise. Fortunately, it is also an unnecessary exercise; for, inasmuch as a reorientation consists of three rotations in sequence, it suffices merely that rotation about one axis not alter the dot product. One proves the proposition in the latter form by setting any two of $\phi$, $\theta$ and $\psi$ to zero before multiplying out and substituting.

Several facets of the cross product draw attention to themselves.

- The cyclic progression

$$\cdots \to x \to y \to z \to x \to y \to z \to x \to y \to \cdots \tag{15.15}$$

  of (15.14) arises again and again in vector analysis. Where the progression is honored, as in $\hat{\mathbf{z}} x_1 y_2$, the associated term bears a + sign, otherwise a − sign, due to § 11.6's parity principle and the right-hand rule.

- The cross product is not commutative. In fact,

$$\mathbf{v}_2 \times \mathbf{v}_1 = -\mathbf{v}_1 \times \mathbf{v}_2, \tag{15.16}$$

  which is a direct consequence of the previous point regarding parity, or which can be seen more prosaically in (15.14) by swapping the places of $\mathbf{v}_1$ and $\mathbf{v}_2$.

- The cross product is not associative. That is,

$$(\mathbf{v}_1 \times \mathbf{v}_2) \times \mathbf{v}_3 \neq \mathbf{v}_1 \times (\mathbf{v}_2 \times \mathbf{v}_3),$$

  as is proved by a suitable counterexample like $\mathbf{v}_1 = \mathbf{v}_2 = \hat{\mathbf{x}}$, $\mathbf{v}_3 = \hat{\mathbf{y}}$.

- The cross product runs perpendicularly to each of its two factors if the vectors involved are real. That is,

$$\mathbf{v}_1 \cdot (\mathbf{v}_1 \times \mathbf{v}_2) = 0 = \mathbf{v}_2 \cdot (\mathbf{v}_1 \times \mathbf{v}_2), \tag{15.17}$$

  as is seen by substituting (15.14) into (15.10) with an appropriate change of variables and simplifying.

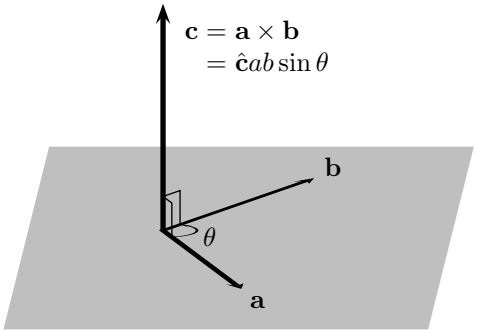
- Unlike the dot product, the cross product is closely tied to three-dimensional space. Two-dimensional space (a plane) can have a cross product so long as one does not mind that the product points off into the third dimension, but to speak of a cross product in four-dimensional space would require arcane definitions and would otherwise make little sense. Fortunately, the physical world is three-dimensional (or, at least, the space in which we model all but a few, exotic physical phenomena is three-dimensional), so the cross product's limitation as defined here to three dimensions will seldom if ever disturb us.

- Section 15.2.1 has related the cosine of the angle between vectors to the dot product. One can similarly relate the angle's sine to the cross product if the vectors involved are real, as

$$\begin{aligned} |\mathbf{v}_1 \times \mathbf{v}_2| &= v_1 v_2 \sin\theta, \\ |\hat{\mathbf{v}}_1 \times \hat{\mathbf{v}}_2| &= \sin\theta, \end{aligned} \tag{15.18}$$

  demonstrated by reorienting axes such that $\hat{\mathbf{v}}_1 = \hat{\mathbf{x}}'$, that $\hat{\mathbf{v}}_2$ has no component in the $\hat{\mathbf{z}}'$ direction, and that $\hat{\mathbf{v}}_2$ has only a nonnegative component in the $\hat{\mathbf{y}}'$ direction; by remembering that reorientation cannot

Figure 15.3: The cross product.




alter a cross product; and finally by applying (15.14) and comparing the result against Fig. 3.1's sine. (If the vectors involved are complex then nothing prevents the operation $|\mathbf{v}_1^* \times \mathbf{v}_2|$ by analogy with eqn. 15.12—in fact the operation $\mathbf{v}_1^* \times \mathbf{v}_2$ without the magnitude sign is used routinely to calculate electromagnetic power flow[7]—but each of the cross product's three rectangular components has its own complex phase which the magnitude operation flattens, so the result's relationship to the sine of an angle is not immediately clear.)

Fig. 15.3 illustrates the cross product.

## 15.3 Orthogonal bases

A vector exists independently of the components by which one expresses it, for, whether $\mathbf{q} = \hat{\mathbf{x}}x + \hat{\mathbf{y}}y + \hat{\mathbf{z}}z$ or $\mathbf{q} = \hat{\mathbf{x}}'x' + \hat{\mathbf{y}}'y' + \hat{\mathbf{z}}'z'$, it remains the same vector $\mathbf{q}$. However, where a model involves a circle, a cylinder or a sphere, where a model involves a contour or a curved surface of some kind, to choose $\hat{\mathbf{x}}'$, $\hat{\mathbf{y}}'$ and $\hat{\mathbf{z}}'$ wisely can immensely simplify the model's analysis. Normally one requires that $\hat{\mathbf{x}}'$, $\hat{\mathbf{y}}'$ and $\hat{\mathbf{z}}'$ each retain unit length, run perpendiclarly to one another, and obey the right-hand rule (§ 3.3), but otherwise any $\hat{\mathbf{x}}'$, $\hat{\mathbf{y}}'$ and $\hat{\mathbf{z}}'$ can serve. Moreover, a model can specify various $\hat{\mathbf{x}}'$, $\hat{\mathbf{y}}'$ and $\hat{\mathbf{z}}'$ under various conditons, for nothing requires the three to be constant.

[7] [28, eqn. 1-51]

Recalling the constants and variables of § 2.7, such a concept is flexible enough to confuse the uninitiated severely and soon. As in § 2.7, here too an example affords perspective. Imagine driving your automobile down a winding road, where $q$ represented your speed[8] and $\hat{\boldsymbol{\ell}}$ represented the direction the road ran, not generally, but just at the spot along the road at which your automobile momentarily happened to be. That your velocity were $\hat{\boldsymbol{\ell}}q$ meant that you kept skilfully to your lane; on the other hand, that your velocity were $(\hat{\boldsymbol{\ell}}\cos\psi + \hat{\mathbf{v}}\sin\psi)q$—where $\hat{\mathbf{v}}$, at right angles to $\hat{\boldsymbol{\ell}}$, represented the direction right-to-left across the road—would have you drifting out of your lane at an angle $\psi$. A headwind had velocity $-\hat{\boldsymbol{\ell}}q_{\text{wind}}$; a crosswind, $\pm\hat{\mathbf{v}}q_{\text{wind}}$. A car a mile ahead of you had velocity $\hat{\boldsymbol{\ell}}_2 q_2 = (\hat{\boldsymbol{\ell}}\cos\beta + \hat{\mathbf{v}}\sin\beta)q_2$, where $\beta$ represented the difference (assuming that the other driver kept skilfully to his own lane) between the road's direction a mile ahead and its direction at your spot. For all these purposes the unit vector $\hat{\boldsymbol{\ell}}$ would remain constant. However, fifteen seconds later, after you had rounded a bend in the road, the symbols $\hat{\boldsymbol{\ell}}$ and $\hat{\mathbf{v}}$ would by definition represent different vectors than before, with respect to which one would express your new velocity as $\hat{\boldsymbol{\ell}}q$ but would no longer express the headwind's velocity as $-\hat{\boldsymbol{\ell}}q_{\text{wind}}$ because, since the road had turned while the wind had not, the wind would no longer be a headwind. And this is where confusion can arise: your own velocity had changed while the expression representing it had not; whereas the wind's velocity had not changed while the expression representing *it* had. This is not because $\hat{\boldsymbol{\ell}}$ differs from place to place at a given moment, for like any other vector the vector $\hat{\boldsymbol{\ell}}$ (as defined in this particular example) is the same vector everywhere. Rather, it is because $\hat{\boldsymbol{\ell}}$ is defined relative to the road at your automobile's location, which location changes as you drive.

If a third unit vector $\hat{\mathbf{w}}$ were defined, perpendicular both to $\hat{\boldsymbol{\ell}}$ and to $\hat{\mathbf{v}}$ such that $[\hat{\boldsymbol{\ell}}\ \hat{\mathbf{v}}\ \hat{\mathbf{w}}]$ obeyed the right-hand rule, then the three together would constitute an *orthogonal basis.* Any three real,[9] right-handedly mutually perpendicular unit vectors $[\hat{\mathbf{x}}'\ \hat{\mathbf{y}}'\ \hat{\mathbf{z}}']$ in three dimensions, whether constant

[8] Conventionally one would prefer the letter $v$ to represent speed, with velocity as $\mathbf{v}$ which in the present example would happen to be $\mathbf{v} = \hat{\boldsymbol{\ell}}v$. However, this section will require the letter $v$ for an unrelated purpose.

[9] A complex orthogonal basis is also theoretically possible but is normally unnecessary in geometrical applications and involves subtleties in the cross product. This chapter, which specifically concerns three-dimensional geometrical vectors rather than the general, $n$-dimensional vectors of Ch. 11, is content to consider real bases only. Note that one can express a complex vector in a real basis.

or variable, for which

$$\begin{array}{lll} \hat{\mathbf{y}}' \cdot \hat{\mathbf{z}}' = 0, & \hat{\mathbf{y}}' \times \hat{\mathbf{z}}' = \hat{\mathbf{x}}', & \Im(\hat{\mathbf{x}}') = 0, \\ \hat{\mathbf{z}}' \cdot \hat{\mathbf{x}}' = 0, & \hat{\mathbf{z}}' \times \hat{\mathbf{x}}' = \hat{\mathbf{y}}', & \Im(\hat{\mathbf{y}}') = 0, \\ \hat{\mathbf{x}}' \cdot \hat{\mathbf{y}}' = 0, & \hat{\mathbf{x}}' \times \hat{\mathbf{y}}' = \hat{\mathbf{z}}', & \Im(\hat{\mathbf{z}}') = 0, \end{array} \tag{15.19}$$

constitutes such an orthogonal basis, from which other vectors can be built. The geometries of some models suggest no particular basis, when one usually just uses a constant $[\hat{\mathbf{x}}\ \hat{\mathbf{y}}\ \hat{\mathbf{z}}]$. The geometries of other models however do suggest a particular basis, often a variable one.

- Where the model features a contour like the example's winding road, an $[\hat{\boldsymbol{\ell}}\ \hat{\mathbf{v}}\ \hat{\mathbf{w}}]$ basis (or a $[\hat{\mathbf{u}}\ \hat{\mathbf{v}}\ \hat{\boldsymbol{\ell}}]$ basis or even a $[\hat{\mathbf{u}}\ \hat{\boldsymbol{\ell}}\ \hat{\mathbf{w}}]$ basis) can be used, where $\hat{\boldsymbol{\ell}}$ locally follows the contour. The variable unit vectors $\hat{\mathbf{v}}$ and $\hat{\mathbf{w}}$ (or $\hat{\mathbf{u}}$ and $\hat{\mathbf{v}}$, etc.) can be defined in any convenient way so long as they remain perpendicular to one another and to $\hat{\boldsymbol{\ell}}$—such that $(\hat{\mathbf{z}} \times \hat{\boldsymbol{\ell}}) \cdot \hat{\mathbf{w}} = 0$ for instance (that is, such that $\hat{\mathbf{w}}$ lay in the plane of $\hat{\mathbf{z}}$ and $\hat{\boldsymbol{\ell}}$)—but if the geometry suggests a particular $\hat{\mathbf{v}}$ or $\hat{\mathbf{w}}$ (or $\hat{\mathbf{u}}$), like the direction right-to-left across the example's road, then that $\hat{\mathbf{v}}$ or $\hat{\mathbf{w}}$ should probably be used. The letter $\ell$ here stands for "longitudinal."[10]

- Where the model features a curved surface like the surface of a wavy sea,[11] a $[\hat{\mathbf{u}}\ \hat{\mathbf{v}}\ \hat{\mathbf{n}}]$ basis (or a $[\hat{\mathbf{u}}\ \hat{\mathbf{n}}\ \hat{\mathbf{w}}]$ basis, etc.) can be used, where $\hat{\mathbf{n}}$ points locally perpendicularly to the surface. The letter $n$ here stands for "normal," a synonym for "perpendicular." Observe, incidentally but significantly, that such a unit normal $\hat{\mathbf{n}}$ tells one everything one needs to know about its surface's local orientation.

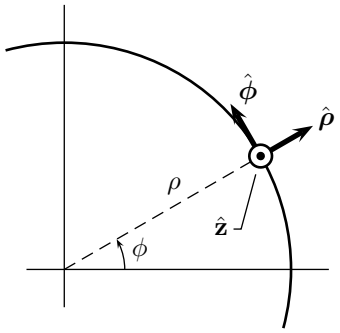
- Combining the last two, where the model features a contour along a curved surface, an $[\hat{\boldsymbol{\ell}}\ \hat{\mathbf{v}}\ \hat{\mathbf{n}}]$ basis can be used. One need not worry about choosing a direction for $\hat{\mathbf{v}}$ in this case since necessarily $\hat{\mathbf{v}} = \hat{\mathbf{n}} \times \hat{\boldsymbol{\ell}}$.

- Where the model features a circle or cylinder, a $[\hat{\boldsymbol{\rho}}\ \hat{\boldsymbol{\phi}}\ \hat{\mathbf{z}}]$ basis can be used, where $\hat{\mathbf{z}}$ is constant and runs along the cylinder's axis (or perpendicularly through the circle's center), $\hat{\boldsymbol{\rho}}$ is variable and points locally away from the axis, and $\hat{\boldsymbol{\phi}}$ is variable and runs locally along the circle's perimeter in the direction of increasing azimuth $\phi$. Refer to § 3.9 and Fig. 15.4.

[10] The assertion wants a citation, which the author lacks.

[11] [49]

Figure 15.4: The cylindrical basis. (The conventional symbols $\odot$ and $\otimes$ respectively represent vectors pointing out of the page toward the reader and into the page away from the reader. Thus, this figure shows the constant basis vector $\hat{\mathbf{z}}$ pointing out of the page toward the reader. The dot in the middle of the $\odot$ is supposed to look like the tip of an arrowhead.)

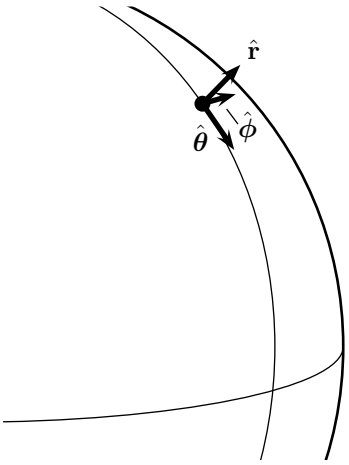



- Where the model features a sphere, an $[\hat{\mathbf{r}}\ \hat{\boldsymbol{\theta}}\ \hat{\boldsymbol{\phi}}]$ basis can be used, where $\hat{\mathbf{r}}$ is variable and points locally away from the sphere's center, $\hat{\boldsymbol{\theta}}$ is variable and runs locally tangentially to the sphere's surface in the direction of increasing elevation $\theta$ (that is, though not usually in the $-\hat{\mathbf{z}}$ direction itself, as nearly as possible to the $-\hat{\mathbf{z}}$ direction without departing from the sphere's surface), and $\hat{\boldsymbol{\phi}}$ is variable and runs locally tangentially to the sphere's surface in the direction of increasing azimuth $\phi$ (that is, along the sphere's surface perpendicularly to $\hat{\mathbf{z}}$). Standing on the earth's surface, with the earth as the sphere, $\hat{\mathbf{r}}$ would be up, $\hat{\boldsymbol{\theta}}$ south, and $\hat{\boldsymbol{\phi}}$ east. Refer to § 3.9 and Fig. 15.5.

- Occasionally a model arises with two circles that share a center but whose axes stand perpendicular to one another. In such a model one conventionally establishes $\hat{\mathbf{z}}$ as the direction of the principal circle's axis but then is left with $\hat{\mathbf{x}}$ or $\hat{\mathbf{y}}$ as the direction of the secondary circle's axis, upon which an $[\hat{\mathbf{x}}\ \hat{\boldsymbol{\rho}}^x\ \hat{\boldsymbol{\phi}}^x]$, $[\hat{\boldsymbol{\phi}}^x\ \hat{\mathbf{r}}\ \hat{\boldsymbol{\theta}}^x]$, $[\hat{\boldsymbol{\phi}}^y\ \hat{\mathbf{y}}\ \hat{\boldsymbol{\rho}}^y]$ or $[\hat{\boldsymbol{\theta}}^y\ \hat{\boldsymbol{\phi}}^y\ \hat{\mathbf{r}}]$ basis can be used locally as appropriate. Refer to § 3.9.

Many other orthogonal bases are possible (as in § 15.7, for instance), but the foregoing are the most common. Whether listed here or not, each orthogonal basis orders its three unit vectors by the right-hand rule (15.19).

Figure 15.5: The spherical basis (see also Fig. 15.1).




Quiz: what does the vector expression $\hat{\boldsymbol{\rho}}3 - \hat{\boldsymbol{\phi}}(1/4) + \hat{\mathbf{z}}2$ mean? Wrong answer: it meant the cylindrical coordinates $(3; -1/4, 2)$; or, it meant the position vector $\hat{\mathbf{x}}3\cos(-1/4) + \hat{\mathbf{y}}3\sin(-1/4) + \hat{\mathbf{z}}2$ associated with those coordinates. Right answer: the expression means nothing certain in itself but acquires a definite meaning only when an azimuthal coordinate $\phi$ is also supplied, after which the expression indicates the ordinary rectangular vector $\hat{\mathbf{x}}'3 - \hat{\mathbf{y}}'(1/4) + \hat{\mathbf{z}}'2$, where $\hat{\mathbf{x}}' = \hat{\boldsymbol{\rho}} = \hat{\mathbf{x}}\cos\phi + \hat{\mathbf{y}}\sin\phi$, $\hat{\mathbf{y}}' = \hat{\boldsymbol{\phi}} = -\hat{\mathbf{x}}\sin\phi + \hat{\mathbf{y}}\cos\phi$, and $\hat{\mathbf{z}}' = \hat{\mathbf{z}}$. But, if this is so—if the cylindrical basis $[\hat{\boldsymbol{\rho}}\ \hat{\boldsymbol{\phi}}\ \hat{\mathbf{z}}]$ is used solely to express *rectangular* vectors—then why should we name this basis "cylindrical"? Answer: only because cylindrical coordinates (supplied somewhere) determine the actual directions of its basis vectors. Once directions are determined such a basis is used purely rectangularly, like any other orthogonal basis.

This can seem confusing until one has grasped what the so-called non-rectangular bases are for. Consider the problem of air flow in a jet engine. It probably suits such a problem that instantaneous local air velocity within the engine cylinder be expressed in cylindrical coordinates, with the $z$ axis oriented along the engine's axle; but this does not mean that the air flow within the engine cylinder were everywhere $\hat{\mathbf{z}}$-directed. On the contrary, a local air velocity of $\mathbf{q} = [-\hat{\boldsymbol{\rho}}5.0 + \hat{\boldsymbol{\phi}}30.0 - \hat{\mathbf{z}}250.0]$ m/s would have air moving

through the point in question at 250.0 m/s aftward along the axle, 5.0 m/s inward toward the axle and 30.0 m/s circulating about the engine cylinder.

In this model, it is true that the basis vectors $\hat{\boldsymbol{\rho}}$ and $\hat{\boldsymbol{\phi}}$ indicate different directions at different positions within the cylinder, but at a particular position the basis vectors are still used rectangularly to express $\mathbf{q}$, the instantaneous local air velocity at that position. It's just that the "rectangle" is rotated locally to line up with the axle.

Naturally, you cannot make full sense of an air-velocity vector $\mathbf{q}$ unless you have also the coordinates $(\rho; \phi, z)$ of the position within the engine cylinder at which the air has the velocity the vector specifies—yet this is when confusion can arise, for besides the air-velocity vector there is also, separately, a position vector $\mathbf{r} = \hat{\mathbf{x}}\rho\cos\phi + \hat{\mathbf{y}}\rho\sin\phi + \hat{\mathbf{z}}z$. One may denote the air-velocity vector as[12] $\mathbf{q}(\mathbf{r})$, a function of position; yet, though the position vector is as much a vector as the velocity vector is, one nonetheless handles it differently. One will not normally express the position vector $\mathbf{r}$ in the cylindrical basis.

It would make little sense to try to express the position vector $\mathbf{r}$ in the cylindrical basis because the position vector is the very thing that *determines* the cylindrical basis. In the cylindrical basis, after all, the position vector is necessarily $\mathbf{r} = \hat{\boldsymbol{\rho}}\rho + \hat{\mathbf{z}}z$ (and consider: in the spherical basis it is the even more cryptic $\mathbf{r} = \hat{\mathbf{r}}r$), and how useful is that, really? Well, maybe it is useful in some situations, but for the most part to express the position vector in the cylindrical basis would be as to say, "My house is zero miles away from home." Or, "The time is presently now." Such statements may be tautologically true, perhaps, but they are confusing because they only seem to give information. The position vector $\mathbf{r}$ determines the basis, after which one expresses things other than position, like instantaneous local air velocity $\mathbf{q}$, in that basis. In fact, the only basis normally suitable to express a position vector is a fixed rectangular basis like $[\hat{\mathbf{x}}\ \hat{\mathbf{y}}\ \hat{\mathbf{z}}]$. Otherwise, one uses cylindrical coordinates $(\rho; \phi, z)$, but not a cylindrical basis $[\hat{\boldsymbol{\rho}}\ \hat{\boldsymbol{\phi}}\ \hat{\mathbf{z}}]$, to express a position $\mathbf{r}$ in a cylindrical geometry.

Maybe the nonrectangular bases were more precisely called "rectangular bases of the nonrectangular coordinate systems," but those are too many words and, anyway, that is not how the usage has evolved. Chapter 16 will elaborate the story by considering spatial derivatives of quantities like air velocity, when one must take the variation in $\hat{\boldsymbol{\rho}}$ and $\hat{\boldsymbol{\phi}}$ from point to point

[12] Conventionally, one is much more likely to denote a velocity vector as $\mathbf{u}(\mathbf{r})$ or $\mathbf{v}(\mathbf{r})$, except that the present chapter is (as footnote 8 has observed) already using the letters $u$ and $v$ for an unrelated purpose. To denote position as $\mathbf{r}$ however is entirely standard.

into account, but the foregoing is the basic story nevertheless.

## 15.4 Notation

The vector notation of §§ 15.1 and 15.2 is correct, familiar and often expedient but sometimes inconveniently prolix. This admittedly difficult section augments the notation to render it much more concise.

### 15.4.1 Components by subscript

The notation

$$\begin{aligned} a_x &\equiv \hat{\mathbf{x}} \cdot \mathbf{a}, & a_\rho &\equiv \hat{\boldsymbol{\rho}} \cdot \mathbf{a}, \\ a_y &\equiv \hat{\mathbf{y}} \cdot \mathbf{a}, & a_r &\equiv \hat{\mathbf{r}} \cdot \mathbf{a}, \\ a_z &\equiv \hat{\mathbf{z}} \cdot \mathbf{a}, & a_\theta &\equiv \hat{\boldsymbol{\theta}} \cdot \mathbf{a}, \\ a_n &\equiv \hat{\mathbf{n}} \cdot \mathbf{a}, & a_\phi &\equiv \hat{\boldsymbol{\phi}} \cdot \mathbf{a}, \end{aligned}$$

and so forth abbreviates the indicated dot product. That is to say, the notation represents the component of a vector $\mathbf{a}$ in the indicated direction. Generically,

$$a_\alpha \equiv \hat{\boldsymbol{\alpha}} \cdot \mathbf{a}. \tag{15.20}$$

Applied mathematicians use subscripts for several unrelated or vaguely related purposes, so the full dot-product notation $\hat{\boldsymbol{\alpha}} \cdot \mathbf{a}$ is often clearer in print than the abbreviation $a_\alpha$ is, but the abbreviation especially helps when several such dot products occur together in the same expression.

Since[13]

$$\begin{aligned} \hat{\mathbf{a}} &= \hat{\mathbf{x}} a_x + \hat{\mathbf{y}} a_y + \hat{\mathbf{z}} a_z, \\ \hat{\mathbf{b}} &= \hat{\mathbf{x}} b_x + \hat{\mathbf{y}} b_y + \hat{\mathbf{z}} b_z, \end{aligned}$$

the abbreviation lends a more amenable notation to the dot and cross products of (15.10) and (15.14):

$$\mathbf{a} \cdot \mathbf{b} = a_x b_x + a_y b_y + a_z b_z; \tag{15.21}$$

$$\mathbf{a} \times \mathbf{b} = \begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ a_x & a_y & a_z \\ b_x & b_y & b_z \end{vmatrix}. \tag{15.22}$$

[13] "Wait!" comes the objection. "I thought that you said that $a_x$ meant $\hat{\mathbf{x}} \cdot \mathbf{a}$. Now you claim that it means the $x$ component of $\mathbf{a}$?"

But there is no difference between $\hat{\mathbf{x}} \cdot \mathbf{a}$ and the $x$ component of $\mathbf{a}$. The two are one and the same.

In fact—because, as we have seen, reorientation of axes cannot alter the dot and cross products—any orthogonal basis $[\hat{\mathbf{x}}'\ \hat{\mathbf{y}}'\ \hat{\mathbf{z}}']$ (§ 15.3) can serve here, so one can write more generally that

$$\mathbf{a}\cdot\mathbf{b} = a_{x'}b_{x'} + a_{y'}b_{y'} + a_{z'}b_{z'}; \tag{15.23}$$

$$\mathbf{a}\times\mathbf{b} = \begin{vmatrix} \hat{\mathbf{x}}' & \hat{\mathbf{y}}' & \hat{\mathbf{z}}' \\ a_{x'} & a_{y'} & a_{z'} \\ b_{x'} & b_{y'} & b_{z'} \end{vmatrix}. \tag{15.24}$$

Because all those prime marks burden the notation and for professional mathematical reasons, the general forms (15.23) and (15.24) are sometimes rendered

$$\mathbf{a}\cdot\mathbf{b} = a_1b_1 + a_2b_2 + a_3b_3,$$

$$\mathbf{a}\times\mathbf{b} = \begin{vmatrix} \hat{\mathbf{e}}_1 & \hat{\mathbf{e}}_2 & \hat{\mathbf{e}}_3 \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix},$$

but you have to be careful about that in applied usage because people are not always sure whether a symbol like $a_3$ means "the third component of the vector $\mathbf{a}$" (as it does here) or "the third vector's component in the $\hat{\mathbf{a}}$ direction" (as it would in eqn. 15.10). Typically, applied mathematicians will write in the manner of (15.21) and (15.22) with the implied understanding that they really mean (15.23) and (15.24) but prefer not to burden the notation with extra little strokes—that is, with the implied understanding that $x$, $y$ and $z$ could just as well be $\rho$, $\phi$ and $z$ or the coordinates of any other orthogonal, right-handed, three-dimensional basis.

Some pretty powerful confusion can afflict the student regarding the roles of the cylindrical symbols $\rho$, $\phi$ and $z$; or, worse, of the spherical symbols $r$, $\theta$ and $\phi$. Such confusion reflects a pardonable but remediable lack of understanding of the relationship between coordinates like $\rho$, $\phi$ and $z$ and their corresponding unit vectors $\hat{\boldsymbol{\rho}}$, $\hat{\boldsymbol{\phi}}$ and $\hat{\mathbf{z}}$. Section 15.3 has already written of the matter; but, further to dispel the confusion, one can now ask the student what the cylindrical coordinates of the vectors $\hat{\boldsymbol{\rho}}$, $\hat{\boldsymbol{\phi}}$ and $\hat{\mathbf{z}}$ are. The correct answer: $(1;\phi,0)$, $(1;\phi+2\pi/4,0)$ and $(0;0,1)$, respectively. Then, to reinforce, one can ask the student which cylindrical coordinates the variable vectors $\hat{\boldsymbol{\rho}}$ and $\hat{\boldsymbol{\phi}}$ are functions of. The correct answer: both are functions of the coordinate $\phi$ only ($\hat{\mathbf{z}}$, a constant vector, is not a function of anything). What the student needs to understand is that, among the cylindrical coordinates, $\phi$ is a different kind of thing than $z$ and $\rho$ are:

- $z$ and $\rho$ are lengths whereas $\phi$ is an angle;
- but $\hat{\boldsymbol{\rho}}$, $\hat{\boldsymbol{\phi}}$ and $\hat{\mathbf{z}}$ are all the same kind of thing, unit vectors;
- and, separately, $a_\rho$, $a_\phi$ and $a_z$ are all the same kind of thing, lengths.

Now to ask the student a harder question: in the cylindrical basis, what is the vector representation of $(\rho_1; \phi_1, z_1)$? The correct answer: $\hat{\boldsymbol{\rho}}\rho_1 \cos(\phi_1 - \phi) + \hat{\boldsymbol{\phi}}\rho_1 \sin(\phi_1 - \phi) + \hat{\mathbf{z}}z_1$. The student that gives this answer probably grasps the cylindrical symbols.

If the reader feels that the notation begins to confuse more than it describes, the writer empathizes but regrets to inform the reader that the rest of the section, far from granting the reader a comfortable respite to absorb the elaborated notation as it stands, will not delay to elaborate the notation yet further! The confusion however is subjective. The trouble with vector work is that one has to learn to abbreviate or the expressions involved grow repetitive and unreadably long. For vectors, the abbreviated notation really is the proper notation. Eventually one accepts the need and takes the trouble to master the conventional vector abbreviation this section presents; and, indeed, the abbreviation is rather elegant once one becomes used to it. So, study closely and take heart! The notation is not actually as impenetrable as it at first will seem.

### 15.4.2 Einstein's summation convention

*Einstein's summation convention* is this: *that repeated indices are implicitly summed over.*[14] For instance, where the convention is in force, the equation[15]

$$\mathbf{a} \cdot \mathbf{b} = a_i b_i \tag{15.25}$$

means that

$$\mathbf{a} \cdot \mathbf{b} = \sum_i a_i b_i$$

or more fully that

$$\mathbf{a} \cdot \mathbf{b} = \sum_{i=x',y',z'} a_i b_i = a_{x'} b_{x'} + a_{y'} b_{y'} + a_{z'} b_{z'},$$

[14] [32]

[15] Some professional mathematicians now write a superscript $a^i$ in certain cases in place of a subscript $a_i$, where the superscript bears some additional semantics [66, "Einstein notation," 05:36, 10 February 2008]. Scientists and engineers however tend to prefer Einstein's original, subscript-only notation.

which is (15.23), except that Einstein's form (15.25) expresses it more succinctly. Likewise,

$$\mathbf{a} \times \mathbf{b} = \hat{\mathbf{i}}(a_{i+1}b_{i-1} - b_{i+1}a_{i-1}) \tag{15.26}$$

is (15.24)—although an experienced applied mathematician would probably apply the Levi-Civita epsilon of § 15.4.3, below, to further abbreviate this last equation to the form of (15.27) before presenting it.

Einstein's summation convention is also called the *Einstein notation,* a term sometimes taken loosely to include also the Kronecker delta and Levi-Civita epsilon of § 15.4.3.

What is important to understand about Einstein's summation convention is that, in and of itself, it brings no new mathematics. It is rather a notational convenience.[16] It asks a reader to regard a repeated index like the $i$ in "$a_ib_i$" as a dummy index (§ 2.3) and thus to read "$a_ib_i$" as "$\sum_i a_ib_i$." It does not magically create a summation where none existed; it just hides the summation sign to keep it from cluttering the page. It is the kind of notational trick an accountant might appreciate. Under the convention, the summational operator $\sum_i$ is implied not written, but the operator is still there. Admittedly confusing on first encounter, the convention's utility and charm are felt after only a little practice.

Incidentally, nothing requires you to invoke Einstein's summation convention everywhere and for all purposes. You can waive the convention, writing the summation symbol out explicitly whenever you like.[17] In contexts outside vector analysis, to invoke the convention at all may make little sense. Nevertheless, you should indeed learn the convention—if only because you must learn it to understand the rest of this chapter—but once having learned it you should naturally use it only where it actually serves to clarify. Fortunately, in vector work, it often does just that.

Quiz:[18] if $\delta_{ij}$ is the Kronecker delta of § 11.2, then what does the symbol $\delta_{ii}$ represent where Einstein's summation convention is in force?

### 15.4.3 The Kronecker delta and the Levi-Civita epsilon

Einstein's summation convention expresses the dot product (15.25) neatly but, as we have seen in (15.26), does not by itself wholly avoid unseemly repetition in the cross product. The *Levi-Civita epsilon*[19] $\epsilon_{ijk}$ mends this,

[16] [64, "Einstein summation"]

[17] [51]

[18] [32]

[19] Also called the *Levi-Civita symbol, tensor,* or *permutor.* For native English speakers who do not speak Italian, the "ci" in Levi-Civita's name is pronounced as the "chi" in

rendering the cross-product as

$$\mathbf{a} \times \mathbf{b} = \epsilon_{ijk} \hat{\imath} a_j b_k, \tag{15.27}$$

where[20]

$$\epsilon_{ijk} \equiv \begin{cases} +1 & \text{if } (i,j,k) = (x',y',z'),\ (y',z',x') \text{ or } (z',x',y'); \\ -1 & \text{if } (i,j,k) = (x',z',y'),\ (y',x',z') \text{ or } (z',y',x'); \\ 0 & \text{otherwise [for instance if } (i,j,k) = (x',x',y')]. \end{cases} \tag{15.28}$$

In the language of § 11.6, the Levi-Civita epsilon quantifies parity. (Chapters 11 and 14 did not use it, but the Levi-Civita notation applies in any number of dimensions, not only three as in the present chapter. In this more general sense the Levi-Civita is the determinant of the permutor whose ones hold the indicated positions—which is a formal way of saying that it's a + sign for even parity and a − sign for odd. For instance, in the four-dimensional, $4 \times 4$ case $\epsilon_{1234} = 1$ whereas $\epsilon_{1243} = -1$: refer to §§ 11.6, 11.7.1 and 14.1. Table 15.1, however, as the rest of this section and chapter, concerns the three-dimensional case only.)

Technically, the Levi-Civita epsilon and Einstein's summation convention are two separate, independent things, but a canny reader takes the Levi-Civita's appearance as a hint that Einstein's convention is probably in force, as in (15.27). The two tend to go together.[21]

The Levi-Civita epsilon $\epsilon_{ijk}$ relates to the Kronecker delta $\delta_{ij}$ of § 11.2 approximately as the cross product relates to the dot product. Both delta and epsilon find use in vector work. For example, one can write (15.25) alternately in the form

$$\mathbf{a} \cdot \mathbf{b} = \delta_{ij} a_i b_j.$$

Table 15.1 lists several relevant properties,[22] each as with Einstein's summation convention in force.[23] Of the table's several properties, the

"children."

[20] [50, "Levi-Civita permutation symbol"]

[21] The writer has heard the apocryphal belief expressed that the letter $\epsilon$, a Greek $e$, stood in this context for "Einstein." As far as the writer knows, $\epsilon$ is merely the letter after $\delta$, which represents the name of Paul Dirac—though the writer does not claim his indirected story to be any less apocryphal than the other one (the capital letter $\Delta$ has a point on top that suggests the pointy nature of the Dirac delta of Fig. 7.10, which makes for yet another plausible story). In any event, one sometimes hears Einstein's summation convention, the Kronecker delta and the Levi-Civita epsilon together referred to as "the Einstein notation," which though maybe not quite terminologically correct is hardly incorrect enough to argue over and is clear enough in practice.

[22] [51]

[23] The table incidentally answers § 15.4.2's quiz.

Table 15.1: Properties of the Kronecker delta and the Levi-Civita epsilon, with Einstein's summation convention in force.

$$\begin{aligned}
\delta_{jk} &= \delta_{kj} \\
\delta_{ij}\delta_{jk} &= \delta_{ik} \\
\delta_{ii} &= 3 \\
\delta_{jk}\epsilon_{ijk} &= 0 \\
\delta_{nk}\epsilon_{ijk} &= \epsilon_{ijn} \\
\epsilon_{ijk} = \epsilon_{jki} = \epsilon_{kij} &= -\epsilon_{ikj} = -\epsilon_{jik} = -\epsilon_{kji} \\
\epsilon_{ijk}\epsilon_{ijk} &= 6 \\
\epsilon_{ijn}\epsilon_{ijk} &= 2\delta_{nk} \\
\epsilon_{imn}\epsilon_{ijk} &= \delta_{mj}\delta_{nk} - \delta_{mk}\delta_{nj}
\end{aligned}$$

property that $\epsilon_{imn}\epsilon_{ijk} = \delta_{mj}\delta_{nk} - \delta_{mk}\delta_{nj}$ is proved by observing that, in the case that $i = x'$, either $(j,k) = (y',z')$ or $(j,k) = (z',y')$, and also either $(m,n) = (y',z')$ or $(m,n) = (z',y')$; and similarly in the cases that $i = y'$ and $i = z'$ (more precisely, in each case the several indices can take any values, but combinations other than the ones listed drive $\epsilon_{imn}$ or $\epsilon_{ijk}$, or both, to zero, thus contributing nothing to the sum). This implies that either $(j,k) = (m,n)$ or $(j,k) = (n,m)$—which, when one takes parity into account, is exactly what the property in question asserts. The property that $\epsilon_{ijn}\epsilon_{ijk} = 2\delta_{nk}$ is proved by observing that, in any given term of the Einstein sum, $i$ is either $x'$ or $y'$ or $z'$ and that $j$ is one of the remaining two, which leaves the third to be shared by both $k$ and $n$. The factor 2 appears because, for $k = n = x'$, an $(i,j) = (y',z')$ term and an $(i,j) = (z',y')$ term both contribute positively to the sum; and similarly for $k = n = y'$ and again for $k = n = z'$.

Unfortunately, the last paragraph likely makes sense to few who do not already know what it means. A concrete example helps. Consider the compound product $\mathbf{c} \times (\mathbf{a} \times \mathbf{b})$. In this section's notation and with the use

of (15.27), the compound product is

$$\begin{aligned}
\mathbf{c} \times (\mathbf{a} \times \mathbf{b}) &= \mathbf{c} \times (\epsilon_{ijk} \hat{\mathbf{i}} a_j b_k) \\
&= \epsilon_{mni} \hat{\mathbf{m}} c_n (\epsilon_{ijk} \hat{\mathbf{i}} a_j b_k)_i \\
&= \epsilon_{mni} \epsilon_{ijk} \hat{\mathbf{m}} c_n a_j b_k \\
&= \epsilon_{imn} \epsilon_{ijk} \hat{\mathbf{m}} c_n a_j b_k \\
&= (\delta_{mj} \delta_{nk} - \delta_{mk} \delta_{nj}) \hat{\mathbf{m}} c_n a_j b_k \\
&= \delta_{mj} \delta_{nk} \hat{\mathbf{m}} c_n a_j b_k - \delta_{mk} \delta_{nj} \hat{\mathbf{m}} c_n a_j b_k \\
&= \hat{\mathbf{j}} c_k a_j b_k - \hat{\mathbf{k}} c_j a_j b_k \\
&= (\hat{\mathbf{j}} a_j)(c_k b_k) - (\hat{\mathbf{k}} b_k)(c_j a_j).
\end{aligned}$$

That is, in light of (15.25),

$$\mathbf{c} \times (\mathbf{a} \times \mathbf{b}) = \mathbf{a}(\mathbf{c} \cdot \mathbf{b}) - \mathbf{b}(\mathbf{c} \cdot \mathbf{a}), \tag{15.29}$$

a useful vector identity. Written without the benefit of Einstein's summation convention, the example's central step would have been

$$\begin{aligned}
\mathbf{c} \times (\mathbf{a} \times \mathbf{b}) &= \sum_{i,j,k,m,n} \epsilon_{imn} \epsilon_{ijk} \hat{\mathbf{m}} c_n a_j b_k \\
&= \sum_{j,k,m,n} (\delta_{mj} \delta_{nk} - \delta_{mk} \delta_{nj}) \hat{\mathbf{m}} c_n a_j b_k,
\end{aligned}$$

which makes sense if you think about it hard enough,[24] and justifies the

[24] If thinking about it hard enough does not work, then here it is in interminable detail:

$$
\begin{aligned}
&\sum_{i,j,k,m,n} \epsilon_{imn}\epsilon_{ijk} f(j,k,m,n) \\
&= \epsilon_{x'y'z'}\epsilon_{x'y'z'} f(y',z',y',z') + \epsilon_{x'y'z'}\epsilon_{x'z'y'} f(y',z',z',y') \\
&\quad + \epsilon_{x'z'y'}\epsilon_{x'y'z'} f(z',y',y',z') + \epsilon_{x'z'y'}\epsilon_{x'z'y'} f(z',y',z',y') \\
&\quad + \epsilon_{y'z'x'}\epsilon_{y'z'x'} f(z',x',z',x') + \epsilon_{y'z'x'}\epsilon_{y'x'z'} f(z',x',x',z') \\
&\quad + \epsilon_{y'x'z'}\epsilon_{y'z'x'} f(x',z',z',x') + \epsilon_{y'x'z'}\epsilon_{y'x'z'} f(x',z',x',z') \\
&\quad + \epsilon_{z'x'y'}\epsilon_{z'x'y'} f(x',y',x',y') + \epsilon_{z'x'y'}\epsilon_{z'y'x'} f(x',y',y',x') \\
&\quad + \epsilon_{z'y'x'}\epsilon_{z'x'y'} f(y',x',x',y') + \epsilon_{z'y'x'}\epsilon_{z'y'x'} f(y',x',y',x') \\
&= f(y',z',y',z') - f(y',z',z',y') - f(z',y',y',z') + f(z',y',z',y') \\
&\quad + f(z',x',z',x') - f(z',x',x',z') - f(x',z',z',x') + f(x',z',x',z') \\
&\quad + f(x',y',x',y') - f(x',y',y',x') - f(y',x',x',y') + f(y',x',y',x') \\
&= \big[f(y',z',y',z') + f(z',x',z',x') + f(x',y',x',y') \\
&\qquad + f(z',y',z',y') + f(x',z',x',z') + f(y',x',y',x')\big] \\
&\quad - \big[f(y',z',z',y') + f(z',x',x',z') + f(x',y',y',x') \\
&\qquad + f(z',y',y',z') + f(x',z',z',x') + f(y',x',x',y')\big] \\
&= \big[f(y',z',y',z') + f(z',x',z',x') + f(x',y',x',y') \\
&\qquad + f(z',y',z',y') + f(x',z',x',z') + f(y',x',y',x') \\
&\qquad + f(x',x',x',x') + f(y',y',y',y') + f(z',z',z',z')\big] \\
&\quad - \big[f(y',z',z',y') + f(z',x',x',z') + f(x',y',y',x') \\
&\qquad + f(z',y',y',z') + f(x',z',z',x') + f(y',x',x',y') \\
&\qquad + f(x',x',x',x') + f(y',y',y',y') + f(z',z',z',z')\big] \\
&= \sum_{j,k,m,n} (\delta_{mj}\delta_{nk} - \delta_{mk}\delta_{nj}) f(j,k,m,n).
\end{aligned}
$$

That is for the property that $\epsilon_{imn}\epsilon_{ijk} = \delta_{mj}\delta_{nk} - \delta_{mk}\delta_{nj}$. For the property that $\epsilon_{ijn}\epsilon_{ijk} = 2\delta_{nk}$, the corresponding calculation is

$$
\begin{aligned}
&\sum_{i,j,k,n} \epsilon_{ijn}\epsilon_{ijk} f(k,n) \\
&= \epsilon_{y'z'x'}\epsilon_{y'z'x'} f(x',x') + \epsilon_{z'y'x'}\epsilon_{z'y'x'} f(x',x') \\
&\quad + \epsilon_{z'x'y'}\epsilon_{z'x'y'} f(y',y') + \epsilon_{x'z'y'}\epsilon_{x'z'y'} f(y',y') \\
&\quad + \epsilon_{x'y'z'}\epsilon_{x'y'z'} f(z',z') + \epsilon_{y'x'z'}\epsilon_{y'x'z'} f(z',z') \\
&= f(x',x') + f(x',x') + f(y',y') + f(y',y') + f(z',z') + f(z',z') \\
&= 2\big[f(x',x') + f(y',y') + f(z',z')\big] \\
&= 2\sum_{k,n} \delta_{nk} f(k,n).
\end{aligned}
$$

For the property that $\epsilon_{ijk}\epsilon_{ijk} = 6$,

$$
\sum_{i,j,k} \epsilon_{ijk}\epsilon_{ijk} = \epsilon^2_{x'y'z'} + \epsilon^2_{y'z'x'} + \epsilon^2_{z'x'y'} + \epsilon^2_{x'z'y'} + \epsilon^2_{y'x'z'} + \epsilon^2_{z'y'x'} = 6.
$$

table's claim that $\epsilon_{imn}\epsilon_{ijk} = \delta_{mj}\delta_{nk} - \delta_{mk}\delta_{nj}$. (Notice that the compound Kronecker operator $\delta_{mj}\delta_{nk}$ includes nonzero terms for the case that $j = k = m = n = x'$, for the case that $j = k = m = n = y'$ and for the case that $j = k = m = n = z'$, whereas the compound Levi-Civita operator $\epsilon_{imn}\epsilon_{ijk}$ does not. However, the compound Kronecker operator $-\delta_{mk}\delta_{nj}$ includes canceling terms for these same three cases. This is why the table's claim is valid as written.)

To belabor the topic further here would serve little purpose. The reader who does not feel entirely sure that he understands what is going on might work out the table's several properties with his own pencil, in something like the style of the example, until he is satisfied that he adequately understands the several properties and their correct use.

Section 16.7 will refine the notation for use when derivatives with respect to angles come into play but, before leaving the present section, we might pause for a moment to appreciate (15.29) in the special case that $\mathbf{b} = \mathbf{c} = \hat{\mathbf{n}}$:

$$-\hat{\mathbf{n}} \times (\hat{\mathbf{n}} \times \mathbf{a}) = \mathbf{a} - \hat{\mathbf{n}}(\hat{\mathbf{n}} \cdot \mathbf{a}). \tag{15.30}$$

The difference $\mathbf{a} - \hat{\mathbf{n}}(\hat{\mathbf{n}} \cdot \mathbf{a})$ evidently projects a vector $\mathbf{a}$ onto the plane whose unit normal is $\hat{\mathbf{n}}$. Equation (15.30) reveals that the double cross product $-\hat{\mathbf{n}} \times (\hat{\mathbf{n}} \times \mathbf{a})$ projects the same vector onto the same plane. Figure 15.6 illustrates.

## 15.5 Algebraic identities

Vector algebra is not in principle very much harder than scalar algebra is, but with three distinct types of product it has more rules controlling the way its products and sums are combined. Table 15.2 lists several of these.[25,26] Most of the table's identities are plain by the formulas (15.9), (15.21) and (15.22) respectively for the scalar, dot and cross products, and two were proved as (15.29) and (15.30). The remaining identity is proved in the notation of § 15.4 as

$$\begin{aligned}
\epsilon_{ijk}c_i a_j b_k &= \epsilon_{ijk}c_i a_j b_k &&= \epsilon_{kij}c_k a_i b_j &&= \epsilon_{jki}c_j a_k b_i \\
&= \epsilon_{ijk}c_i a_j b_k &&= \epsilon_{ijk}a_i b_j c_k &&= \epsilon_{ijk}b_i c_j a_k \\
&= \mathbf{c} \cdot (\epsilon_{ijk}\hat{\mathbf{i}} a_j b_k) &&= \mathbf{a} \cdot (\epsilon_{ijk}\hat{\mathbf{i}} b_j c_k) &&= \mathbf{b} \cdot (\epsilon_{ijk}\hat{\mathbf{i}} c_j a_k).
\end{aligned}$$

It is precisely to encapsulate such interminable detail that we use the Kronecker delta, the Levi-Civita epsilon and the properties of Table 15.1.

[25] [60, Appendix II][28, Appendix A]

[26] Nothing in any of the table's identities requires the vectors involved to be real. The table is equally as valid when vectors are complex.

Figure 15.6: A vector projected onto a plane.

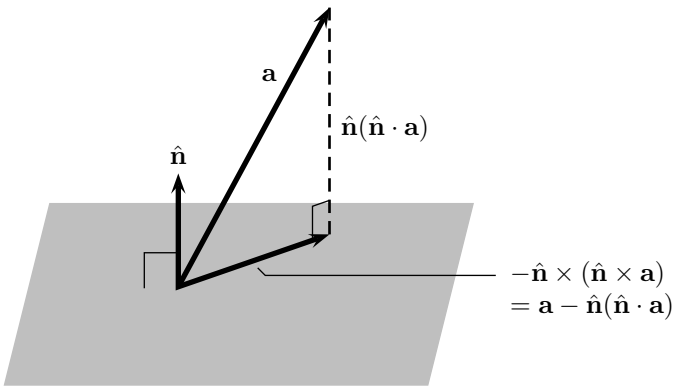



Table 15.2: Algebraic vector identities.

$$
\begin{array}{rclcrcl}
\psi\mathbf{a} & = & \hat{\mathbf{i}}\psi a_i & \qquad \mathbf{a}\cdot\mathbf{b} & \equiv & a_i b_i & \qquad \mathbf{a}\times\mathbf{b} \equiv \epsilon_{ijk}\hat{\mathbf{i}} a_j b_k \\
\mathbf{a}^*\cdot\mathbf{a} & = & |a|^2 & & & (\psi)(\mathbf{a}+\mathbf{b}) & = & \psi\mathbf{a}+\psi\mathbf{b} \\
\mathbf{b}\cdot\mathbf{a} & = & \mathbf{a}\cdot\mathbf{b} & & & \mathbf{b}\times\mathbf{a} & = & -\mathbf{a}\times\mathbf{b} \\
\mathbf{c}\cdot(\mathbf{a}+\mathbf{b}) & = & \mathbf{c}\cdot\mathbf{a}+\mathbf{c}\cdot\mathbf{b} & & & \mathbf{c}\times(\mathbf{a}+\mathbf{b}) & = & \mathbf{c}\times\mathbf{a}+\mathbf{c}\times\mathbf{b} \\
\mathbf{a}\cdot(\psi\mathbf{b}) & = & (\psi)(\mathbf{a}\cdot\mathbf{b}) & & & \mathbf{a}\times(\psi\mathbf{b}) & = & (\psi)(\mathbf{a}\times\mathbf{b})
\end{array}
$$

$$
\begin{aligned}
\mathbf{c}\cdot(\mathbf{a}\times\mathbf{b}) &= \mathbf{a}\cdot(\mathbf{b}\times\mathbf{c}) = \mathbf{b}\cdot(\mathbf{c}\times\mathbf{a}) \\
\mathbf{c}\times(\mathbf{a}\times\mathbf{b}) &= \mathbf{a}(\mathbf{c}\cdot\mathbf{b}) - \mathbf{b}(\mathbf{c}\cdot\mathbf{a}) \\
-\hat{\mathbf{n}}\times(\hat{\mathbf{n}}\times\mathbf{a}) &= \mathbf{a} - \hat{\mathbf{n}}(\hat{\mathbf{n}}\cdot\mathbf{a})
\end{aligned}
$$

That is,

$$\mathbf{c} \cdot (\mathbf{a} \times \mathbf{b}) = \mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = \mathbf{b} \cdot (\mathbf{c} \times \mathbf{a}). \tag{15.31}$$

Besides the several vector identities, the table also includes the three vector products in Einstein notation.[27]

Each definition and identity of Table 15.2 is invariant under reorientation of axes.

## 15.6 Isotropy

A real,[28] three-dimensional coordinate system[29] $(\alpha; \beta; \gamma)$ is *isotropic* at a point $\mathbf{r} = \mathbf{r}_1$ if and only if

$$\begin{aligned} \hat{\boldsymbol{\beta}}(\mathbf{r}_1) \cdot \hat{\boldsymbol{\gamma}}(\mathbf{r}_1) &= 0, \\ \hat{\boldsymbol{\gamma}}(\mathbf{r}_1) \cdot \hat{\boldsymbol{\alpha}}(\mathbf{r}_1) &= 0, \\ \hat{\boldsymbol{\alpha}}(\mathbf{r}_1) \cdot \hat{\boldsymbol{\beta}}(\mathbf{r}_1) &= 0, \end{aligned} \tag{15.32}$$

and

$$\left| \frac{\partial \mathbf{r}}{\partial \alpha} \right|_{\mathbf{r}=\mathbf{r}_1} = \left| \frac{\partial \mathbf{r}}{\partial \beta} \right|_{\mathbf{r}=\mathbf{r}_1} = \left| \frac{\partial \mathbf{r}}{\partial \gamma} \right|_{\mathbf{r}=\mathbf{r}_1}. \tag{15.33}$$

That is, a three-dimensional system is isotropic if its three coordinates advance locally at right angles to one another but at the same rate.

Of the three basic three-dimensional coordinate systems—indeed, of all the three-dimensional coordinate systems this book treats—only the rectangular is isotropic according to (15.32) and (15.33).[30] Isotropy admittedly would not be a very interesting property if that were all there were to it. However, there is also *two-dimensional isotropy,* more interesting because it arises oftener.

[27] If the reader's native language is English, then he is likely to have heard of the unfortunate "back cab rule," which actually is not a rule but an unhelpful mnemonic for one of Table 15.2's identities. The mnemonic is mildly orthographically clever but, when learned, significantly impedes real understanding of the vector. The writer recommends that the reader forget the rule if he has heard of it for, in mathematics, spelling-based mnemonics are seldom if ever a good idea.

[28] The reader is reminded that one can licitly express a complex vector in a real basis.

[29] This chapter's footnote 31 and Ch. 16's footnote 21 explain the usage of semicolons as coordinate delimiters.

[30] Whether it is even possible to construct an isotropic, nonrectangular coordinate system in three dimensions is a question we will leave to the professional mathematician. The author has not encountered such a system.

A real, two-dimensional coordinate system $(\alpha;\beta)$ is isotropic at a point $\boldsymbol{\rho}^\gamma = \boldsymbol{\rho}_1^\gamma$ if and only if

$$\hat{\boldsymbol{\alpha}}(\boldsymbol{\rho}_1^\gamma) \cdot \hat{\boldsymbol{\beta}}(\boldsymbol{\rho}_1^\gamma) = 0 \tag{15.34}$$

and

$$\left| \frac{\partial \boldsymbol{\rho}^\gamma}{\partial \alpha} \right|_{\boldsymbol{\rho}^\gamma = \boldsymbol{\rho}_1^\gamma} = \left| \frac{\partial \boldsymbol{\rho}^\gamma}{\partial \beta} \right|_{\boldsymbol{\rho}^\gamma = \boldsymbol{\rho}_1^\gamma}, \tag{15.35}$$

where $\boldsymbol{\rho}^\gamma = \hat{\boldsymbol{\alpha}}\alpha + \hat{\boldsymbol{\beta}}\beta$ represents position in the $\alpha$-$\beta$ plane. (If the $\alpha$-$\beta$ plane happens to be the $x$-$y$ plane, as is often the case, then $\boldsymbol{\rho}^\gamma = \boldsymbol{\rho}^z = \boldsymbol{\rho}$ and per eqn. 3.20 one can omit the superscript.) The two-dimensional rectangular system $(x, y)$ naturally is isotropic. Because $|\partial\boldsymbol{\rho}/\partial\phi| = (\rho)\,|\partial\boldsymbol{\rho}/\partial\rho|$ the standard two-dimensional cylindrical system $(\rho;\phi)$ as such is nonisotropic, but the change of coordinate

$$\lambda \equiv \ln \frac{\rho}{\rho_o}, \tag{15.36}$$

where $\rho_o$ is some arbitrarily chosen reference radius, converts the system straightforwardly into the *logarithmic cylindrical* system $(\lambda;\phi)$ which is isotropic everywhere in the plane except at the origin $\rho = 0$. Further two-dimensionally isotropic coordinate systems include the parabolic system of § 15.7.2, to follow.

## 15.7 Parabolic coordinates

Scientists and engineers find most spatial-geometrical problems they encounter in practice to fall into either of two categories. The first category comprises problems of simple geometry conforming to any one of the three basic coordinate systems—rectangular, cylindrical or spherical. The second category comprises problems of complicated geometry, analyzed in the rectangular system not because the problems' geometries fit that system but rather because they fit no system and thus give one little reason to depart from the rectangular. One however occasionally encounters problems of a third category, whose geometries are simple but, though simple, nevertheless fit none of the three basic coordinate systems. Then it may fall to the scientist or engineer to devise a special coordinate system congenial to the problem.

This section will treat the *parabolic* coordinate systems which, besides being arguably the most useful of the various special systems, serve as good examples of the kind of special system a scientist or engineer might be

called upon to devise. The two three-dimensional parabolic systems are the *parabolic cylindrical* system $(\sigma, \tau, z)$ of § 15.7.4 and the *circular paraboloidal* system[31] $(\eta; \phi, \xi)$ of § 15.7.5, where the angle $\phi$ and the length $z$ are familiar to us but $\sigma$, $\tau$, $\eta$ and $\xi$—neither angles nor lengths but root-lengths (that is, coordinates having dimensions of [length]$^{1/2}$)—are new.[32] Both three-dimensional parabolic systems derive from the two-dimensional parabolic system $(\sigma, \tau)$ of § 15.7.2.[33]

However, before handling any parabolic system we ought formally to introduce the parabola itself, next.

### 15.7.1 The parabola

Parabolic coordinates are based on a useful geometrical curve called the *parabola,* which many or most readers will have met long before opening this book's covers. The parabola, simple but less obvious than the circle, may however not be equally well known to all readers, and even readers already acquainted with it might appreciate a reëxamination. This subsection reviews the parabola.

*Given a point, called the* focus, *and a line, called the* directrix,[34] *plus the plane in which the focus and the directrix both lie, the associated* parabola *is that curve which lies in the plane everywhere equidistant from both focus and directrix.*[35] See Fig. 15.7.

Referring to the figure, if rectangular coordinates are established such that $\hat{\mathbf{x}}$ and $\hat{\mathbf{y}}$ lie in the plane, that the parabola's focus lies at $(x, y) = (0, k)$, and that the equation $y = k - \sigma^2$ describes the parabola's directrix, then the equation

$$x^2 + (y - k)^2 = (y - k + \sigma^2)^2$$

[31] The reader probably will think nothing of it now, but later may wonder why the circular paraboloidal coordinates are $(\eta; \phi, \xi)$ rather than $(\xi; \phi, \eta)$ or $(\eta, \xi; \phi)$. The peculiar ordering is to honor the right-hand rule (§ 3.3 and eqn. 15.19), since $\hat{\boldsymbol{\eta}} \times \hat{\boldsymbol{\xi}} = -\hat{\boldsymbol{\phi}}$ rather than $+\hat{\boldsymbol{\phi}}$. See § 15.7.5. (Regarding the semicolon ";" delimiter, it doesn't mean much. This book arbitrarily uses a semicolon when the following coordinate happens to be an angle, which helps to distinguish rectangular coordinates from cylindrical from spherical. Admittedly, such a notational convention ceases to help much when parabolic coordinates arrive, but we will continue to use it for inertia's sake. See also Ch. 16's footnote 21.)

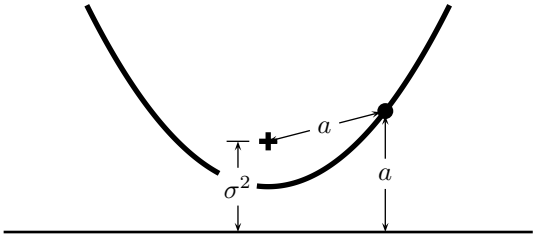
[32] The letters $\sigma$, $\tau$, $\eta$ and $\xi$ are available letters this section happens to use, not necessarily standard parabolic symbols. See Appendix B.

[33] [66, "Parabolic coordinates," 09:59, 19 July 2008]

[34] Whether the parabola's definition ought to forbid the directrix to pass through the focus is a stylistic question this book will leave unanswered.

[35] [55, § 12-1]

Figure 15.7: The parabola.




evidently expresses the equidistance rule of the parabola's definition. Solving for $y - k$ and then, from that solution, for $y$, we have that

$$y = \frac{x^2}{2\sigma^2} + \left(k - \frac{\sigma^2}{2}\right). \tag{15.37}$$

With the definitions that

$$\begin{aligned} \mu &\equiv \frac{1}{2\sigma^2}, \\ \kappa &\equiv k - \frac{\sigma^2}{2}, \end{aligned} \tag{15.38}$$

given which

$$\begin{aligned} \sigma^2 &= \frac{1}{2\mu}, \\ k &= \kappa + \frac{1}{4\mu}, \end{aligned} \tag{15.39}$$

eqn. (15.37) becomes

$$y = \mu x^2 + \kappa. \tag{15.40}$$

Equations fitting the general form (15.40) arise extremely commonly in applications. To choose a particularly famous example, the equation that describes a projectile's flight in the absence of air resistance fits the form. Any equation that fits the form can be plotted as a parabola, which for example is why projectiles fly in parabolic arcs.

Observe that the parabola's definition does not actually require the directrix to be $\hat{\mathbf{y}}$-oriented: the directrix can be $\hat{\mathbf{x}}$-oriented or, indeed, oriented

any way (though naturally in that case eqns. 15.37 and 15.40 would have to be modified). Observe also the geometrical fact that *the parabola's track necessarily bisects the angle between the two line segments labeled "a" in Fig. 15.7.* One of the consequences of this geometrical fact—a fact it seems better to let the reader visualize and ponder than to try to justify in so many words[36]—is that a parabolic mirror reflects precisely[37] toward its focus all light rays that arrive perpendicularly to its directrix (which for instance is why satellite dish antennas have parabolic cross-sections).

### 15.7.2 Parabolic coordinates in two dimensions

Parabolic coordinates are most easily first explained in the two-dimensional case that $z = 0$. In two dimensions, the parabolic coordinates $(\sigma, \tau)$ represent the point in the $x$-$y$ plane that lies equidistant

- from the line $y = -\sigma^2$,
- from the line $y = +\tau^2$, and
- from the point $\rho = 0$,

where the parameter $k$ of § 15.7.1 has been set to $k = 0$. Figure 15.8 depicts the construction described. In the figure are two dotted curves, one of which represents the point's parabolic track if $\sigma$ were varied while $\tau$ were held constant and the other of which represents the point's parabolic track if $\tau$ were varied while $\sigma$ were held constant. Observe according to § 15.7.1's bisection finding that each parabola necessarily bisects the angle between two of the three line segments labeled $a$ in the figure. Observe further that the two angles' sum is the straight angle $2\pi/2$, from which one can

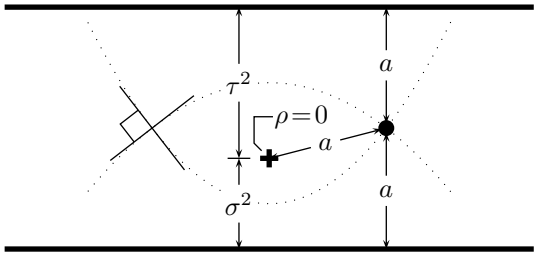
[36] If it helps nevertheless, some words: Consider that the two line segments labeled $a$ in the figure run in the directions of increasing distance respectively from the focus and from the directrix. If you want to draw away from the directrix at the same rate as you draw away from the focus, thus maintaining equal distances, then your track cannot but exactly bisect the angle between the two segments.

Once you grasp the idea, the bisection is obvious, though to grasp the idea can take some thought.

*To bisect* a thing, incidentally—if the context has not already made the meaning plain—is to divide the thing at its middle into two equal parts.

[37] Well, actually, physically, the ray model of light implied here is valid only insofar as $\lambda \ll \sigma^2$, where $\lambda$ represents the light's characteristic wavelength. Also, regardless of $\lambda$, the ray model breaks down in the immediate neighborhood of the mirror's focus. Such wave-mechanical considerations are confined to a footnote not because they were untrue but rather because they do not concern the present geometrical discussion. Insofar as rays are concerned, the focusing is precise.

Figure 15.8: Locating a point in two dimensions by parabolic construction.




conclude, significantly, that *the two parabolas cross precisely at right angles to one another.*

Figure 15.9 lays out the parabolic coordinate grid. Notice in the figure that one of the grid's several cells is subdivided at its quarter-marks for illustration's sake, to show how one can subgrid at need to locate points like, for example, $(\sigma, \tau) = (\frac{7}{2}, -\frac{9}{4})$ visually. (That the subgrid's cells approach square shape implies that the parabolic system is isotropic, a significant fact § 15.7.3 will demonstrate formally.)

Using the Pythagorean theorem, one can symbolically express the equidistant construction rule above as

$$\begin{aligned} a &= \sigma^2 + y = \tau^2 - y, \\ a^2 &= \rho^2 = x^2 + y^2. \end{aligned} \tag{15.41}$$

From the first line of (15.41),

$$y = \frac{\tau^2 - \sigma^2}{2}. \tag{15.42}$$

On the other hand, combining the two lines of (15.41),

$$(\sigma^2 + y)^2 = x^2 + y^2 = (\tau^2 - y)^2,$$

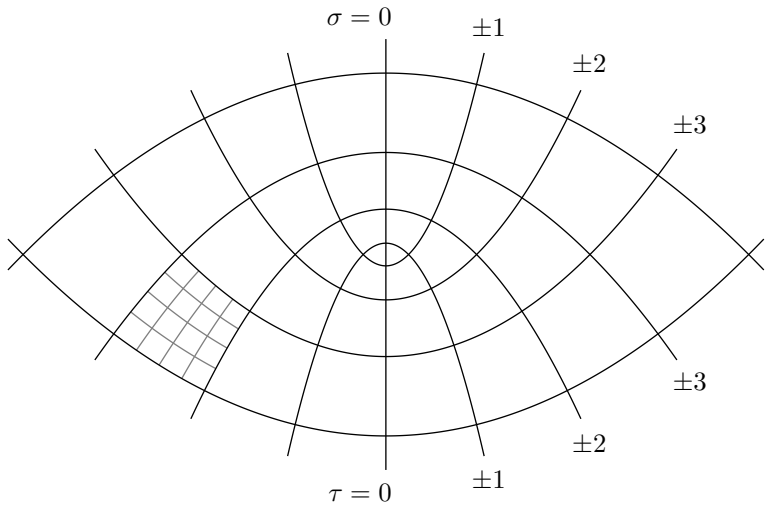
or, subtracting $y^2$,

$$\sigma^4 + 2\sigma^2 y = x^2 = \tau^4 - 2\tau^2 y.$$

Substituting (15.42)'s expression for $y$,

$$x^2 = (\sigma\tau)^2.$$

Figure 15.9: The parabolic coordinate grid in two dimensions.




That either $x = +\sigma\tau$ or $x = -\sigma\tau$ would satisfy this equation. Arbitrarily choosing the + sign gives us that

$$x = \sigma\tau. \tag{15.43}$$

Also, since $\rho^2 = x^2 + y^2$, (15.42) and (15.43) together imply that

$$\rho = \frac{\tau^2 + \sigma^2}{2}. \tag{15.44}$$

Combining (15.42) and (15.44) to isolate $\sigma^2$ and $\tau^2$ yields

$$\begin{aligned} \sigma^2 &= \rho - y, \\ \tau^2 &= \rho + y. \end{aligned} \tag{15.45}$$

### 15.7.3 Properties

The derivatives of (15.43), (15.42) and (15.44) are

$$\begin{aligned} dx &= \sigma\, d\tau + \tau\, d\sigma, \\ dy &= \tau\, d\tau - \sigma\, d\sigma, \\ d\rho &= \tau\, d\tau + \sigma\, d\sigma. \end{aligned} \tag{15.46}$$

Solving the first two lines of (15.46) simultaneously for $d\sigma$ and $d\tau$ and then collapsing the resultant subexpression $\tau^2 + \sigma^2$ per (15.44) yields

$$\begin{aligned} d\sigma &= \frac{\tau\,dx - \sigma\,dy}{2\rho}, \\ d\tau &= \frac{\sigma\,dx + \tau\,dy}{2\rho}, \end{aligned} \tag{15.47}$$

from which it is apparent that

$$\begin{aligned} \hat{\boldsymbol{\sigma}} &= \frac{\hat{\mathbf{x}}\tau - \hat{\mathbf{y}}\sigma}{\sqrt{\tau^2 + \sigma^2}}, \\ \hat{\boldsymbol{\tau}} &= \frac{\hat{\mathbf{x}}\sigma + \hat{\mathbf{y}}\tau}{\sqrt{\tau^2 + \sigma^2}}; \end{aligned}$$

or, collapsing again per (15.44), that

$$\begin{aligned} \hat{\boldsymbol{\sigma}} &= \frac{\hat{\mathbf{x}}\tau - \hat{\mathbf{y}}\sigma}{\sqrt{2\rho}}, \\ \hat{\boldsymbol{\tau}} &= \frac{\hat{\mathbf{x}}\sigma + \hat{\mathbf{y}}\tau}{\sqrt{2\rho}}, \end{aligned} \tag{15.48}$$

of which the dot product

$$\hat{\boldsymbol{\sigma}} \cdot \hat{\boldsymbol{\tau}} = 0 \text{ if } \rho \neq 0 \tag{15.49}$$

is null, confirming our earlier finding that the various grid parabolas cross always at right angles to one another. Solving (15.48) simultaneously for $\hat{\mathbf{x}}$ and $\hat{\mathbf{y}}$ then produces

$$\begin{aligned} \hat{\mathbf{x}} &= \frac{\hat{\boldsymbol{\tau}}\sigma + \hat{\boldsymbol{\sigma}}\tau}{\sqrt{2\rho}}, \\ \hat{\mathbf{y}} &= \frac{\hat{\boldsymbol{\tau}}\tau - \hat{\boldsymbol{\sigma}}\sigma}{\sqrt{2\rho}}. \end{aligned} \tag{15.50}$$

One can express an infinitesimal change in position in the plane as

$$\begin{aligned} d\boldsymbol{\rho} &= \hat{\mathbf{x}}\,dx + \hat{\mathbf{y}}\,dy \\ &= \hat{\mathbf{x}}(\sigma\,d\tau + \tau\,d\sigma) + \hat{\mathbf{y}}(\tau\,d\tau - \sigma\,d\sigma) \\ &= (\hat{\mathbf{x}}\tau - \hat{\mathbf{y}}\sigma)\,d\sigma + (\hat{\mathbf{x}}\sigma + \hat{\mathbf{y}}\tau)\,d\tau, \end{aligned}$$

in which (15.46) has expanded the differentials and from which

$$\begin{aligned} \frac{\partial \boldsymbol{\rho}}{\partial \sigma} &= \hat{\mathbf{x}}\tau - \hat{\mathbf{y}}\sigma, \\ \frac{\partial \boldsymbol{\rho}}{\partial \tau} &= \hat{\mathbf{x}}\sigma + \hat{\mathbf{y}}\tau, \end{aligned}$$

Table 15.3: Parabolic coordinate properties.

$$
\begin{aligned}
x &= \sigma\tau \\
y &= \frac{\tau^2 - \sigma^2}{2} \\
\rho &= \frac{\tau^2 + \sigma^2}{2} \\
\rho^2 &= x^2 + y^2 \\
\sigma^2 &= \rho - y \\
\tau^2 &= \rho + y
\end{aligned}
\qquad\qquad
\begin{aligned}
\hat{\mathbf{x}} &= \frac{\hat{\boldsymbol{\tau}}\sigma + \hat{\boldsymbol{\sigma}}\tau}{\sqrt{2\rho}} \\
\hat{\mathbf{y}} &= \frac{\hat{\boldsymbol{\tau}}\tau - \hat{\boldsymbol{\sigma}}\sigma}{\sqrt{2\rho}} \\
\hat{\boldsymbol{\sigma}} &= \frac{\hat{\mathbf{x}}\tau - \hat{\mathbf{y}}\sigma}{\sqrt{2\rho}} \\
\hat{\boldsymbol{\tau}} &= \frac{\hat{\mathbf{x}}\sigma + \hat{\mathbf{y}}\tau}{\sqrt{2\rho}} \\
\hat{\boldsymbol{\sigma}} \times \hat{\boldsymbol{\tau}} &= \hat{\mathbf{z}} \\
\hat{\boldsymbol{\sigma}} \cdot \hat{\boldsymbol{\tau}} &= 0 \\
\left|\frac{\partial \boldsymbol{\rho}}{\partial \sigma}\right| &= \left|\frac{\partial \boldsymbol{\rho}}{\partial \tau}\right|
\end{aligned}
$$

and thus

$$\left|\frac{\partial \boldsymbol{\rho}}{\partial \sigma}\right| = \left|\frac{\partial \boldsymbol{\rho}}{\partial \tau}\right|. \tag{15.51}$$

Equations (15.49) and (15.51) respectively meet the requirements (15.34) and (15.35), implying that *the two-dimensional parabolic coordinate system is isotropic* except at $\rho = 0$.

Table 15.3 summarizes, gathering parabolic coordinate properties from this subsection and § 15.7.2.

### 15.7.4 The parabolic cylindrical coordinate system

Two-dimensional parabolic coordinates are trivially extended to three dimensions by adding a $z$ coordinate, thus constituting the *parabolic cylindrical* coordinate system $(\sigma, \tau, z)$. The surfaces of constant $\sigma$ and of constant $\tau$ in this system are *parabolic cylinders* (and the surfaces of constant $z$ naturally are planes). All the properties of Table 15.3 apply. Observe however that the system is isotropic only in two dimensions not three.

The orthogonal parabolic cylindrical basis is $[\hat{\boldsymbol{\sigma}}\ \hat{\boldsymbol{\tau}}\ \hat{\mathbf{z}}]$.

Table 15.4: Circular paraboloidal coordinate properties.

$$
\begin{aligned}
\rho &= \eta\xi \\
z &= \frac{\xi^2 - \eta^2}{2} \\
r &= \frac{\xi^2 + \eta^2}{2} \\
r^2 &= \rho^2 + z^2 = x^2 + y^2 + z^2 \\
\eta^2 &= r - z \\
\xi^2 &= r + z
\end{aligned}
\qquad\qquad
\begin{aligned}
\hat{\boldsymbol{\rho}} &= \frac{\hat{\boldsymbol{\xi}}\eta + \hat{\boldsymbol{\eta}}\xi}{\sqrt{2r}} \\
\hat{\mathbf{z}} &= \frac{\hat{\boldsymbol{\xi}}\xi - \hat{\boldsymbol{\eta}}\eta}{\sqrt{2r}} \\
\hat{\boldsymbol{\eta}} &= \frac{\hat{\boldsymbol{\rho}}\xi - \hat{\mathbf{z}}\eta}{\sqrt{2r}} \\
\hat{\boldsymbol{\xi}} &= \frac{\hat{\boldsymbol{\rho}}\eta + \hat{\mathbf{z}}\xi}{\sqrt{2r}} \\
\hat{\boldsymbol{\eta}} \times \hat{\boldsymbol{\xi}} &= -\hat{\boldsymbol{\phi}} \\
\hat{\boldsymbol{\eta}} \cdot \hat{\boldsymbol{\xi}} &= 0 \\
\left|\frac{\partial \mathbf{r}}{\partial \eta}\right| &= \left|\frac{\partial \mathbf{r}}{\partial \xi}\right|
\end{aligned}
$$

### 15.7.5 The circular paraboloidal coordinate system

Sometimes one would like to extend the parabolic system to three dimensions by adding an azimuth $\phi$ rather than a height $z$. This is possible, but then one tends to prefer the parabolas, foci and directrices of Figs. 15.8 and 15.9 to run in the $\rho$-$z$ plane rather than in the $x$-$y$. Therefore, one defines the coordinates $\eta$ and $\xi$ to represent in the $\rho$-$z$ plane what the letters $\sigma$ and $\tau$ have represented in the $x$-$y$. The properties of Table 15.4 result, which are just the properties of Table 15.3 with coordinates changed. The system is the *circular paraboloidal system* $(\eta; \phi, \xi)$.

The surfaces of constant $\eta$ and of constant $\xi$ in the circular paraboloidal system are *paraboloids,* parabolas rotated about the $z$ axis (and the surfaces of constant $\phi$ are planes, or half planes if you like, just as in the cylindrical system). Like the parabolic cylindrical system, the circular paraboloidal system too is isotropic in two dimensions.

Notice that, given the usual definition of the $\hat{\boldsymbol{\phi}}$ unit basis vector, $\hat{\boldsymbol{\eta}} \times \hat{\boldsymbol{\xi}} = -\hat{\boldsymbol{\phi}}$ rather than $+\hat{\boldsymbol{\phi}}$ as one might first guess. The correct, right-handed sequence of the orthogonal circular paraboloidal basis therefore would be

$[\hat{\boldsymbol{\eta}}\ \hat{\boldsymbol{\phi}}\ \hat{\boldsymbol{\xi}}]$.[38]

This concludes the present chapter on the algebra of vector analysis. Chapter 16, next, will venture hence into the larger and even more interesting realm of vector calculus.

[38] See footnote 31.

# Chapter 16

# Vector calculus

Chapter 15 has introduced the algebra of the three-dimensional geometrical vector. Like the scalar, the vector is a continuous quantity and as such has not only an algebra but also a calculus. This chapter develops the calculus of the vector.

## 16.1 Fields and their derivatives

A scalar quantity $\sigma(t)$ or vector quantity $\mathbf{f}(t)$ whose value varies over time is "a function of time $t$." We can likewise call a scalar quantity[1] $\psi(\mathbf{r})$ or vector quantity $\mathbf{a}(\mathbf{r})$ whose value varies over space "a function of position $\mathbf{r}$," but there is a special, alternate name for such a quantity. We call it a *field.*

A field is a quantity distributed over space or, if you prefer, a function in which spatial position serves as independent variable. Air pressure $p(\mathbf{r})$ is an example of a *scalar field,* whose value at a given location $\mathbf{r}$ has amplitude but no direction. Wind velocity[2] $\mathbf{q}(\mathbf{r})$ is an example of a *vector field,* whose value at a given location $\mathbf{r}$ has both amplitude and direction. These are typical examples. Tactically, a vector field can be thought of as composed of three scalar fields

$$\mathbf{q}(\mathbf{r}) = \hat{\mathbf{x}}q_x(\mathbf{r}) + \hat{\mathbf{y}}q_y(\mathbf{r}) + \hat{\mathbf{z}}q_z(\mathbf{r});$$

[1] This $\psi(\mathbf{r})$ is unrelated to the Tait-Bryan and Euler roll angles $\psi$ of § 15.1, an unfortunate but tolerable overloading of the Greek letter $\psi$ in the conventional notation of vector analysis. In the unlikely event of confusion, you can use an alternate letter like $\eta$ for the roll angle. See Appendix B.

[2] As § 15.3, this section also uses the letter $q$ for velocity in place of the conventional $v$ [7, § 18.4], which it needs for another purpose.

but, since

$$\mathbf{q}(\mathbf{r}) = \hat{\mathbf{x}}'q_{x'}(\mathbf{r}) + \hat{\mathbf{y}}'q_{y'}(\mathbf{r}) + \hat{\mathbf{z}}'q_{z'}(\mathbf{r})$$

for any orthogonal basis $[\mathbf{x}'\ \mathbf{y}'\ \mathbf{z}']$ as well, the specific scalar fields $q_x(\mathbf{r})$, $q_y(\mathbf{r})$ and $q_z(\mathbf{r})$ are no more essential to the vector field $\mathbf{q}(\mathbf{r})$ than the specific scalars $b_x$, $b_y$ and $b_z$ are to a vector $\mathbf{b}$. As we said, the three components come tactically; typically, such components are uninteresting in themselves. The field $\mathbf{q}(\mathbf{r})$ as a whole is the interesting thing.

Scalar and vector fields are of utmost use in the modeling of physical phenomena.

As one can take the derivative $d\sigma/dt$ or $d\mathbf{f}/dt$ with respect to time $t$ of a function $\sigma(t)$ or $\mathbf{f}(t)$, one can likewise take the derivative with respect to position $\mathbf{r}$ of a field $\psi(\mathbf{r})$ or $\mathbf{a}(\mathbf{r})$. However, derivatives with respect to position create a notational problem, for it is not obvious what symbology like $d\psi/d\mathbf{r}$ or $d\mathbf{a}/d\mathbf{r}$ would actually mean. The notation $d\sigma/dt$ means "the rate of $\sigma$ as time $t$ advances," but if the notation $d\psi/d\mathbf{r}$ likewise meant "the rate of $\psi$ as position $\mathbf{r}$ advances" then it would necessarily prompt one to ask, "advances in which direction?" The notation offers no hint. In fact $d\psi/d\mathbf{r}$ and $d\mathbf{a}/d\mathbf{r}$ mean nothing very distinct in most contexts and we shall avoid such notation. If we will speak of a field's derivative with respect to position $\mathbf{r}$ then we shall be more precise.

Section 15.2 has given the vector three distinct kinds of product. This section gives the field no fewer than four distinct kinds of derivative: the directional derivative; the gradient; the divergence; and the curl.[3]

So many derivatives bring the student a conceptual difficulty one could call "the caveman problem." Imagine a caveman. Suppose that you tried to describe to the caveman a house or building of more than one floor. He might not understand. You and I who already grasp the concept of upstairs and downstairs do not find a building of two floors, or three or even thirty, especially hard to envision, but our caveman is used to thinking of the ground and the floor as more or less the same thing. To try to think of upstairs and downstairs might confuse him with partly false images of sitting in a tree or of clambering onto (and breaking) the roof of his hut. "There are many trees and antelopes but only one sky and floor. How can one speak of many skies or many floors?" The student's principal conceptual difficulty with the several vector derivatives is of this kind.

[3]Vector veterans may notice that the Laplacian is not listed. This is not because the Laplacian were uninteresting but rather because the Laplacian is actually a second-order derivative—a derivative of a derivative. We will address the Laplacian in § 16.4.

### 16.1.1 The $\nabla$ operator

Consider a vector

$$\mathbf{a} = \hat{\mathbf{x}}a_x + \hat{\mathbf{y}}a_y + \hat{\mathbf{z}}a_z.$$

Then consider a "vector"

$$\mathbf{c} = \hat{\mathbf{x}}[\text{Tuesday}] + \hat{\mathbf{y}}[\text{Wednesday}] + \hat{\mathbf{z}}[\text{Thursday}].$$

If you think that the latter does not look very much like a vector, then the writer thinks as you do, but consider:

$$\mathbf{c} \cdot \mathbf{a} = [\text{Tuesday}]a_x + [\text{Wednesday}]a_y + [\text{Thursday}]a_z.$$

The writer does not know how to interpret a nonsensical term like "[Tuesday]$a_x$" any more than the reader does, but the point is that $\mathbf{c}$ behaves as though it were a vector insofar as vector operations like the dot product are concerned. What matters in this context is not that $\mathbf{c}$ have amplitude and direction (it has neither) but rather that it have the three orthonormal components it needs to participate formally in relevant vector operations. It has these. That the components' amplitudes seem nonsensical is beside the point. Maybe there exists a model in which "[Tuesday]" knows how to operate on a scalar like $a_x$. (Operate on? Yes. Nothing in the dot product's definition requires the component amplitudes of $\mathbf{c}$ *to multiply* those of $\mathbf{a}$. Multiplication is what the component amplitudes of true vectors do, but $\mathbf{c}$ is not a true vector, so "[Tuesday]" might do something to $a_x$ other than to multiply it. Section 16.1.2 elaborates the point.) If there did exist such a model, then the dot product $\mathbf{c} \cdot \mathbf{a}$ could be licit in that model. As if this were not enough, the cross product $\mathbf{c} \times \mathbf{a}$ too could be licit in that model, composed according to the usual rule for cross products. The model might allow it. The dot and cross products in and of themselves do not forbid it.

Now consider a "vector"

$$\nabla = \hat{\mathbf{x}}\frac{\partial}{\partial x} + \hat{\mathbf{y}}\frac{\partial}{\partial y} + \hat{\mathbf{z}}\frac{\partial}{\partial z}. \tag{16.1}$$

This $\nabla$ is not a true vector any more than $\mathbf{c}$ is, maybe, but if we treat it as one then we have that

$$\nabla \cdot \mathbf{a} = \frac{\partial a_x}{\partial x} + \frac{\partial a_y}{\partial y} + \frac{\partial a_z}{\partial z}.$$

Such a dot product might or might not prove useful; but, unlike the terms in the earlier dot product, at least we know what this one's terms mean.

Well, days of the week, partial derivatives, ersatz vectors—it all seems rather abstract. What's the point? The answer is that there wouldn't be any point if the only nonvector "vectors" in question were of $\mathbf{c}$'s nonsensical kind. The operator $\nabla$ however shares more in common with a true vector than merely having $x$, $y$ and $z$ components; for, like a true vector, the operator $\nabla$ is amenable to having its axes reoriented by (15.1), (15.2), (15.7) and (15.8). This is easier to see at first with respect the true vector $\mathbf{a}$, as follows. Consider rotating the $x$ and $y$ axes through an angle $\phi$ about the $z$ axis. There ensues

$$\begin{aligned}
\mathbf{a} &= \hat{\mathbf{x}}a_x + \hat{\mathbf{y}}a_y + \hat{\mathbf{z}}a_z \\
&= (\hat{\mathbf{x}}' \cos\phi - \hat{\mathbf{y}}' \sin\phi)(a_{x'} \cos\phi - a_{y'} \sin\phi) \\
&\quad + (\hat{\mathbf{x}}' \sin\phi + \hat{\mathbf{y}}' \cos\phi)(a_{x'} \sin\phi + a_{y'} \cos\phi) + \hat{\mathbf{z}}'a_{z'} \\
&= \hat{\mathbf{x}}'[a_{x'} \cos^2\phi - a_{y'} \cos\phi \sin\phi + a_{x'} \sin^2\phi + a_{y'} \cos\phi \sin\phi] \\
&\quad + \hat{\mathbf{y}}'[-a_{x'} \cos\phi \sin\phi + a_{y'} \sin^2\phi + a_{x'} \cos\phi \sin\phi + a_{y'} \cos^2\phi] \\
&\quad + \hat{\mathbf{z}}'a_{z'} \\
&= \hat{\mathbf{x}}'a_{x'} + \hat{\mathbf{y}}'a_{y'} + \hat{\mathbf{z}}'a_{z'},
\end{aligned}$$

where the final expression has different axes than the original but, relative to those axes, exactly the same form. Further rotation about other axes would further reorient but naturally also would not alter the form. Now consider $\nabla$. The partial differential operators $\partial/\partial x$, $\partial/\partial y$ and $\partial/\partial z$ change no differently under reorientation than the component amplitudes $a_x$, $a_y$ and $a_z$ do. Hence,

$$\nabla = \hat{\mathbf{i}}\frac{\partial}{\partial i} = \hat{\mathbf{x}}'\frac{\partial}{\partial x'} + \hat{\mathbf{y}}'\frac{\partial}{\partial y'} + \hat{\mathbf{z}}'\frac{\partial}{\partial z'}, \tag{16.2}$$

evidently the same operator regardless of the choice of basis $[\hat{\mathbf{x}}'\ \hat{\mathbf{y}}'\ \hat{\mathbf{z}}']$. It is this invariance under reorientation that makes the $\nabla$ operator useful.

If $\nabla$ takes the place of the ambiguous $d/d\mathbf{r}$, then what takes the place of the ambiguous $d/d\mathbf{r}'$, $d/d\mathbf{r}_o$, $d/d\tilde{\mathbf{r}}$, $d/d\mathbf{r}^\dagger$ and so on? Answer: $\nabla'$, $\nabla_o$, $\tilde{\nabla}$, $\nabla^\dagger$ and so on. Whatever mark distinguishes the special $\mathbf{r}$, the same mark distinguishes the corresponding special $\nabla$. For example, where $\mathbf{r}_o = \hat{\mathbf{i}}i_o$, there $\nabla_o = \hat{\mathbf{i}}\,\partial/\partial i_o$. That is the convention.[4]

Introduced by Oliver Heaviside, informally pronounced "del" (in the author's country at least), the vector differential operator $\nabla$ finds extensive use in the modeling of physical phenomena. After a brief digression to

[4]A few readers not fully conversant with the material of Ch. 15, to whom this chapter had been making sense until the last two sentences, may suddenly find the notation

discuss operator notation, the subsections that follow will use the operator to develop and present the four basic kinds of vector derivative.

### 16.1.2 Operator notation

Section 16.1.1 has introduced *operator notation* without explaining what it is or even what it concerns. This subsection digresses to explain.

Operator notation concerns the representation of unary operators and the operations they specify. Section 7.3 has already broadly introduced the notion of the operator. A *unary operator* is a mathematical agent that transforms a single discrete quantity, a single distributed quantity, a single field, a single function or another single mathematical object in some definite way. For example, $J \equiv \int_0^t dt$ is a unary operator, more fully written $J \equiv \int_0^t \cdot\, dt$ where the "$\cdot$" holds the place of the thing operated upon,[5] whose effect is such that, for instance, $Jt = t^2/2$ and $J \cos \omega t = (\sin \omega t)/\omega$. Any letter might serve as well as the example's $J$; but what distinguishes operator notation is that, like the matrix row operator $A$ in matrix notation's product $A\mathbf{x}$ (§ 11.1.1), the operator $J$ in operator notation's operation $Jt$ attacks from the left. Thus, generally, $Jt \neq tJ$ if $J$ is a unary operator, though the notation $Jt$ usually formally resembles multiplication in other respects as we shall see.

The matrix actually is a type of unary operator and matrix notation is a specialization of operator notation, so we have met operator notation before. And, in fact, we have met operator notation much earlier than that. The product $5t$ can if you like be regarded as the unary operator "5 times," operating on $t$. Perhaps you did not know that 5 was an operator—and, indeed, the scalar 5 itself is no operator but just a number—but where no other operation is defined operator notation implies scalar multiplication by default. Seen in this way, $5t$ and $t5$ actually mean two different things; though naturally in the specific case of scalar multiplication, which happens to be commutative, it is true that $5t = t5$.

The $\mathbf{a}\cdot$ in the dot product $\mathbf{a} \cdot \mathbf{b}$ and the $\mathbf{a}\times$ in the cross product $\mathbf{a} \times \mathbf{b}$ can profitably be regarded as unary operators.

Whether operator notation can licitly represent any unary operation

---

incomprehensible. The notation is Einstein's. It means

$$\hat{\mathbf{i}} i_o = \sum_{i=x',y',z'} \hat{\mathbf{i}} i_o = \hat{\mathbf{x}}' x'_o + \hat{\mathbf{y}}' y'_o + \hat{\mathbf{z}}' z'_o,$$

in the leftmost form of which the summation sign is implied not written. Refer to § 15.4.

[5] [10]

whatsoever is a definitional question we will leave for the professional mathematician to answer, but in normal usage operator notation represents only *linear unary operations,* unary operations that honor § 7.3.3's rule of linearity. The operators $J$ and $A$ above are examples of linear unary operators; the operator $K \equiv \cdot + 3$ is not linear and almost certainly should never be represented in operator notation as here, lest an expression like $Kt$ mislead an understandably unsuspecting audience. Linear unary operators often do not commute, so $J_1 J_2 \neq J_2 J_1$ generally; but otherwise linear unary operators follow familiar rules of multiplication like $(J_2 + J_3)J_1 = J_2 J_1 + J_3 J_1$. Linear unary operators obey a definite algebra, the same algebra matrices obey. It is precisely this algebra that makes operator notation so useful.

Operators associate from right to left (§ 2.1.1) so that, in operator notation, $J\omega t = J(\omega t)$, not $(J\omega)t$. Observe however that the perceived need for parentheses comes only of the purely notational ambiguity as to whether $\omega t$ bears the semantics of "the product of $\omega$ and $t$" or those of "the unary operator '$\omega$ times,' operating on $t$." The perceived need and any associated confusion would vanish if $\Omega \equiv (\omega)(\cdot)$ were unambiguously an operator, in which case the product $J\Omega$ would itself be an operator, whereupon $(J\Omega)t = J\Omega t = J(\Omega t) = J(\omega t)$. Indeed, one can compare the distinction in § 11.3.2 between $\lambda$ and $\lambda I$ against the distinction between $\omega$ and $\Omega$ here, for a linear unary operator enjoys the same associativity (11.5) a matrix enjoys, and for the same reason. Still, rather than go to the trouble of defining extra symbols like $\Omega$, it is usually easier just to write the parentheses, which take little space on the page and are universally understood; or, better, to rely on the right-to-left convention that $J\omega t = J(\omega t)$. Modern conventional applied mathematical notation though generally excellent remains imperfect; so, notationally, when it matters, operators associate from right to left except where parentheses group otherwise.

One can speak of a unary operator like $J$, $A$ or $\Omega$ without giving it anything in particular to operate upon. One can leave an operation unresolved. For example, $tJ$ is itself a unary operator—it is the operator $t \int_0^t dt$—though one can assign no particular value to it until it actually operates on something. The operator $\nabla$ of (16.2) is an unresolved unary operator of the same kind.

### 16.1.3 The directional derivative and the gradient

In the calculus of vector fields, the derivative notation $d/d\mathbf{r}$ is ambiguous because, as the section's introduction has observed, the notation gives $\mathbf{r}$ no specific direction in which to advance. In operator notation, however,

given (16.2) and accorded a reference vector $\mathbf{b}$ to supply a direction and a scale, one can compose the *directional derivative* operator

$$(\mathbf{b} \cdot \nabla) = b_i \frac{\partial}{\partial i} \tag{16.3}$$

to express the derivative unambiguously. This operator applies equally to the scalar field,

$$(\mathbf{b} \cdot \nabla)\psi(\mathbf{r}) = b_i \frac{\partial \psi}{\partial i},$$

as to the vector field,

$$(\mathbf{b} \cdot \nabla)\mathbf{a}(\mathbf{r}) = b_i \frac{\partial \mathbf{a}}{\partial i} = \hat{\mathbf{j}} b_i \frac{\partial a_j}{\partial i}. \tag{16.4}$$

For the scalar field the parentheses are unnecessary and conventionally are omitted, as

$$\mathbf{b} \cdot \nabla \psi(\mathbf{r}) = b_i \frac{\partial \psi}{\partial i}. \tag{16.5}$$

In the case (16.4) of the vector field, however, $\nabla \mathbf{a}(\mathbf{r})$ itself means nothing coherent[6] so the parentheses usually are retained. Equations (16.4) and (16.5) define the directional derivative.

Note that the directional derivative is the derivative not of the reference vector $\mathbf{b}$ but only of the field $\psi(\mathbf{r})$ or $\mathbf{a}(\mathbf{r})$. The vector $\mathbf{b}$ just directs and scales the derivative; it is not the object of it. Nothing requires $\mathbf{b}$ to be constant, though. It can be a vector field $\mathbf{b}(\mathbf{r})$ that varies from place to place; the directional derivative does not care.

Within (16.5), the quantity

$$\nabla \psi(\mathbf{r}) = \hat{\mathbf{i}} \frac{\partial \psi}{\partial i} \tag{16.6}$$

is called the *gradient* of the scalar field $\psi(\mathbf{r})$. Though both scalar and vector fields have directional derivatives, only scalar fields have gradients. The gradient represents the amplitude and direction of a scalar field's locally steepest rate.

Formally a dot product, the directional derivative operator $\mathbf{b} \cdot \nabla$ is invariant under reorientation of axes, whereupon the directional derivative is invariant, too. The result of a $\nabla$ operation, the gradient $\nabla \psi(\mathbf{r})$ is likewise invariant.

[6] Well, it does mean something coherent in *dyadic analysis* [9, Appendix B], but this book won't treat that.

### 16.1.4 Divergence

There exist other vector derivatives than the directional derivative and gradient of § 16.1.3. One of these is divergence. It is not easy to motivate divergence directly, however, so we will approach it indirectly, through the concept of flux as follows.

The *flux* of a vector field $\mathbf{a}(\mathbf{r})$ outward from a region in space is

$$\Phi \equiv \oint_S \mathbf{a}(\mathbf{r}) \cdot d\mathbf{s}, \tag{16.7}$$

where

$$d\mathbf{s} \equiv \hat{\mathbf{n}} \cdot ds \tag{16.8}$$

is a vector infinitesimal of amplitude $ds$, directed normally outward from the closed surface bounding the region—$ds$ being the area of an infinitesimal element of the surface, the area of a tiny patch. Flux is flow through a surface: in this case, net flow outward from the region in question. (The paragraph says much in relatively few words. If it seems opaque then try to visualize eqn. 16.7's dot product $\mathbf{a}[\mathbf{r}] \cdot d\mathbf{s}$, in which the vector $d\mathbf{s}$ represents the area and orientation of a patch of the region's enclosing surface. When something like air flows through any surface—not necessarily a physical barrier but an imaginary surface like the goal line's vertical plane in a football game[7]—what matters is not the surface's area as such but rather the area the surface presents to the flow. The surface presents its full area to a perpendicular flow, but otherwise the flow sees a foreshortened surface, *as though the surface were projected onto a plane perpendicular to the flow.* Refer to Fig. 15.2. Now realize that eqn. 16.7 actually describes flux not through an open surface but through a closed—it could be the imaginary rectangular box enclosing the region of football play to goal-post height; where wind blowing through the region, entering and leaving, would constitute zero net flux; but where a positive net flux would have barometric pressure falling and air leaving the region maybe because a storm is coming—and you've got the idea.)

A region of positive flux is a *source;* of negative flux, a *sink.* One can contemplate the flux $\Phi_{\text{open}} = \int_S \mathbf{a}(\mathbf{r}) \cdot d\mathbf{s}$ through an open surface as well as through a closed, but it is the outward flux (16.7) through a closed surface that will concern us here.

[7]The author has American football in mind but other football games have goal lines and goal posts, too. Pick your favorite brand of football.

The outward flux $\Phi$ of a vector field $\mathbf{a}(\mathbf{r})$ through a closed surface bounding some definite region in space is evidently

$$\Phi = \iint \Delta a_x(y,z)\,dy\,dz + \iint \Delta a_y(z,x)\,dz\,dx + \iint \Delta a_z(x,y)\,dx\,dy,$$

where

$$\begin{aligned}
\Delta a_x(y,z) &= \int_{x_{\min}(y,z)}^{x_{\max}(y,z)} \frac{\partial a_x}{\partial x}\,dx, \\
\Delta a_y(z,x) &= \int_{y_{\min}(z,x)}^{y_{\max}(z,x)} \frac{\partial a_y}{\partial y}\,dy, \\
\Delta a_z(x,y) &= \int_{z_{\min}(x,y)}^{z_{\max}(x,y)} \frac{\partial a_z}{\partial z}\,dz
\end{aligned}$$

represent the increase across the region respectively of $a_x$, $a_y$ or $a_z$ along an $\hat{\mathbf{x}}$-, $\hat{\mathbf{y}}$- or $\hat{\mathbf{z}}$-directed line.[8] If the field has constant derivatives $\partial\mathbf{a}/\partial i$, or equivalently if the region in question is small enough that the derivatives are practically constant through it, then these increases are simply

$$\begin{aligned}
\Delta a_x(y,z) &= \frac{\partial a_x}{\partial x}\,\Delta x(y,z), \\
\Delta a_y(z,x) &= \frac{\partial a_y}{\partial y}\,\Delta y(z,x), \\
\Delta a_z(x,y) &= \frac{\partial a_z}{\partial z}\,\Delta z(x,y),
\end{aligned}$$

upon which

$$\begin{aligned}
\Phi \;=\; & \frac{\partial a_x}{\partial x} \iint \Delta x(y,z)\,dy\,dz + \frac{\partial a_y}{\partial y} \iint \Delta y(z,x)\,dz\,dx \\
& + \frac{\partial a_z}{\partial z} \iint \Delta z(x,y)\,dx\,dy.
\end{aligned}$$

But each of the last equation's three integrals represents the region's volume $V$, so

$$\Phi = (V)\left(\frac{\partial a_x}{\partial x} + \frac{\partial a_y}{\partial y} + \frac{\partial a_z}{\partial z}\right);$$

[8] Naturally, if the region's boundary happens to be concave, then some lines might enter and exit the region more than once, but this merely elaborates the limits of integration along those lines. It changes the problem in no essential way.

or, dividing through by the volume,

$$\frac{\Phi}{V} = \frac{\partial a_x}{\partial x} + \frac{\partial a_y}{\partial y} + \frac{\partial a_z}{\partial z} = \frac{\partial a_i}{\partial i} = \nabla \cdot \mathbf{a}(\mathbf{r}). \tag{16.9}$$

We give this ratio of outward flux to volume,

$$\nabla \cdot \mathbf{a}(\mathbf{r}) = \frac{\partial a_i}{\partial i}, \tag{16.10}$$

the name *divergence,* representing the intensity of a source or sink.

Formally a dot product, divergence is invariant under reorientation of axes.

### 16.1.5 Curl

Curl is to divergence as the cross product is to the dot product. Curl is a little trickier to visualize, though. It needs first the concept of circulation as follows.

The *circulation* of a vector field $\mathbf{a}(\mathbf{r})$ about a closed contour in space is

$$\Gamma \equiv \oint \mathbf{a}(\mathbf{r}) \cdot d\boldsymbol{\ell}, \tag{16.11}$$

where, unlike the $\oint_S$ of (16.7) which represented a double integration over a surface, the $\oint$ here represents only a single integration. One can in general contemplate circulation about any closed contour, but it suits our purpose here to consider specifically a closed contour that happens not to depart from a single, flat plane in space.

Let $[\hat{\mathbf{u}}\ \hat{\mathbf{v}}\ \hat{\mathbf{n}}]$ be an orthogonal basis with $\hat{\mathbf{n}}$ normal to the contour's plane such that travel positively along the contour tends from $\hat{\mathbf{u}}$ toward $\hat{\mathbf{v}}$ rather than the reverse. The circulation $\Gamma$ of a vector field $\mathbf{a}(\mathbf{r})$ about this contour is evidently

$$\Gamma = \int \Delta a_v(v)\, dv - \int \Delta a_u(u)\, du,$$

where

$$\Delta a_v(v) = \int_{u_{\min}(v)}^{u_{\max}(v)} \frac{\partial a_v}{\partial u}\, du,$$
$$\Delta a_u(u) = \int_{v_{\min}(u)}^{v_{\max}(u)} \frac{\partial a_u}{\partial v}\, dv$$

represent the increase across the contour's interior respectively of $a_v$ or $a_u$ along a $\hat{\mathbf{u}}$- or $\hat{\mathbf{v}}$-directed line. If the field has constant derivatives $\partial\mathbf{a}/\partial i$, or

equivalently if the contour in question is short enough that the derivatives are practically constant over it, then these increases are simply

$$\Delta a_v(v) = \frac{\partial a_v}{\partial u}\,\Delta u(v),$$
$$\Delta a_u(u) = \frac{\partial a_u}{\partial v}\,\Delta v(u),$$

upon which

$$\Gamma = \frac{\partial a_v}{\partial u}\int \Delta u(v)\,dv - \frac{\partial a_u}{\partial v}\int \Delta v(u)\,du.$$

But each of the last equation's two integrals represents the area $A$ within the contour, so

$$\Gamma = (A)\left(\frac{\partial a_v}{\partial u} - \frac{\partial a_u}{\partial v}\right);$$

or, dividing through by the area,

$$\begin{aligned}\frac{\Gamma}{A} &= \frac{\partial a_v}{\partial u} - \frac{\partial a_u}{\partial v} \\ &= \hat{\mathbf{n}}\cdot\left[\epsilon_{ijk}\hat{\mathbf{i}}\frac{\partial a_k}{\partial j}\right] = \hat{\mathbf{n}}\cdot\begin{vmatrix} \hat{\mathbf{u}} & \hat{\mathbf{v}} & \hat{\mathbf{n}} \\ \partial/\partial u & \partial/\partial v & \partial/\partial n \\ a_u & a_v & a_n \end{vmatrix} \\ &= \hat{\mathbf{n}}\cdot[\nabla\times\mathbf{a}(\mathbf{r})]. \end{aligned} \tag{16.12}$$

We give this ratio of circulation to area,

$$\hat{\mathbf{n}}\cdot[\nabla\times\mathbf{a}(\mathbf{r})] = \hat{\mathbf{n}}\cdot\left[\epsilon_{ijk}\hat{\mathbf{i}}\frac{\partial a_k}{\partial j}\right] = \frac{\partial a_v}{\partial u} - \frac{\partial a_u}{\partial v}, \tag{16.13}$$

the name *directional curl,* representing the intensity of circulation, the degree of twist so to speak, about a specified axis. The cross product in (16.13),

$$\nabla\times\mathbf{a}(\mathbf{r}) = \epsilon_{ijk}\hat{\mathbf{i}}\frac{\partial a_k}{\partial j}, \tag{16.14}$$

we call *curl.*

Curl (16.14) is an interesting quantity. Although it emerges from directional curl (16.13) and although we have developed directional curl with respect to a contour in some specified plane, curl (16.14) itself turns out to be altogether independent of any particular plane. We might have chosen another plane and though $\hat{\mathbf{n}}$ would then be different the same (16.14) would necessarily result. Directional curl, a scalar, is a property of the field

and the plane. Curl, a vector, unexpectedly is a property of the field only. Directional curl evidently cannot exceed curl in magnitude, but will equal it when $\hat{\mathbf{n}}$ points in its direction, so it may be said that curl is the locally greatest directional curl, oriented normally to the locally greatest directional curl's plane.

We have needed $\hat{\mathbf{n}}$ and (16.13) to motivate and develop the concept (16.14) of curl. Once developed, however, the concept of curl stands on its own, whereupon one can return to define directional curl more generally than (16.13) has defined it. As in (16.4) here too any reference vector $\mathbf{b}$ or vector field $\mathbf{b}(\mathbf{r})$ can serve to direct the curl, not only $\hat{\mathbf{n}}$. Hence,

$$\mathbf{b} \cdot [\nabla \times \mathbf{a}(\mathbf{r})] = \mathbf{b} \cdot \left[ \epsilon_{ijk} \hat{\mathbf{i}} \frac{\partial a_k}{\partial j} \right] = \epsilon_{ijk} b_i \frac{\partial a_k}{\partial j}. \tag{16.15}$$

This would be the actual definition of *directional curl*. Note however that directional curl so defined is not a distinct kind of derivative but rather is just curl, dot-multiplied by a reference vector.

Formally a cross product, curl is invariant under reorientation of axes. An ordinary dot product, directional curl is likewise invariant.

### 16.1.6 Cross-directional derivatives

The several directional derivatives of the $\mathbf{b} \cdot \nabla$ class, including the scalar (16.5) and vector (16.4) directional derivatives themselves and also including directional curl (16.15), compute rates with reference to some direction $\mathbf{b}$. Another class of directional derivatives however is possible, that of the *cross-directional derivatives*.[9] These compute rates perpendicularly to $\mathbf{b}$. Unlike the vector directional derivative (16.4), the cross-directional derivatives are not actually new derivatives but are cross products of $\mathbf{b}$ with derivatives already familiar to us. The cross-directional derivatives are

$$\begin{aligned} \mathbf{b} \times \nabla \psi &= \epsilon_{ijk} \hat{\mathbf{i}} b_j \frac{\partial \psi}{\partial k}, \\ \mathbf{b} \times \nabla \times \mathbf{a} &= \hat{\mathbf{j}} b_i \left( \frac{\partial a_i}{\partial j} - \frac{\partial a_j}{\partial i} \right). \end{aligned} \tag{16.16}$$

[9] The author is unaware of a conventional name for these derivatives. The name *cross-directional* seems as apt as any.

We call these respectively the *cross-directional derivative* (itself) and *cross-directional curl,* the latter derived as

$$\begin{aligned}
\mathbf{b}\times\nabla\times\mathbf{a} &= \mathbf{b}\times\left(\epsilon_{ijk}\hat{\mathbf{i}}\frac{\partial a_k}{\partial j}\right) \\
&= \epsilon_{mni}\hat{\mathbf{m}}b_n\left(\epsilon_{ijk}\hat{\mathbf{i}}\frac{\partial a_k}{\partial j}\right)_i = \epsilon_{mni}\epsilon_{ijk}\hat{\mathbf{m}}b_n\frac{\partial a_k}{\partial j} \\
&= (\delta_{mj}\delta_{nk}-\delta_{mk}\delta_{nj})\hat{\mathbf{m}}b_n\frac{\partial a_k}{\partial j} \\
&= \hat{\mathbf{j}}b_k\frac{\partial a_k}{\partial j}-\hat{\mathbf{k}}b_j\frac{\partial a_k}{\partial j} = \hat{\mathbf{j}}b_i\frac{\partial a_i}{\partial j}-\hat{\mathbf{j}}b_i\frac{\partial a_j}{\partial i}
\end{aligned}$$

where the Levi-Civita identity that $\epsilon_{mni}\epsilon_{ijk} = \epsilon_{imn}\epsilon_{ijk} = \delta_{mj}\delta_{nk}-\delta_{mk}\delta_{nj}$ comes from Table 15.1.

## 16.2 Integral forms

The vector field's distinctive maneuver is the shift between integral forms, which we are now prepared to treat. This shift comes in two kinds. The two subsections that follow explain.

### 16.2.1 The divergence theorem

Section 16.1.4 has contemplated the flux of a vector field $\mathbf{a}(\mathbf{r})$ from a volume small enough that the divergence $\nabla\cdot\mathbf{a}$ were practically constant through the volume. One would like to treat the flux from larger, more general volumes as well. According to the definition (16.7), the flux from any volume is

$$\Phi = \oint_S \mathbf{a}\cdot d\mathbf{s}.$$

If one subdivides a large volume into infinitesimal volume elements $dv$, then the flux from a single volume element is

$$\Phi_{\text{element}} = \oint_{S_{\text{element}}} \mathbf{a}\cdot d\mathbf{s}.$$

Even a single volume element however can have two distinct kinds of surface area: inner surface area shared with another element; and outer surface area shared with no other element because it belongs to the surface of the larger, overall volume. Interior elements naturally have only the former kind but

boundary elements have both kinds of surface area, so one can elaborate the last equation to read

$$\Phi_{\text{element}} = \int_{S_{\text{inner}}} \mathbf{a} \cdot d\mathbf{s} + \int_{S_{\text{outer}}} \mathbf{a} \cdot d\mathbf{s}$$

for a single element, where $\oint_{S_{\text{element}}} = \int_{S_{\text{inner}}} + \int_{S_{\text{outer}}}$. Adding all the elements together, we have that

$$\sum_{\text{elements}} \Phi_{\text{element}} = \sum_{\text{elements}} \int_{S_{\text{inner}}} \mathbf{a} \cdot d\mathbf{s} + \sum_{\text{elements}} \int_{S_{\text{outer}}} \mathbf{a} \cdot d\mathbf{s};$$

but the inner sum is null because it includes each interior surface twice, because each interior surface is shared by two elements such that $d\mathbf{s}_2 = -d\mathbf{s}_1$ (in other words, such that the one volume element's $d\mathbf{s}$ on the surface the two elements share points oppositely to the other volume element's $d\mathbf{s}$ on the same surface), so

$$\sum_{\text{elements}} \Phi_{\text{element}} = \sum_{\text{elements}} \int_{S_{\text{outer}}} \mathbf{a} \cdot d\mathbf{s} = \oint_S \mathbf{a} \cdot d\mathbf{s}.$$

In this equation, the last integration is over the surface of the larger, overall volume, which surface after all consists of nothing other than the several boundary elements' outer surface patches. Applying (16.9) to the equation's left side to express the flux $\Phi_{\text{element}}$ from a single volume element yields

$$\sum_{\text{elements}} \nabla \cdot \mathbf{a}\, dv = \oint_S \mathbf{a} \cdot d\mathbf{s}.$$

That is,

$$\int_V \nabla \cdot \mathbf{a}\, dv = \oint_S \mathbf{a} \cdot d\mathbf{s}. \tag{16.17}$$

Equation (16.17) is the *divergence theorem.*[10] The divergence theorem, the vector's version of the fundamental theorem of calculus (7.2), neatly relates the divergence within a volume to the flux from it. It is an important result. The integral on the equation's left and the one on its right each arise in vector analysis more often than one might expect. When they do, (16.17) swaps the one integral for the other, often a profitable maneuver.[11]

[10] [60, eqn. 1.2.8]

[11] Where a wave propagates through a material interface, the associated field can be discontinuous and, consequently, the field's divergence can be infinite, which would seem

### 16.2.2 Stokes' theorem

Corresponding to the divergence theorem of § 16.2.1 is a second, related theorem for directional curl, developed as follows. If an open surface, whether the surface be confined to a plane or be warped in three dimensions (as for example in bowl shape), is subdivided into infinitesimal surface elements $d\mathbf{s}$—each element small enough not only to experience essentially constant curl but also to be regarded as planar—then according to (16.11) the circulation about the entire surface is

$$\Gamma = \oint \mathbf{a} \cdot d\boldsymbol{\ell}$$

and the circulation about any one surface element is

$$\Gamma_{\text{element}} = \oint_{\text{element}} \mathbf{a} \cdot d\boldsymbol{\ell}.$$

From this equation, reasoning parallel to that of § 16.2.1—only using (16.12) in place of (16.9)—concludes that

$$\int_S (\nabla \times \mathbf{a}) \cdot d\mathbf{s} = \oint \mathbf{a} \cdot d\boldsymbol{\ell}. \tag{16.18}$$

Equation (16.18) is *Stokes' theorem,*[12,13] neatly relating the directional curl over a (possibly nonplanar) surface to the circulation about it. Like the divergence theorem (16.17), Stokes' theorem (16.18) serves to swap one vector integral for another where such a maneuver is needed.

## 16.3 Summary of definitions and identities of vector calculus

Table 16.1 lists useful definitions and identities of vector calculus,[14] the first several of which it gathers from §§ 16.1 and 16.2, the last several of

to call assumptions underlying (16.17) into question. However, the infinite divergence at a material interface is normally *integrable* in the same way the Dirac delta of § 7.7, though infinite, is integrable. One can integrate finitely through either infinity. If one can conceive of an interface not as a sharp layer of zero thickness but rather as a thin layer of thickness $\epsilon$, through which the associated field varies steeply but continuously, then the divergence theorem necessarily remains valid in the limit $\epsilon \to 0$.

[12] [60, eqn. 1.4.20]

[13] If (16.17) is "the divergence theorem," then should (16.18) not be "the curl theorem"? Answer: maybe it should be, but no one calls it that. Sir George Gabriel Stokes evidently is not to be denied his fame!

[14] [5, Appendix II.3][60, Appendix II][28, Appendix A]

which (exhibiting heretofore unfamiliar symbols like $\nabla^2$) it gathers from § 16.4 to follow. Of the identities in the middle of the table, a few are statements of the $\nabla$ operator's distributivity over summation. The rest are vector derivative product rules (§ 4.5.2).

The product rules resemble the triple products of Table 15.2, only with the $\nabla$ operator in place of the vector $\mathbf{c}$. However, since $\nabla$ is a differential operator for which, for instance, $\mathbf{b} \cdot \nabla \neq \nabla \cdot \mathbf{b}$, its action differs from a vector's in some cases, and there are more distinct ways in which it can act. Among the several product rules the easiest to prove is that

$$\nabla(\psi\omega) = \hat{\mathbf{i}}\frac{\partial(\psi\omega)}{\partial i} = \omega\hat{\mathbf{i}}\frac{\partial\psi}{\partial i} + \psi\hat{\mathbf{i}}\frac{\partial\omega}{\partial i} = \omega\nabla\psi + \psi\nabla\omega.$$

The hardest to prove is that

$$\begin{aligned}
\nabla(\mathbf{a}\cdot\mathbf{b}) &= \nabla(a_i b_i) = \hat{\mathbf{j}}\frac{\partial(a_i b_i)}{\partial j} = \hat{\mathbf{j}} b_i \frac{\partial a_i}{\partial j} + \hat{\mathbf{j}} a_i \frac{\partial b_i}{\partial j} \\
&= \hat{\mathbf{j}} b_i \frac{\partial a_j}{\partial i} + \hat{\mathbf{j}} b_i \left( \frac{\partial a_i}{\partial j} - \frac{\partial a_j}{\partial i} \right) + \hat{\mathbf{j}} a_i \frac{\partial b_j}{\partial i} + \hat{\mathbf{j}} a_i \left( \frac{\partial b_i}{\partial j} - \frac{\partial b_j}{\partial i} \right) \\
&= (\mathbf{b}\cdot\nabla)\mathbf{a} + \mathbf{b}\times\nabla\times\mathbf{a} + (\mathbf{a}\cdot\nabla)\mathbf{b} + \mathbf{a}\times\nabla\times\mathbf{b} \\
&= (\mathbf{b}\cdot\nabla + \mathbf{b}\times\nabla\times)\mathbf{a} + (\mathbf{a}\cdot\nabla + \mathbf{a}\times\nabla\times)\mathbf{b},
\end{aligned}$$

because to prove it one must recognize in it the cross-directional curl of (16.16). Also nontrivial to prove is that

$$\begin{aligned}
\nabla\times(\mathbf{a}\times\mathbf{b}) &= \nabla\times(\epsilon_{ijk}\hat{\mathbf{i}} a_j b_k) \\
&= \epsilon_{mni}\hat{\mathbf{m}}\frac{\partial(\epsilon_{ijk}\hat{\mathbf{i}} a_j b_k)_i}{\partial n} = \epsilon_{mni}\epsilon_{ijk}\hat{\mathbf{m}}\frac{\partial(a_j b_k)}{\partial n} \\
&= (\delta_{mj}\delta_{nk} - \delta_{mk}\delta_{nj})\hat{\mathbf{m}}\frac{\partial(a_j b_k)}{\partial n} \\
&= \hat{\mathbf{j}}\frac{\partial(a_j b_k)}{\partial k} - \hat{\mathbf{k}}\frac{\partial(a_j b_k)}{\partial j} = \hat{\mathbf{j}}\frac{\partial(a_j b_i)}{\partial i} - \hat{\mathbf{j}}\frac{\partial(a_i b_j)}{\partial i} \\
&= \left( \hat{\mathbf{j}} b_i \frac{\partial a_j}{\partial i} + \hat{\mathbf{j}} a_j \frac{\partial b_i}{\partial i} \right) - \left( \hat{\mathbf{j}} a_i \frac{\partial b_j}{\partial i} + \hat{\mathbf{j}} b_j \frac{\partial a_i}{\partial i} \right) \\
&= (\mathbf{b}\cdot\nabla + \nabla\cdot\mathbf{b})\mathbf{a} - (\mathbf{a}\cdot\nabla + \nabla\cdot\mathbf{a})\mathbf{b}.
\end{aligned}$$

Table 16.1: Definitions and identities of vector calculus (see also Table 15.2 on page 433).

$$
\begin{aligned}
\nabla &\equiv \hat{\mathbf{i}}\frac{\partial}{\partial i} & \mathbf{b}\cdot\nabla &= b_i\frac{\partial}{\partial i}\\
\nabla\psi &= \hat{\mathbf{i}}\frac{\partial\psi}{\partial i} & \mathbf{b}\cdot\nabla\psi &= b_i\frac{\partial\psi}{\partial i}\\
\nabla\cdot\mathbf{a} &= \frac{\partial a_i}{\partial i} & (\mathbf{b}\cdot\nabla)\mathbf{a} &= b_i\frac{\partial\mathbf{a}}{\partial i} = \hat{\mathbf{j}}b_i\frac{\partial a_j}{\partial i}\\
\nabla\times\mathbf{a} &= \epsilon_{ijk}\hat{\mathbf{i}}\frac{\partial a_k}{\partial j} & \mathbf{b}\cdot\nabla\times\mathbf{a} &= \epsilon_{ijk}b_i\frac{\partial a_k}{\partial j}\\
\mathbf{b}\times\nabla\psi &= \epsilon_{ijk}\hat{\mathbf{i}}b_j\frac{\partial\psi}{\partial k} & \mathbf{b}\times\nabla\times\mathbf{a} &= \hat{\mathbf{j}}b_i\left(\frac{\partial a_i}{\partial j}-\frac{\partial a_j}{\partial i}\right)\\
\Phi &\equiv \int_S \mathbf{a}\cdot d\mathbf{s} & \int_V \nabla\cdot\mathbf{a}\,dv &= \oint_S \mathbf{a}\cdot d\mathbf{s}\\
\Gamma &\equiv \int_C \mathbf{a}\cdot d\boldsymbol{\ell} & \int_S(\nabla\times\mathbf{a})\cdot d\mathbf{s} &= \oint \mathbf{a}\cdot d\boldsymbol{\ell}
\end{aligned}
$$

$$
\begin{aligned}
\nabla\cdot(\mathbf{a}+\mathbf{b}) &= \nabla\cdot\mathbf{a}+\nabla\cdot\mathbf{b}\\
\nabla\times(\mathbf{a}+\mathbf{b}) &= \nabla\times\mathbf{a}+\nabla\times\mathbf{b}\\
\nabla(\psi+\omega) &= \nabla\psi+\nabla\omega\\
\nabla(\psi\omega) &= \omega\nabla\psi+\psi\nabla\omega\\
\nabla\cdot(\psi\mathbf{a}) &= \mathbf{a}\cdot\nabla\psi+\psi\nabla\cdot\mathbf{a}\\
\nabla\times(\psi\mathbf{a}) &= \psi\nabla\times\mathbf{a}-\mathbf{a}\times\nabla\psi\\
\nabla(\mathbf{a}\cdot\mathbf{b}) &= (\mathbf{b}\cdot\nabla+\mathbf{b}\times\nabla\times)\mathbf{a}+(\mathbf{a}\cdot\nabla+\mathbf{a}\times\nabla\times)\mathbf{b}\\
\nabla\cdot(\mathbf{a}\times\mathbf{b}) &= \mathbf{b}\cdot\nabla\times\mathbf{a}-\mathbf{a}\cdot\nabla\times\mathbf{b}\\
\nabla\times(\mathbf{a}\times\mathbf{b}) &= (\mathbf{b}\cdot\nabla+\nabla\cdot\mathbf{b})\mathbf{a}-(\mathbf{a}\cdot\nabla+\nabla\cdot\mathbf{a})\mathbf{b}
\end{aligned}
$$

$$
\begin{aligned}
\nabla^2 &\equiv \frac{\partial^2}{\partial i^2} & \nabla\nabla\cdot\mathbf{a} &= \hat{\mathbf{j}}\frac{\partial^2 a_i}{\partial j\,\partial i}\\
\nabla^2\psi &= \nabla\cdot\nabla\psi = \frac{\partial^2\psi}{\partial i^2} & \nabla^2\mathbf{a} &= \frac{\partial^2\mathbf{a}}{\partial i^2} = \hat{\mathbf{j}}\frac{\partial^2 a_j}{\partial i^2} = \hat{\mathbf{j}}\nabla^2(\hat{\mathbf{j}}\cdot\mathbf{a})
\end{aligned}
$$

$$
\begin{aligned}
\nabla\times\nabla\psi &= 0 & \nabla\cdot\nabla\times\mathbf{a} &= 0\\
\nabla\times\nabla\times\mathbf{a} &= \hat{\mathbf{j}}\frac{\partial}{\partial i}\left(\frac{\partial a_i}{\partial j}-\frac{\partial a_j}{\partial i}\right)\\
\nabla\nabla\cdot\mathbf{a} &= \nabla^2\mathbf{a}+\nabla\times\nabla\times\mathbf{a}
\end{aligned}
$$

The others are less hard:[15]

$$\begin{aligned}
\nabla \cdot (\psi \mathbf{a}) &= \frac{\partial(\psi a_i)}{\partial i} = a_i \frac{\partial \psi}{\partial i} + \psi \frac{\partial a_i}{\partial i} = \mathbf{a} \cdot \nabla \psi + \psi \nabla \cdot \mathbf{a}; \\
\nabla \times (\psi \mathbf{a}) &= \epsilon_{ijk} \hat{\mathbf{i}} \frac{\partial(\psi a_k)}{\partial j} = \epsilon_{ijk} \hat{\mathbf{i}} \psi \frac{\partial a_k}{\partial j} + \epsilon_{ijk} \hat{\mathbf{i}} a_k \frac{\partial \psi}{\partial j} \\
&= \psi \nabla \times \mathbf{a} - \mathbf{a} \times \nabla \psi; \\
\nabla \cdot (\mathbf{a} \times \mathbf{b}) &= \frac{\partial(\epsilon_{ijk} a_j b_k)}{\partial i} = \epsilon_{ijk} b_k \frac{\partial a_j}{\partial i} + \epsilon_{ijk} a_j \frac{\partial b_k}{\partial i} \\
&= \mathbf{b} \cdot \nabla \times \mathbf{a} - \mathbf{a} \cdot \nabla \times \mathbf{b}.
\end{aligned}$$

Inasmuch as none of the derivatives or products within the table's several product rules vary under rotation of axes, the product rules are themselves invariant. That the definitions and identities at the top of the table are invariant, we have already seen; and § 16.4, next, will give invariance to the definitions and identities at the bottom. The whole table therefore is invariant under rotation of axes.

## 16.4 The Laplacian and other second-order derivatives

Table 16.1 ends with second-order vector derivatives. Like vector products and first-order vector derivatives, second-order vector derivatives too come in several kinds, the simplest of which is the *Laplacian*[16]

$$\begin{aligned}
\nabla^2 &\equiv \frac{\partial^2}{\partial i^2}, \\
\nabla^2 \psi = \nabla \cdot \nabla \psi &= \frac{\partial^2 \psi}{\partial i^2}, \\
\nabla^2 \mathbf{a} = \frac{\partial^2 \mathbf{a}}{\partial i^2} &= \hat{\mathbf{j}} \frac{\partial^2 a_j}{\partial i^2} = \hat{\mathbf{j}} \nabla^2 (\hat{\mathbf{j}} \cdot \mathbf{a}).
\end{aligned} \tag{16.19}$$

Other second-order vector derivatives include

$$\begin{aligned}
\nabla \nabla \cdot \mathbf{a} &= \hat{\mathbf{j}} \frac{\partial^2 a_i}{\partial j \, \partial i}, \\
\nabla \times \nabla \times \mathbf{a} &= \hat{\mathbf{j}} \frac{\partial}{\partial i} \left( \frac{\partial a_i}{\partial j} - \frac{\partial a_j}{\partial i} \right),
\end{aligned} \tag{16.20}$$

[15] And probably should have been left as exercises, except that this book is not actually an instructional textbook. The reader who wants exercises might hide the page from sight and work the three identities out with his own pencil.

[16] Though seldom seen in applied usage in the author's country, the alternate symbol $\Delta$ replaces $\nabla^2$ in some books.

the latter of which is derived as

$$\begin{aligned}
\nabla \times \nabla \times \mathbf{a} &= \nabla \times \left( \epsilon_{ijk} \hat{\mathbf{i}} \frac{\partial a_k}{\partial j} \right) \\
&= \epsilon_{mni} \hat{\mathbf{m}} \frac{\partial}{\partial n} \left( \epsilon_{ijk} \hat{\mathbf{i}} \frac{\partial a_k}{\partial j} \right)_i = \epsilon_{mni} \epsilon_{ijk} \hat{\mathbf{m}} \frac{\partial^2 a_k}{\partial n \, \partial j} \\
&= (\delta_{mj} \delta_{nk} - \delta_{mk} \delta_{nj}) \hat{\mathbf{m}} \frac{\partial^2 a_k}{\partial n \, \partial j} \\
&= \hat{\mathbf{j}} \frac{\partial^2 a_k}{\partial k \, \partial j} - \hat{\mathbf{k}} \frac{\partial^2 a_k}{\partial j^2} = \hat{\mathbf{j}} \frac{\partial^2 a_i}{\partial i \, \partial j} - \hat{\mathbf{j}} \frac{\partial^2 a_j}{\partial i^2}.
\end{aligned}$$

Combining the various second-order vector derivatives yields the useful identity that

$$\nabla \nabla \cdot \mathbf{a} = \nabla^2 \mathbf{a} + \nabla \times \nabla \times \mathbf{a}. \tag{16.21}$$

Table 16.1 summarizes.

The table includes two curious null identities,

$$\begin{aligned}
\nabla \times \nabla \psi &= 0, \\
\nabla \cdot \nabla \times \mathbf{a} &= 0.
\end{aligned} \tag{16.22}$$

In words, (16.22) states that *gradients do not curl and curl does not diverge.* This is unexpected but is a direct consequence of the definitions of the gradient, curl and divergence:

$$\begin{aligned}
\nabla \times \nabla \psi = \nabla \times \left( \hat{\mathbf{i}} \frac{\partial \psi}{\partial i} \right) = \epsilon_{mni} \hat{\mathbf{m}} \frac{\partial^2 \psi}{\partial n \, \partial i} &= 0; \\
\nabla \cdot \nabla \times \mathbf{a} = \nabla \cdot \left( \epsilon_{ijk} \hat{\mathbf{i}} \frac{\partial a_k}{\partial j} \right) = \epsilon_{ijk} \frac{\partial^2 a_k}{\partial i \, \partial j} &= 0.
\end{aligned}$$

A field like $\nabla \psi$ that does not curl is called an *irrotational* field. A field like $\nabla \times \mathbf{a}$ that does not diverge is called a *solenoidal, source-free* or (prosaically) *divergenceless* field.[17]

[17] In the writer's country, the United States, there has been a mistaken belief afoot that, if two fields $\mathbf{b}_1(\mathbf{r})$ and $\mathbf{b}_2(\mathbf{r})$ had everywhere the same divergence and curl, then the two fields could differ only by an additive constant. Even at least one widely distributed textbook expresses this belief, naming it *Helmholtz's theorem;* but it is not just the one textbook, for the writer has heard it verbally from at least two engineers, unacquainted with one other, who had earned Ph.D.s in different eras in different regions of the country. So the belief must be correct, mustn't it?

Well, maybe it is, but the writer remains unconvinced. Consider the admittedly contrived counterexample of $\mathbf{b}_1 = \hat{\mathbf{x}} y + \hat{\mathbf{y}} x$, $\mathbf{b}_2 = 0$.

Each of this section's second-order vector derivatives—including the vector Laplacian $\nabla^2\mathbf{a}$, according to (16.21)—is or can be composed of first-order vector derivatives already familiar to us from § 16.1. Therefore, inasmuch as each of those first-order vector derivatives is invariant under reorientation of axes, each second-order vector derivative is likewise invariant.

## 16.5 Contour derivative product rules

Equation (4.25) gives the derivative product rule for functions of a scalar variable. Fields—that is, functions of a vector variable—obey product rules, too, several of which Table 16.1 lists. The table's product rules however are general product rules that regard full spatial derivatives. What about derivatives along an arbitrary contour? Do they obey product rules, too? That is, one supposes that[18]

$$\begin{aligned}
\frac{\partial}{\partial \ell}(\psi\omega) &= \omega\frac{\partial \psi}{\partial \ell} + \psi\frac{\partial \omega}{\partial \ell}, \\
\frac{\partial}{\partial \ell}(\psi\mathbf{a}) &= \mathbf{a}\frac{\partial \psi}{\partial \ell} + \psi\frac{\partial \mathbf{a}}{\partial \ell}, \\
\frac{\partial}{\partial \ell}(\mathbf{a}\cdot\mathbf{b}) &= \mathbf{b}\cdot\frac{\partial \mathbf{a}}{\partial \ell} + \mathbf{a}\cdot\frac{\partial \mathbf{b}}{\partial \ell}, \\
\frac{\partial}{\partial \ell}(\mathbf{a}\times\mathbf{b}) &= -\mathbf{b}\times\frac{\partial \mathbf{a}}{\partial \ell} + \mathbf{a}\times\frac{\partial \mathbf{b}}{\partial \ell}.
\end{aligned} \tag{16.23}$$

where $\ell$ is the distance along some arbitrary contour in space. As a hypothesis, (16.23) is attractive. But is it true?

That the first line of (16.23) is true is clear, if you think about it in the right way, because, in the restricted case (16.23) represents, one can treat the scalar fields $\psi(\mathbf{r})$ and $\omega(\mathbf{r})$ as ordinary scalar functions $\psi(\ell)$ and

On an applied level, the writer knows of literally no other false theorem so widely believed to be true, which leads the writer to suspect that he himself had somehow erred in judging the theorem false. What the writer really believes however is that Hermann von Helmholtz probably originally had put some appropriate restrictions on $\mathbf{b}_1$ and $\mathbf{b}_2$ which, if obeyed, made his theorem true but which at some time after his death got lost in transcription. That a transcription error would go undetected so many years would tend to suggest that Helmholtz's theorem, though interesting, were not actually very necessary in practical applications. (Some believe the theorem necessary to establish a "gauge" in a wave equation, but if they examine the use of their gauges closely then they will likely discover that one does not logically actually need to invoke the theorem to use the gauges.)

Corrections by readers are invited.

[18] The $-$ sign in (16.23)'s last line is an artifact of ordering the line's factors in the style of Table 16.1. Before proving the line, the narrative will reverse the order to kill the sign. See below.

$\omega(\ell)$ of the scalar distance $\ell$ along the contour, whereupon (4.25) applies—for (16.23) never evaluates $\psi(\mathbf{r})$ or $\omega(\mathbf{r})$ but along the contour. The same naturally goes for the vector fields $\mathbf{a}(\mathbf{r})$ and $\mathbf{b}(\mathbf{r})$, which one can treat as vector functions $\mathbf{a}(\ell)$ and $\mathbf{b}(\ell)$ of the scalar distance $\ell$; so the second and third lines of (16.23) are true, too, since one can write the second line in the form

$$\hat{\mathbf{i}}\left[\frac{\partial}{\partial \ell}(\psi a_i)\right] = \hat{\mathbf{i}}\left[a_i \frac{\partial \psi}{\partial \ell} + \psi \frac{\partial a_i}{\partial \ell}\right]$$

and the third line in the form

$$\frac{\partial}{\partial \ell}(a_i b_i) = b_i \frac{\partial a_i}{\partial \ell} + a_i \frac{\partial b_i}{\partial \ell},$$

each of which, according to the first line, is true separately for $i = x$, for $i = y$ and for $i = z$.

The truth of (16.23)'s last line is slightly less obvious. Nevertheless, one can reorder factors to write the line as

$$\frac{\partial}{\partial \ell}(\mathbf{a} \times \mathbf{b}) = \frac{\partial \mathbf{a}}{\partial \ell} \times \mathbf{b} + \mathbf{a} \times \frac{\partial \mathbf{b}}{\partial \ell},$$

the Levi-Civita form (§ 15.4.3) of which is

$$\epsilon_{ijk}\hat{\mathbf{i}}\left[\frac{\partial}{\partial \ell}(a_j b_k)\right] = \epsilon_{ijk}\hat{\mathbf{i}}\left[\frac{\partial a_j}{\partial \ell} b_k + a_j \frac{\partial b_k}{\partial \ell}\right].$$

The Levi-Civita form is true separately for $(i, j, k) = (x, y, z)$, for $(i, j, k) = (x, z, y)$, and so forth, so (16.23)'s last line as a whole is true, too, which completes the proof of (16.23).

## 16.6 Metric coefficients

A scalar field $\psi(\mathbf{r})$ is the same field whether expressed as a function $\psi(x, y, z)$ of rectangular coordinates, $\psi(\rho; \phi, z)$ of cylindrical coordinates or $\psi(r; \theta; \phi)$ of spherical coordinates,[19] or indeed of coordinates in any three-dimensional system. However, cylindrical and spherical geometries normally recommend cylindrical and spherical coordinate systems, systems which make some of the identities of Table 16.1 hard to use.

The reason a cylindrical or spherical system makes some of the table's identities hard to use is that some of the table's identities involve derivatives

[19] For example, the field $\psi = x^2 + y^2$ in rectangular coordinates is $\psi = \rho^2$ in cylindrical coordinates. Refer to Table 3.4.

Table 16.2: The metric coefficients of the rectangular, cylindrical and spherical coordinate systems.

| RECT. | CYL. | SPHER. |
|---|---|---|
| $h_x = 1$ | $h_\rho = 1$ | $h_r = 1$ |
| $h_y = 1$ | $h_\phi = \rho$ | $h_\theta = r$ |
| $h_z = 1$ | $h_z = 1$ | $h_\phi = r \sin\theta$ |

$d/di$, notation which per § 15.4.2 stands for $d/dx'$, $d/dy'$ or $d/dz'$ where the coordinates $x'$, $y'$ and $z'$ represent lengths. But among the cylindrical and spherical coordinates are $\theta$ and $\phi$, angles rather than lengths. Because one cannot use an angle as though it were a length, the notation $d/di$ cannot stand for $d/d\theta$ or $d/d\phi$ and, thus, one cannot use the table in cylindrical or spherical coordinates as the table stands.

We therefore want factors to convert the angles in question to lengths (or, more generally, when special coordinate systems like the parabolic systems of § 15.7 come into play, to convert coordinates other than lengths to lengths). Such factors are called *metric coefficients* and Table 16.2 lists them.[20] The use of the table is this: that for any metric coefficient $h_\alpha$ a change $d\alpha$ in its coordinate $\alpha$ sweeps out a length $h_\alpha\, d\alpha$. For example, in cylindrical coordinates $h_\phi = \rho$ according to table, so a change $d\phi$ in the azimuthal coordinate $\phi$ sweeps out a length $\rho\, d\phi$—a fact we have already informally observed as far back as § 7.4.1, which the table now formalizes.

In the table, incidentally, the metric coefficient $h_\phi$ seems to have two different values, one value in cylindrical coordinates and another in spherical. The two are really the same value, though, since $\rho = r \sin\theta$ per Table 3.4.

### 16.6.1 Displacements, areas and volumes

In any orthogonal, right-handed, three-dimensional coordinate system $(\alpha; \beta; \gamma)$—whether the symbols $(\alpha; \beta; \gamma)$ stand for $(x, y, z)$, $(y, z, x)$, $(z, x, y)$, $(x', y', z')$, $(\rho; \phi, z)$, $(r; \theta; \phi)$, $(\phi^x, r; \theta^x)$, etc.,[21] or even something exotic like

[20] [12, § 2-4]

[21] The book's admittedly clumsy usage of semicolons ";" and commas "," to delimit coordinate triplets, whereby a semicolon precedes an angle (or, in this section's case, precedes a generic coordinate like $\alpha$ that could stand for an angle), serves well enough to distinguish the three principal coordinate systems $(x, y, z)$, $(\rho; \phi, z)$ and $(r; \theta; \phi)$ visually from one another but ceases to help much when further coordinate systems such as $(\phi^x, r; \theta^x)$ come into play. Logically, maybe, it would make more sense to write in the

the parabolic $(\sigma, \tau, z)$ of § 15.7—the product

$$d\mathbf{s} = \hat{\boldsymbol{\alpha}} h_\beta h_\gamma \, d\beta \, d\gamma \tag{16.24}$$

represents an area infinitesimal normal to $\hat{\boldsymbol{\alpha}}$. For example, the area infinitesimal on a spherical surface of radius $r$ is $d\mathbf{s} = \hat{\mathbf{r}} h_\theta h_\phi \, d\theta \, d\phi = \hat{\mathbf{r}} r^2 \sin\theta \, d\theta \, d\phi$.

Again in any orthogonal, right-handed, three-dimensional coordinate system $(\alpha; \beta; \gamma)$, the product

$$dv = h_\alpha h_\beta h_\gamma \, d\alpha \, d\beta \, d\gamma \tag{16.25}$$

represents a volume infinitesimal. For example, the volume infinitesimal in a spherical geometry is $dv = h_r h_\theta h_\phi \, dr \, d\theta \, d\phi = r^2 \sin\theta \, dr \, d\theta \, d\phi$.

Notice that § 7.4 has already calculated several integrals involving area and volume infinitesimals of these kinds.

A volume infinitesimal (16.25) cannot meaningfully in three dimensions be expressed as a vector as an area infinitesimal (16.24) can, since in three dimensions a volume has no orientation. Naturally however, a length or *displacement* infinitesimal can indeed be expressed as a vector, as

$$d\boldsymbol{\ell} = \hat{\boldsymbol{\alpha}} h_\alpha \, d\alpha. \tag{16.26}$$

Section 16.10 will have more to say about vector infinitesimals in non-rectangular coordinates.

### 16.6.2 The vector field and its scalar components

Like a scalar field $\psi(\mathbf{r})$, a vector field $\mathbf{a}(\mathbf{r})$ too is the same field whether expressed as a function $\mathbf{a}(x, y, z)$ of rectangular coordinates, $\mathbf{a}(\rho; \phi, z)$ of cylindrical coordinates or $\mathbf{a}(r; \theta; \phi)$ of spherical coordinates, or indeed of coordinates in any three-dimensional system. A vector field however is the sum of three scalar fields, each scaling an appropriate unit vector. In rectangular coordinates,

$$\mathbf{a}(\mathbf{r}) = \hat{\mathbf{x}} a_x(\mathbf{r}) + \hat{\mathbf{y}} a_y(\mathbf{r}) + \hat{\mathbf{z}} a_z(\mathbf{r});$$

in cylindrical coordinates,

$$\mathbf{a}(\mathbf{r}) = \hat{\boldsymbol{\rho}} a_\rho(\mathbf{r}) + \hat{\boldsymbol{\phi}} a_\phi(\mathbf{r}) + \hat{\mathbf{z}} a_z(\mathbf{r});$$

manner of $(, x, y, z)$, but to do so seems overwrought and fortunately no one the author knows of does it in that way. The delimiters just are not that important.

The book adheres to the semicolon convention not for any deep reason but only for lack of a better convention. See also Ch. 15's footnote 31.

and in spherical coordinates,

$$\mathbf{a}(\mathbf{r}) = \hat{\mathbf{r}} a_r(\mathbf{r}) + \hat{\boldsymbol{\theta}} a_\theta(\mathbf{r}) + \hat{\boldsymbol{\phi}} a_\phi(\mathbf{r}).$$

The scalar fields $a_\rho(\mathbf{r})$, $a_r(\mathbf{r})$, $a_\theta(\mathbf{r})$ and $a_\phi(\mathbf{r})$ in and of themselves do not differ in nature from $a_x(\mathbf{r})$, $a_y(\mathbf{r})$, $a_z(\mathbf{r})$, $\psi(\mathbf{r})$ or any other scalar field. One does tend to use them differently, though, because constant unit vectors $\hat{\mathbf{x}}$, $\hat{\mathbf{y}}$ and $\hat{\mathbf{z}}$ exist to combine the scalar fields $a_x(\mathbf{r})$, $a_y(\mathbf{r})$, $a_z(\mathbf{r})$ to compose the vector field $\mathbf{a}(\mathbf{r})$ whereas no such constant unit vectors exist to combine the scalar fields $a_\rho(\mathbf{r})$, $a_r(\mathbf{r})$, $a_\theta(\mathbf{r})$ and $a_\phi(\mathbf{r})$. Of course there are the variable unit vectors $\hat{\boldsymbol{\rho}}(\mathbf{r})$, $\hat{\mathbf{r}}(\mathbf{r})$, $\hat{\boldsymbol{\theta}}(\mathbf{r})$ and $\hat{\boldsymbol{\phi}}(\mathbf{r})$, but the practical and philosophical differences between these and the constant unit vectors is greater than it might seem. For instance, it is true that $\hat{\boldsymbol{\rho}} \cdot \hat{\boldsymbol{\phi}} = 0$, so long as what is meant by this is that $\hat{\boldsymbol{\rho}}(\mathbf{r}) \cdot \hat{\boldsymbol{\phi}}(\mathbf{r}) = 0$. However, $\hat{\boldsymbol{\rho}}(\mathbf{r}_1) \cdot \hat{\boldsymbol{\phi}}(\mathbf{r}_2) \neq 0$, an algebraic error fairly easy to commit. On the other hand, that $\hat{\mathbf{x}} \cdot \hat{\mathbf{y}} = 0$ is always true.

(One might ask why such a subsection as this would appear in a section on metric coefficients. The subsection is here because no obviously better spot for it presents itself, but moreover because we shall need the understanding the subsection conveys to apply metric coefficients consistently and correctly in § 16.9 to come.)

## 16.7 Nonrectangular notation

Section 15.4 has introduced Einstein's summation convention, the Kronecker delta $\delta_{ij}$ and the Levi-Civita epsilon $\epsilon_{ijk}$ together as notation for use in the definition of vector operations and in the derivation of vector identities. The notation relies on symbols like $i$, $j$ and $k$ to stand for unspecified coordinates, and Tables 15.2 and 16.1 use it extensively. Unfortunately, the notation fails in the nonrectangular coordinate systems when derivatives come into play, as they do in Table 16.1, because $\partial/\partial i$ is taken to represent a derivative specifically with respect to a length whereas nonrectangular coordinates like $\theta$ and $\phi$ are not lengths. Fortunately, this failure is not hard to redress.

Whereas the standard Einstein symbols $i$, $j$ and $k$ can stand only for lengths, the modified Einstein symbols $\tilde{\imath}$, $\tilde{\jmath}$ and $\tilde{k}$, which this section now introduces, can stand for any coordinates, even for coordinates like $\theta$ and $\phi$ that are not lengths. The tilde "˜" atop the symbol $\tilde{\imath}$ warns readers that the coordinate it represents is not necessarily a length and that, if one wants a length, one must multiply $\tilde{\imath}$ by an appropriate metric coefficient $h_{\tilde{\imath}}$ (§ 16.6). The products $h_{\tilde{\imath}}\tilde{\imath}$, $h_{\tilde{\jmath}}\tilde{\jmath}$ and $h_{\tilde{k}}\tilde{k}$ always represent lengths.

The symbols $\hat{\mathbf{i}}$, $\hat{\mathbf{j}}$ and $\hat{\mathbf{k}}$ need no modification even when modified symbols like $\tilde{\imath}$, $\tilde{\jmath}$ and $\tilde{k}$ are in use, since $\hat{\mathbf{i}}$, $\hat{\mathbf{j}}$ and $\hat{\mathbf{k}}$ are taken to represent unit vectors and $[\hat{\mathbf{i}}\ \hat{\mathbf{j}}\ \hat{\mathbf{k}}]$, a proper orthogonal basis irrespective of the coordinate system—so long, naturally, as the coordinate system is an orthogonal, right-handed coordinate system as are all the coordinate systems in this book.

The modified notation will find use in § 16.9.3.

## 16.8 Derivatives of the basis vectors

The derivatives of the various unit basis vectors with respect to the several coordinates of their respective coordinate systems are not hard to compute. In fact, looking at Fig. 15.1 on page 410, Fig. 15.4 on page 421, and Fig. 15.5 on page 422, one can just write them down. Table 16.3 records them.

Naturally, one can compute the table's derivatives symbolically, instead, as for example

$$\frac{\partial \hat{\boldsymbol{\rho}}}{\partial \phi} = \frac{\partial}{\partial \phi}(\hat{\mathbf{x}} \cos\phi + \hat{\mathbf{y}} \sin\phi) = -\hat{\mathbf{x}} \sin\phi + \hat{\mathbf{y}} \cos\phi = +\hat{\boldsymbol{\phi}}.$$

Such an approach prospers in special coordinate systems like the parabolic systems of Tables 15.3 and 15.4, but in cylindrical and spherical coordinates it is probably easier just to look at the figures.

## 16.9 Derivatives in the nonrectangular systems

This section develops vector derivatives in cylindrical and spherical coordinates.

### 16.9.1 Derivatives in cylindrical coordinates

According to Table 16.1,

$$\nabla\psi = \hat{\mathbf{i}} \frac{\partial \psi}{\partial i},$$

but as § 16.6 has observed Einstein's symbol $i$ must stand for a length not an angle, whereas one of the three cylindrical coordinates—the azimuth $\phi$—is an angle. The cylindrical metric coefficients of Table 16.2 make the necessary conversion, the result of which is

$$\nabla\psi = \hat{\boldsymbol{\rho}} \frac{\partial \psi}{\partial \rho} + \hat{\boldsymbol{\phi}} \frac{\partial \psi}{\rho\, \partial \phi} + \hat{\mathbf{z}} \frac{\partial \psi}{\partial z}. \tag{16.27}$$

Table 16.3: Derivatives of the basis vectors.

RECTANGULAR

$$\begin{array}{lll} \dfrac{\partial \hat{\mathbf{x}}}{\partial x} = 0 & \dfrac{\partial \hat{\mathbf{x}}}{\partial y} = 0 & \dfrac{\partial \hat{\mathbf{x}}}{\partial z} = 0 \\ \dfrac{\partial \hat{\mathbf{y}}}{\partial x} = 0 & \dfrac{\partial \hat{\mathbf{y}}}{\partial y} = 0 & \dfrac{\partial \hat{\mathbf{y}}}{\partial z} = 0 \\ \dfrac{\partial \hat{\mathbf{z}}}{\partial x} = 0 & \dfrac{\partial \hat{\mathbf{z}}}{\partial y} = 0 & \dfrac{\partial \hat{\mathbf{z}}}{\partial z} = 0 \end{array}$$

CYLINDRICAL

$$\begin{array}{lll} \dfrac{\partial \hat{\boldsymbol{\rho}}}{\partial \rho} = 0 & \dfrac{\partial \hat{\boldsymbol{\rho}}}{\partial \phi} = +\hat{\boldsymbol{\phi}} & \dfrac{\partial \hat{\boldsymbol{\rho}}}{\partial z} = 0 \\ \dfrac{\partial \hat{\boldsymbol{\phi}}}{\partial \rho} = 0 & \dfrac{\partial \hat{\boldsymbol{\phi}}}{\partial \phi} = -\hat{\boldsymbol{\rho}} & \dfrac{\partial \hat{\boldsymbol{\phi}}}{\partial z} = 0 \\ \dfrac{\partial \hat{\mathbf{z}}}{\partial \rho} = 0 & \dfrac{\partial \hat{\mathbf{z}}}{\partial \phi} = 0 & \dfrac{\partial \hat{\mathbf{z}}}{\partial z} = 0 \end{array}$$

SPHERICAL

$$\begin{array}{lll} \dfrac{\partial \hat{\mathbf{r}}}{\partial r} = 0 & \dfrac{\partial \hat{\mathbf{r}}}{\partial \theta} = +\hat{\boldsymbol{\theta}} & \dfrac{\partial \hat{\mathbf{r}}}{\partial \phi} = +\hat{\boldsymbol{\phi}} \sin\theta \\ \dfrac{\partial \hat{\boldsymbol{\theta}}}{\partial r} = 0 & \dfrac{\partial \hat{\boldsymbol{\theta}}}{\partial \theta} = -\hat{\mathbf{r}} & \dfrac{\partial \hat{\boldsymbol{\theta}}}{\partial \phi} = +\hat{\boldsymbol{\phi}} \cos\theta \\ \dfrac{\partial \hat{\boldsymbol{\phi}}}{\partial r} = 0 & \dfrac{\partial \hat{\boldsymbol{\phi}}}{\partial \theta} = 0 & \dfrac{\partial \hat{\boldsymbol{\phi}}}{\partial \phi} = -\hat{\boldsymbol{\rho}} = -\hat{\mathbf{r}} \sin\theta - \hat{\boldsymbol{\theta}} \cos\theta \end{array}$$

Again according to Table 16.1,

$$(\mathbf{b}\cdot\nabla)\mathbf{a} = b_i\frac{\partial\mathbf{a}}{\partial i}.$$

Applying the cylindrical metric coefficients, we have that

$$(\mathbf{b}\cdot\nabla)\mathbf{a} = b_\rho\frac{\partial\mathbf{a}}{\partial\rho} + b_\phi\frac{\partial\mathbf{a}}{\rho\,\partial\phi} + b_z\frac{\partial\mathbf{a}}{\partial z}. \tag{16.28}$$

Expanding the vector field $\mathbf{a}$ in the cylindrical basis,

$$(\mathbf{b}\cdot\nabla)\mathbf{a} = \left\{b_\rho\frac{\partial}{\partial\rho} + b_\phi\frac{\partial}{\rho\,\partial\phi} + b_z\frac{\partial}{\partial z}\right\}\left(\hat{\boldsymbol{\rho}}a_\rho + \hat{\boldsymbol{\phi}}a_\phi + \hat{\mathbf{z}}a_z\right).$$

Here are three derivatives of three terms, each term of two factors. Evaluating the derivatives according to the contour derivative product rule (16.23) yields (3)(3)(2) = 0x12 (eighteen) terms in the result. Half the 0x12 terms involve derivatives of the basis vectors, which Table 16.3 computes. Some of the 0x12 terms turn out to be null. The result is that

$$\begin{aligned}(\mathbf{b}\cdot\nabla)\mathbf{a} \;=\;& b_\rho\left[\hat{\boldsymbol{\rho}}\frac{\partial a_\rho}{\partial\rho} + \hat{\boldsymbol{\phi}}\frac{\partial a_\phi}{\partial\rho} + \hat{\mathbf{z}}\frac{\partial a_z}{\partial\rho}\right] \\ &+ \frac{b_\phi}{\rho}\left[\hat{\boldsymbol{\rho}}\left(\frac{\partial a_\rho}{\partial\phi} - a_\phi\right) + \hat{\boldsymbol{\phi}}\left(\frac{\partial a_\phi}{\partial\phi} + a_\rho\right) + \hat{\mathbf{z}}\frac{\partial a_z}{\partial\phi}\right] \\ &+ b_z\left[\hat{\boldsymbol{\rho}}\frac{\partial a_\rho}{\partial z} + \hat{\boldsymbol{\phi}}\frac{\partial a_\phi}{\partial z} + \hat{\mathbf{z}}\frac{\partial a_z}{\partial z}\right].\end{aligned} \tag{16.29}$$

To evaluate divergence and curl wants more care. It also wants a constant basis to work in, whereas $[\hat{\mathbf{x}}\;\hat{\mathbf{y}}\;\hat{\mathbf{z}}]$ is awkward in a cylindrical geometry and $[\hat{\boldsymbol{\rho}}\;\hat{\boldsymbol{\phi}}\;\hat{\mathbf{z}}]$ is not constant. Fortunately, nothing prevents us from defining a constant basis $[\hat{\boldsymbol{\rho}}_o\;\hat{\boldsymbol{\phi}}_o\;\hat{\mathbf{z}}]$ such that $[\hat{\boldsymbol{\rho}}\;\hat{\boldsymbol{\phi}}\;\hat{\mathbf{z}}] = [\hat{\boldsymbol{\rho}}_o\;\hat{\boldsymbol{\phi}}_o\;\hat{\mathbf{z}}]$ *at the point* $\mathbf{r} = \mathbf{r}_o$ *at which the derivative is evaluated.* If this is done, then the basis $[\hat{\boldsymbol{\rho}}_o\;\hat{\boldsymbol{\phi}}_o\;\hat{\mathbf{z}}]$ is constant like $[\hat{\mathbf{x}}\;\hat{\mathbf{y}}\;\hat{\mathbf{z}}]$ but not awkward like it.

According to Table 16.1,

$$\nabla\cdot\mathbf{a} = \frac{\partial a_i}{\partial i}$$

In cylindrical coordinates and the $[\hat{\boldsymbol{\rho}}_o\ \hat{\boldsymbol{\phi}}_o\ \hat{\mathbf{z}}]$ basis, this is[22]

$$\nabla \cdot \mathbf{a} = \frac{\partial(\hat{\boldsymbol{\rho}}_o \cdot \mathbf{a})}{\partial \rho} + \frac{\partial(\hat{\boldsymbol{\phi}}_o \cdot \mathbf{a})}{\rho\,\partial \phi} + \frac{\partial(\hat{\mathbf{z}} \cdot \mathbf{a})}{\partial z}.$$

Applying the contour derivative product rule (16.23),

$$\nabla \cdot \mathbf{a} = \hat{\boldsymbol{\rho}}_o \cdot \frac{\partial \mathbf{a}}{\partial \rho} + \frac{\partial \hat{\boldsymbol{\rho}}_o}{\partial \rho} \cdot \mathbf{a} + \hat{\boldsymbol{\phi}}_o \cdot \frac{\partial \mathbf{a}}{\rho\,\partial \phi} + \frac{\partial \hat{\boldsymbol{\phi}}_o}{\rho\,\partial \phi} \cdot \mathbf{a} + \hat{\mathbf{z}} \cdot \frac{\partial \mathbf{a}}{\partial z} + \frac{\partial \hat{\mathbf{z}}}{\partial z} \cdot \mathbf{a}.$$

But $[\hat{\boldsymbol{\rho}}_o\ \hat{\boldsymbol{\phi}}_o\ \mathbf{z}]$ are constant unit vectors, so

$$\nabla \cdot \mathbf{a} = \hat{\boldsymbol{\rho}}_o \cdot \frac{\partial \mathbf{a}}{\partial \rho} + \hat{\boldsymbol{\phi}}_o \cdot \frac{\partial \mathbf{a}}{\rho\,\partial \phi} + \hat{\mathbf{z}} \cdot \frac{\partial \mathbf{a}}{\partial z}.$$

That is,

$$\nabla \cdot \mathbf{a} = \hat{\boldsymbol{\rho}} \cdot \frac{\partial \mathbf{a}}{\partial \rho} + \hat{\boldsymbol{\phi}} \cdot \frac{\partial \mathbf{a}}{\rho\,\partial \phi} + \hat{\mathbf{z}} \cdot \frac{\partial \mathbf{a}}{\partial z}.$$

Expanding the field in the cylindrical basis,

$$\nabla \cdot \mathbf{a} = \left\{ \hat{\boldsymbol{\rho}} \cdot \frac{\partial}{\partial \rho} + \hat{\boldsymbol{\phi}} \cdot \frac{\partial}{\rho\,\partial \phi} + \hat{\mathbf{z}} \cdot \frac{\partial}{\partial z} \right\} \left( \hat{\boldsymbol{\rho}} a_\rho + \hat{\boldsymbol{\phi}} a_\phi + \hat{\mathbf{z}} a_z \right).$$

As above, here again the expansion yields 0x12 terms. Fortunately, this time most of the terms turn out to be null. The result is that

$$\nabla \cdot \mathbf{a} = \frac{\partial a_\rho}{\partial \rho} + \frac{a_\rho}{\rho} + \frac{\partial a_\phi}{\rho\,\partial \phi} + \frac{\partial a_z}{\partial z},$$

or, expressed more cleverly in light of (4.28), that

$$\nabla \cdot \mathbf{a} = \frac{\partial(\rho a_\rho)}{\rho\,\partial \rho} + \frac{\partial a_\phi}{\rho\,\partial \phi} + \frac{\partial a_z}{\partial z}. \tag{16.30}$$

Again according to Table 16.1,

$$\begin{aligned} \nabla \times \mathbf{a} &= \epsilon_{ijk} \hat{\mathbf{i}} \frac{\partial a_k}{\partial j} \\ &= \hat{\boldsymbol{\rho}}_o \left[ \frac{\partial(\hat{\mathbf{z}} \cdot \mathbf{a})}{\rho\,\partial \phi} - \frac{\partial(\hat{\boldsymbol{\phi}}_o \cdot \mathbf{a})}{\partial z} \right] + \hat{\boldsymbol{\phi}}_o \left[ \frac{\partial(\hat{\boldsymbol{\rho}}_o \cdot \mathbf{a})}{\partial z} - \frac{\partial(\hat{\mathbf{z}} \cdot \mathbf{a})}{\partial \rho} \right] \\ &\quad + \hat{\mathbf{z}} \left[ \frac{\partial(\hat{\boldsymbol{\phi}}_o \cdot \mathbf{a})}{\partial \rho} - \frac{\partial(\hat{\boldsymbol{\rho}}_o \cdot \mathbf{a})}{\rho\,\partial \phi} \right]. \end{aligned}$$

[22] Mistakenly to write here that

$$\nabla \cdot \mathbf{a} = \frac{\partial a_\rho}{\partial \rho} + \frac{\partial a_\phi}{\rho\,\partial \phi} + \frac{\partial a_z}{\partial z},$$

which is not true, would be a ghastly error, leading to any number of hard-to-detect false conclusions. Refer to § 16.6.2.

That is,

$$\begin{aligned}\nabla \times \mathbf{a} &= \hat{\boldsymbol{\rho}}\left[\hat{\mathbf{z}} \cdot \frac{\partial \mathbf{a}}{\rho\,\partial\phi} - \hat{\boldsymbol{\phi}} \cdot \frac{\partial \mathbf{a}}{\partial z}\right] + \hat{\boldsymbol{\phi}}\left[\hat{\boldsymbol{\rho}} \cdot \frac{\partial \mathbf{a}}{\partial z} - \hat{\mathbf{z}} \cdot \frac{\partial \mathbf{a}}{\partial \rho}\right] \\ &\quad + \hat{\mathbf{z}}\left[\hat{\boldsymbol{\phi}} \cdot \frac{\partial \mathbf{a}}{\partial \rho} - \hat{\boldsymbol{\rho}} \cdot \frac{\partial \mathbf{a}}{\rho\,\partial\phi}\right].\end{aligned}$$

Expanding the field in the cylindrical basis,

$$\begin{aligned}\nabla \times \mathbf{a} &= \left\{\hat{\boldsymbol{\rho}}\left[\hat{\mathbf{z}} \cdot \frac{\partial}{\rho\,\partial\phi} - \hat{\boldsymbol{\phi}} \cdot \frac{\partial}{\partial z}\right] + \hat{\boldsymbol{\phi}}\left[\hat{\boldsymbol{\rho}} \cdot \frac{\partial}{\partial z} - \hat{\mathbf{z}} \cdot \frac{\partial}{\partial \rho}\right]\right. \\ &\quad \left. + \hat{\mathbf{z}}\left[\hat{\boldsymbol{\phi}} \cdot \frac{\partial}{\partial \rho} - \hat{\boldsymbol{\rho}} \cdot \frac{\partial}{\rho\,\partial\phi}\right]\right\}\left(\hat{\boldsymbol{\rho}} a_\rho + \hat{\boldsymbol{\phi}} a_\phi + \hat{\mathbf{z}} a_z\right).\end{aligned}$$

Here the expansion yields 0x24 terms, but fortunately as last time this time most of the terms again turn out to be null. The result is that

$$\nabla \times \mathbf{a} = \hat{\boldsymbol{\rho}}\left[\frac{\partial a_z}{\rho\,\partial\phi} - \frac{\partial a_\phi}{\partial z}\right] + \hat{\boldsymbol{\phi}}\left[\frac{\partial a_\rho}{\partial z} - \frac{\partial a_z}{\partial \rho}\right] + \hat{\mathbf{z}}\left[\frac{\partial a_\phi}{\partial \rho} + \frac{a_\phi}{\rho} - \frac{\partial a_\rho}{\rho\,\partial\phi}\right],$$

or, expressed more cleverly, that

$$\nabla \times \mathbf{a} = \hat{\boldsymbol{\rho}}\left[\frac{\partial a_z}{\rho\,\partial\phi} - \frac{\partial a_\phi}{\partial z}\right] + \hat{\boldsymbol{\phi}}\left[\frac{\partial a_\rho}{\partial z} - \frac{\partial a_z}{\partial \rho}\right] + \frac{\hat{\mathbf{z}}}{\rho}\left[\frac{\partial(\rho a_\phi)}{\partial \rho} - \frac{\partial a_\rho}{\partial \phi}\right]. \quad (16.31)$$

Table 16.4 summarizes.

One can compute a second-order vector derivative in cylindrical coordinates as a sequence of two first-order cylindrical vector derivatives. For example, because Table 16.1 gives the scalar Laplacian as $\nabla^2\psi = \nabla \cdot \nabla\psi$, one can calculate $\nabla^2\psi$ in cylindrical coordinates by taking the divergence of $\psi$'s gradient.[23] To calculate the vector Laplacian $\nabla^2\mathbf{a}$ in cylindrical coordinates is tedious but nonetheless can with care be done accurately by means of Table 16.1's identity that $\nabla^2\mathbf{a} = \nabla\nabla \cdot \mathbf{a} - \nabla \times \nabla \times \mathbf{a}$. (This means that to calculate the vector Laplacian $\nabla^2\mathbf{a}$ in cylindrical coordinates takes not just two but actually four first-order cylindrical vector derivatives, for the author regrettably knows of no valid shortcut—the clumsy alternative,

[23] A concrete example: if $\psi(\mathbf{r}) = e^{i\phi}/\rho$, then $\nabla\psi = (-\hat{\boldsymbol{\rho}} + i\hat{\boldsymbol{\phi}})e^{i\phi}/\rho^2$ per Table 16.4, whereupon

$$\nabla^2\psi = \nabla \cdot \left[\left(-\hat{\boldsymbol{\rho}} + i\hat{\boldsymbol{\phi}}\right)\frac{e^{i\phi}}{\rho^2}\right] = \left(-\hat{\boldsymbol{\rho}} + i\hat{\boldsymbol{\phi}}\right) \cdot \nabla\left(\frac{e^{i\phi}}{\rho^2}\right) + \frac{e^{i\phi}}{\rho^2}\nabla \cdot \left(-\hat{\boldsymbol{\rho}} + i\hat{\boldsymbol{\phi}}\right).$$

To finish the example is left as an exercise.

Table 16.4: Vector derivatives in cylindrical coordinates.

$$
\begin{aligned}
\nabla\psi &= \hat{\boldsymbol{\rho}}\frac{\partial\psi}{\partial\rho} + \hat{\boldsymbol{\phi}}\frac{\partial\psi}{\rho\,\partial\phi} + \hat{\mathbf{z}}\frac{\partial\psi}{\partial z} \\
(\mathbf{b}\cdot\nabla)\mathbf{a} &= b_\rho\frac{\partial\mathbf{a}}{\partial\rho} + b_\phi\frac{\partial\mathbf{a}}{\rho\,\partial\phi} + b_z\frac{\partial\mathbf{a}}{\partial z} \\
&= b_\rho\left[\hat{\boldsymbol{\rho}}\frac{\partial a_\rho}{\partial\rho} + \hat{\boldsymbol{\phi}}\frac{\partial a_\phi}{\partial\rho} + \hat{\mathbf{z}}\frac{\partial a_z}{\partial\rho}\right] \\
&\quad + \frac{b_\phi}{\rho}\left[\hat{\boldsymbol{\rho}}\left(\frac{\partial a_\rho}{\partial\phi} - a_\phi\right) + \hat{\boldsymbol{\phi}}\left(\frac{\partial a_\phi}{\partial\phi} + a_\rho\right) + \hat{\mathbf{z}}\frac{\partial a_z}{\partial\phi}\right] \\
&\quad + b_z\left[\hat{\boldsymbol{\rho}}\frac{\partial a_\rho}{\partial z} + \hat{\boldsymbol{\phi}}\frac{\partial a_\phi}{\partial z} + \hat{\mathbf{z}}\frac{\partial a_z}{\partial z}\right] \\
\nabla\cdot\mathbf{a} &= \frac{\partial(\rho a_\rho)}{\rho\,\partial\rho} + \frac{\partial a_\phi}{\rho\,\partial\phi} + \frac{\partial a_z}{\partial z} \\
\nabla\times\mathbf{a} &= \hat{\boldsymbol{\rho}}\left[\frac{\partial a_z}{\rho\,\partial\phi} - \frac{\partial a_\phi}{\partial z}\right] + \hat{\boldsymbol{\phi}}\left[\frac{\partial a_\rho}{\partial z} - \frac{\partial a_z}{\partial\rho}\right] + \frac{\hat{\mathbf{z}}}{\rho}\left[\frac{\partial(\rho a_\phi)}{\partial\rho} - \frac{\partial a_\rho}{\partial\phi}\right]
\end{aligned}
$$

less proper, less insightful, even more tedious and not recommended, being to take the Laplacian in rectangular coordinates and then to convert back to the cylindrical domain; for to work cylindrical problems directly in cylindrical coordinates is almost always advisable.)

### 16.9.2 Derivatives in spherical coordinates

One can compute vector derivatives in spherical coordinates as in cylindrical coordinates (§ 16.9.1), only the spherical details though not essentially more complicated are messier. According to Table 16.1,

$$\nabla\psi = \hat{\mathbf{i}}\frac{\partial\psi}{\partial i}.$$

Applying the spherical metric coefficients of Table 16.2, we have that

$$\nabla\psi = \hat{\mathbf{r}}\frac{\partial\psi}{\partial r} + \hat{\boldsymbol{\theta}}\frac{\partial\psi}{r\,\partial\theta} + \hat{\boldsymbol{\phi}}\frac{\partial\psi}{(r\sin\theta)\,\partial\phi}. \tag{16.32}$$

Again according to Table 16.1,

$$(\mathbf{b}\cdot\nabla)\mathbf{a} = b_i\frac{\partial\mathbf{a}}{\partial i}.$$

Applying the cylindrical metric coefficients, we have that

$$(\mathbf{b}\cdot\nabla)\mathbf{a} = b_r\frac{\partial\mathbf{a}}{\partial r} + b_\theta\frac{\partial\mathbf{a}}{r\,\partial\theta} + b_\phi\frac{\partial\mathbf{a}}{(r\sin\theta)\,\partial\phi}. \tag{16.33}$$

Expanding the vector field $\mathbf{a}$ in the spherical basis,

$$(\mathbf{b}\cdot\nabla)\mathbf{a} = \left\{b_r\frac{\partial}{\partial r} + b_\theta\frac{\partial}{r\,\partial\theta} + b_\phi\frac{\partial}{(r\sin\theta)\,\partial\phi}\right\}\left(\hat{\mathbf{r}}a_r + \hat{\boldsymbol{\theta}}a_\theta + \hat{\boldsymbol{\phi}}a_\phi\right).$$

Evaluating the derivatives,

$$\begin{aligned}(\mathbf{b}\cdot\nabla)\mathbf{a} &= b_r\left[\hat{\mathbf{r}}\frac{\partial a_r}{\partial r} + \hat{\boldsymbol{\theta}}\frac{\partial a_\theta}{\partial r} + \hat{\boldsymbol{\phi}}\frac{\partial a_\phi}{\partial r}\right]\\ &\quad + \frac{b_\theta}{r}\left[\hat{\mathbf{r}}\left(\frac{\partial a_r}{\partial\theta} - a_\theta\right) + \hat{\boldsymbol{\theta}}\left(\frac{\partial a_\theta}{\partial\theta} + a_r\right) + \hat{\boldsymbol{\phi}}\frac{\partial a_\phi}{\partial\theta}\right]\\ &\quad + \frac{b_\phi}{r\sin\theta}\left[\hat{\mathbf{r}}\left(\frac{\partial a_r}{\partial\phi} - a_\phi\sin\theta\right) + \hat{\boldsymbol{\theta}}\left(\frac{\partial a_\theta}{\partial\phi} - a_\phi\cos\theta\right)\right.\\ &\qquad \left. + \hat{\boldsymbol{\phi}}\left(\frac{\partial a_\phi}{\partial\phi} + a_r\sin\theta + a_\theta\cos\theta\right)\right].\end{aligned} \tag{16.34}$$

According to Table 16.1, reasoning as in § 16.9.1,

$$\nabla\cdot\mathbf{a} = \frac{\partial a_i}{\partial i} = \hat{\mathbf{r}}\cdot\frac{\partial\mathbf{a}}{\partial r} + \hat{\boldsymbol{\theta}}\cdot\frac{\partial\mathbf{a}}{r\,\partial\theta} + \hat{\boldsymbol{\phi}}\cdot\frac{\partial\mathbf{a}}{(r\sin\theta)\,\partial\phi}.$$

Expanding the field in the spherical basis,

$$\nabla\cdot\mathbf{a} = \left\{\hat{\mathbf{r}}\cdot\frac{\partial}{\partial r} + \hat{\boldsymbol{\theta}}\cdot\frac{\partial}{r\,\partial\theta} + \hat{\boldsymbol{\phi}}\cdot\frac{\partial}{(r\sin\theta)\,\partial\phi}\right\}\left(\hat{\mathbf{r}}a_r + \hat{\boldsymbol{\theta}}a_\theta + \hat{\boldsymbol{\phi}}a_\phi\right).$$

Evaluating the derivatives, the result is that

$$\nabla\cdot\mathbf{a} = \frac{\partial a_r}{\partial r} + \frac{2a_r}{r} + \frac{\partial a_\theta}{r\,\partial\theta} + \frac{a_\theta}{r\tan\theta} + \frac{\partial a_\phi}{(r\sin\theta)\,\partial\phi},$$

or, expressed more cleverly, that

$$\nabla\cdot\mathbf{a} = \frac{1}{r}\left[\frac{\partial(r^2a_r)}{r\,\partial r} + \frac{\partial(a_\theta\sin\theta)}{(\sin\theta)\,\partial\theta} + \frac{\partial a_\phi}{(\sin\theta)\,\partial\phi}\right]. \tag{16.35}$$

Again according to Table 16.1, reasoning as in § 16.9.1,

$$\begin{aligned}\nabla\times\mathbf{a} &= \epsilon_{ijk}\hat{\mathbf{i}}\frac{\partial a_k}{\partial j}\\ &= \hat{\mathbf{r}}\left[\hat{\boldsymbol{\phi}}\cdot\frac{\partial\mathbf{a}}{r\,\partial\theta} - \hat{\boldsymbol{\theta}}\cdot\frac{\partial\mathbf{a}}{(r\sin\theta)\,\partial\phi}\right] + \hat{\boldsymbol{\theta}}\left[\hat{\mathbf{r}}\cdot\frac{\partial\mathbf{a}}{(r\sin\theta)\,\partial\phi} - \hat{\boldsymbol{\phi}}\cdot\frac{\partial\mathbf{a}}{\partial r}\right]\\ &\quad + \hat{\boldsymbol{\phi}}\left[\hat{\boldsymbol{\theta}}\cdot\frac{\partial\mathbf{a}}{\partial r} - \hat{\mathbf{r}}\cdot\frac{\partial\mathbf{a}}{r\,\partial\theta}\right].\end{aligned}$$

Expanding the field in the spherical basis,

$$\begin{aligned}\nabla \times \mathbf{a} &= \left\{ \hat{\mathbf{r}} \left[ \hat{\boldsymbol{\phi}} \cdot \frac{\partial}{r\,\partial\theta} - \hat{\boldsymbol{\theta}} \cdot \frac{\partial}{(r\sin\theta)\,\partial\phi} \right] + \hat{\boldsymbol{\theta}} \left[ \hat{\mathbf{r}} \cdot \frac{\partial}{(r\sin\theta)\,\partial\phi} - \hat{\boldsymbol{\phi}} \cdot \frac{\partial}{\partial r} \right] \right. \\ &\quad \left. + \hat{\boldsymbol{\phi}} \left[ \hat{\boldsymbol{\theta}} \cdot \frac{\partial}{\partial r} - \hat{\mathbf{r}} \cdot \frac{\partial}{r\,\partial\theta} \right] \right\} \left( \hat{\mathbf{r}} a_r + \hat{\boldsymbol{\theta}} a_\theta + \hat{\boldsymbol{\phi}} a_\phi \right).\end{aligned}$$

Evaluating the derivatives, the result is that

$$\begin{aligned}\nabla \times \mathbf{a} &= \hat{\mathbf{r}} \left[ \frac{\partial a_\phi}{r\,\partial\theta} + \frac{a_\phi}{r\tan\theta} - \frac{\partial a_\theta}{(r\sin\theta)\,\partial\phi} \right] + \hat{\boldsymbol{\theta}} \left[ \frac{\partial a_r}{(r\sin\theta)\,\partial\phi} - \frac{\partial a_\phi}{\partial r} - \frac{a_\phi}{r} \right] \\ &\quad + \hat{\boldsymbol{\phi}} \left[ \frac{\partial a_\theta}{\partial r} + \frac{a_\theta}{r} - \frac{\partial a_r}{r\,\partial\theta} \right],\end{aligned}$$

or, expressed more cleverly, that

$$\begin{aligned}\nabla \times \mathbf{a} &= \frac{\hat{\mathbf{r}}}{r\sin\theta} \left[ \frac{\partial(a_\phi \sin\theta)}{\partial\theta} - \frac{\partial a_\theta}{\partial\phi} \right] + \frac{\hat{\boldsymbol{\theta}}}{r} \left[ \frac{\partial a_r}{(\sin\theta)\,\partial\phi} - \frac{\partial(r a_\phi)}{\partial r} \right] \\ &\quad + \frac{\hat{\boldsymbol{\phi}}}{r} \left[ \frac{\partial(r a_\theta)}{\partial r} - \frac{\partial a_r}{\partial\theta} \right].\end{aligned} \tag{16.36}$$

Table 16.5 summarizes.

One can compute a second-order vector derivative in spherical coordinates as in cylindrical coordinates, as a sequence of two first-order vector derivatives. Refer to § 16.9.1.

### 16.9.3 Finding the derivatives geometrically

The method of §§ 16.9.1 and 16.9.2 is general, reliable and correct, but there exists an alternate, arguably neater method to derive nonrectangular formulas for most vector derivatives. Adapting the notation to this subsection's purpose we can write (16.9) as

$$\nabla \cdot \mathbf{a}(\mathbf{r}) \equiv \lim_{\Delta V \to 0} \frac{\Phi}{\Delta V}, \tag{16.37}$$

thus defining a vector's divergence fundamentally as in § 16.1.4, geometrically, as the ratio of flux $\Phi$ from a vanishing test volume $\Delta V$ to the volume itself; where per (16.7) $\Phi = \oint_S \mathbf{a}(\mathbf{r}') \cdot d\mathbf{s}$, where $\mathbf{r}'$ is a position on the test volume's surface, and where $d\mathbf{s} = d\mathbf{s}(\mathbf{r}')$ is the corresponding surface patch.

Table 16.5: Vector derivatives in spherical coordinates.

$$
\begin{aligned}
\nabla\psi &= \hat{\mathbf{r}}\frac{\partial\psi}{\partial r} + \hat{\boldsymbol{\theta}}\frac{\partial\psi}{r\,\partial\theta} + \hat{\boldsymbol{\phi}}\frac{\partial\psi}{(r\sin\theta)\,\partial\phi} \\
(\mathbf{b}\cdot\nabla)\mathbf{a} &= b_r\frac{\partial\mathbf{a}}{\partial r} + b_\theta\frac{\partial\mathbf{a}}{r\,\partial\theta} + b_\phi\frac{\partial\mathbf{a}}{(r\sin\theta)\,\partial\phi} \\
&= b_r\left[\hat{\mathbf{r}}\frac{\partial a_r}{\partial r} + \hat{\boldsymbol{\theta}}\frac{\partial a_\theta}{\partial r} + \hat{\boldsymbol{\phi}}\frac{\partial a_\phi}{\partial r}\right] \\
&\quad + \frac{b_\theta}{r}\left[\hat{\mathbf{r}}\left(\frac{\partial a_r}{\partial\theta} - a_\theta\right) + \hat{\boldsymbol{\theta}}\left(\frac{\partial a_\theta}{\partial\theta} + a_r\right) + \hat{\boldsymbol{\phi}}\frac{\partial a_\phi}{\partial\theta}\right] \\
&\quad + \frac{b_\phi}{r\sin\theta}\left[\hat{\mathbf{r}}\left(\frac{\partial a_r}{\partial\phi} - a_\phi\sin\theta\right) + \hat{\boldsymbol{\theta}}\left(\frac{\partial a_\theta}{\partial\phi} - a_\phi\cos\theta\right)\right. \\
&\qquad\qquad \left. + \hat{\boldsymbol{\phi}}\left(\frac{\partial a_\phi}{\partial\phi} + a_r\sin\theta + a_\theta\cos\theta\right)\right] \\
\nabla\cdot\mathbf{a} &= \frac{1}{r}\left[\frac{\partial(r^2 a_r)}{r\,\partial r} + \frac{\partial(a_\theta\sin\theta)}{(\sin\theta)\,\partial\theta} + \frac{\partial a_\phi}{(\sin\theta)\,\partial\phi}\right] \\
\nabla\times\mathbf{a} &= \frac{\hat{\mathbf{r}}}{r\sin\theta}\left[\frac{\partial(a_\phi\sin\theta)}{\partial\theta} - \frac{\partial a_\theta}{\partial\phi}\right] + \frac{\hat{\boldsymbol{\theta}}}{r}\left[\frac{\partial a_r}{(\sin\theta)\,\partial\phi} - \frac{\partial(r a_\phi)}{\partial r}\right] \\
&\quad + \frac{\hat{\boldsymbol{\phi}}}{r}\left[\frac{\partial(r a_\theta)}{\partial r} - \frac{\partial a_r}{\partial\theta}\right]
\end{aligned}
$$

So long as the test volume $\Delta V$ includes the point $\mathbf{r}$ and is otherwise infinitesimal in extent, we remain free to shape the volume as we like,[24] so let us give it six sides and shape it as an almost rectangular box that conforms precisely to the coordinate system $(\alpha;\beta;\gamma)$ in use:

$$\begin{aligned}
\alpha - \frac{\Delta\alpha}{2} \le \alpha' \le \alpha + \frac{\Delta\alpha}{2};\\
\beta - \frac{\Delta\beta}{2} \le \beta' \le \beta + \frac{\Delta\beta}{2};\\
\gamma - \frac{\Delta\gamma}{2} \le \gamma' \le \gamma + \frac{\Delta\gamma}{2}.
\end{aligned}$$

The fluxes outward through the box's $+\alpha$- and $-\alpha$-ward sides will then be[25]

$$\begin{aligned}
\Phi_{+\alpha} &= (+a_\alpha)(h_\beta h_\gamma\,\Delta\beta\,\Delta\gamma)\big|_{\mathbf{r}'=\mathbf{r}(\alpha+\Delta\alpha/2;\beta;\gamma)}\\
&= +a_\alpha h_\beta h_\gamma\big|_{\mathbf{r}'=\mathbf{r}(\alpha+\Delta\alpha/2;\beta;\gamma)}\,\Delta\beta\,\Delta\gamma,\\
\Phi_{-\alpha} &= (-a_\alpha)(h_\beta h_\gamma\,\Delta\beta\,\Delta\gamma)\big|_{\mathbf{r}'=\mathbf{r}(\alpha-\Delta\alpha/2;\beta;\gamma)}\\
&= -a_\alpha h_\beta h_\gamma\big|_{\mathbf{r}'=\mathbf{r}(\alpha-\Delta\alpha/2;\beta;\gamma)}\,\Delta\beta\,\Delta\gamma,
\end{aligned}$$

products of the outward-directed field components and the areas (16.24) of the sides through which the fields pass. Thence by successive steps, the net

[24] A professional mathematician would probably enjoin the volume's shape to obey certain technical restrictions, such as that it remain wholly enclosed within a sphere of vanishing radius, but we will not try for such a level of rigor here.

[25] More rigorously, one might digress from this point to expand the field in a three-dimensional Taylor series (§ 8.16) to account for the field's variation over a single side of the test volume. So lengthy a digression however would only formalize what we already knew; namely, that one can approximate to first order the integral of a well-behaved quantity over an infinitesimal domain by the quantity's value at the domain's midpoint. If you will believe that $\lim_{\Delta\tau\to 0}\int_{\tau-\Delta\tau/2}^{\tau+\Delta\tau/2} f(\tau')\,d\tau' = f(\tau)\,\Delta\tau$ for any $\tau$ in the neighborhood of which $f(\tau)$ is well behaved, then you will probably also believe its three-dimensional analog in the narrative. (If the vagueness in this context of the adjective "well-behaved" deeply troubles any reader then that reader may possess the worthy temperament of a professional mathematician; he might review Ch. 8 and then seek further illumination in the professional mathematical literature. Other readers, of more practical temperament, are advised to visualize test volumes in rectangular, cylindrical and spherical coordinates and to ponder the matter a while. Consider: if the field grows in strength across a single side of the test volume and if the test volume is small enough that second-order effects can be ignored, then what single value ought one to choose to represent the field over the whole side but its value at the side's midpoint? Such visualization should soon clear up any confusion and is what the writer recommends. Incidentally, the contrast between the two modes of thought this footnote reveals is exactly the sort of thing Courant and Hilbert were talking about in § 1.2.1.)

flux outward through the pair of opposing sides will be

$$\begin{aligned}
\Phi_\alpha &= \Phi_{+\alpha} + \Phi_{-\alpha} \\
&= \left[ a_\alpha h_\beta h_\gamma \big|_{\mathbf{r}'=\mathbf{r}(\alpha+\Delta\alpha/2;\beta;\gamma)} - a_\alpha h_\beta h_\gamma \big|_{\mathbf{r}'=\mathbf{r}(\alpha-\Delta\alpha/2;\beta;\gamma)} \right] \Delta\beta\,\Delta\gamma \\
&= \left[ \frac{\partial(a_\alpha h_\beta h_\gamma)}{\partial\alpha}\,\Delta\alpha \right] \Delta\beta\,\Delta\gamma = \Delta\alpha\,\Delta\beta\,\Delta\gamma\,\frac{\partial(a_\alpha h_\beta h_\gamma)}{\partial\alpha} \\
&= \frac{\Delta V\,\partial(a_\alpha h_\beta h_\gamma)}{h_\alpha h_\beta h_\gamma\,\partial\alpha}.
\end{aligned}$$

Naturally, the same goes for the other two pairs of sides:

$$\Phi_\beta = \frac{\Delta V\,\partial(a_\beta h_\gamma h_\alpha)}{h_\alpha h_\beta h_\gamma\,\partial\beta};$$
$$\Phi_\gamma = \frac{\Delta V\,\partial(a_\gamma h_\alpha h_\beta)}{h_\alpha h_\beta h_\gamma\,\partial\gamma}.$$

The three equations are better written

$$\Phi_\alpha = \frac{\Delta V\,\partial}{h^3\,\partial\alpha}\left(\frac{h^3 a_\alpha}{h_\alpha}\right),$$
$$\Phi_\beta = \frac{\Delta V\,\partial}{h^3\,\partial\beta}\left(\frac{h^3 a_\beta}{h_\beta}\right),$$
$$\Phi_\gamma = \frac{\Delta V\,\partial}{h^3\,\partial\gamma}\left(\frac{h^3 a_\gamma}{h_\gamma}\right),$$

where

$$h^3 \equiv h_\alpha h_\beta h_\gamma. \tag{16.38}$$

The total flux from the test volume then is

$$\begin{aligned}
\Phi &= \Phi_\alpha + \Phi_\beta + \Phi_\gamma \\
&= \frac{\Delta V}{h^3}\left[\frac{\partial}{\partial\alpha}\left(\frac{h^3 a_\alpha}{h_\alpha}\right) + \frac{\partial}{\partial\beta}\left(\frac{h^3 a_\beta}{h_\beta}\right) + \frac{\partial}{\partial\gamma}\left(\frac{h^3 a_\gamma}{h_\gamma}\right)\right];
\end{aligned}$$

or, invoking Einstein's summation convention in § 16.7's modified style,

$$\Phi = \frac{\Delta V\,\partial}{h^3\,\partial\tilde{\imath}}\left(\frac{h^3 a_{\tilde{\imath}}}{h_{\tilde{\imath}}}\right).$$

Finally, substituting the last equation into (16.37),

$$\nabla\cdot\mathbf{a} = \frac{\partial}{h^3\,\partial\tilde{\imath}}\left(\frac{h^3 a_{\tilde{\imath}}}{h_{\tilde{\imath}}}\right). \tag{16.39}$$

An analogous formula for curl is not much harder to derive but is harder to approach directly, so we will approach it by deriving first the formula for $\hat{\boldsymbol{\gamma}}$-directed directional curl. Equation (16.12) has it that[26]

$$\hat{\boldsymbol{\gamma}} \cdot \nabla \times \mathbf{a}(\mathbf{r}) \equiv \lim_{\Delta A \to 0} \frac{\Gamma}{\Delta A}, \tag{16.40}$$

where per (16.11) $\Gamma = \oint_\gamma \mathbf{a}(\mathbf{r}') \cdot d\boldsymbol{\ell}$ and the notation $\oint_\gamma$ reminds us that the contour of integration lies in the $\alpha$-$\beta$ plane, perpendicular to $\hat{\boldsymbol{\gamma}}$. In this case the contour of integration bounds not a test volume but a test surface, which we give four edges and an almost rectangular shape that conforms precisely to the coordinate system $(\alpha; \beta; \gamma)$ in use:

$$\begin{aligned} \alpha - \frac{\Delta\alpha}{2} \le \alpha' \le \alpha + \frac{\Delta\alpha}{2}; \\ \beta - \frac{\Delta\beta}{2} \le \beta' \le \beta + \frac{\Delta\beta}{2}; \\ \gamma' = \gamma. \end{aligned}$$

The circulations along the $+\alpha$- and $-\alpha$-ward edges will be

$$\begin{aligned} \Gamma_{+\alpha} &= +h_\beta a_\beta \big|_{\mathbf{r}'=\mathbf{r}(\alpha+\Delta\alpha/2;\beta;\gamma)} \, \Delta\beta, \\ \Gamma_{-\alpha} &= -h_\beta a_\beta \big|_{\mathbf{r}'=\mathbf{r}(\alpha-\Delta\alpha/2;\beta;\gamma)} \, \Delta\beta, \end{aligned}$$

and likewise the circulations along the $-\beta$- and $+\beta$-ward edges will be

$$\begin{aligned} \Gamma_{-\beta} &= +h_\alpha a_\alpha \big|_{\mathbf{r}'=\mathbf{r}(\alpha;\beta-\Delta\beta/2;\gamma)} \, \Delta\alpha, \\ \Gamma_{+\beta} &= -h_\alpha a_\alpha \big|_{\mathbf{r}'=\mathbf{r}(\alpha;\beta+\Delta\beta/2;\gamma)} \, \Delta\alpha, \end{aligned}$$

whence the total circulation about the contour is

$$\begin{aligned} \Gamma &= \frac{\partial(h_\beta a_\beta)}{\partial\alpha} \Delta\alpha \, \Delta\beta - \frac{\partial(h_\alpha a_\alpha)}{\partial\beta} \Delta\beta \, \Delta\alpha \\ &= \frac{h_\gamma \, \Delta A}{h^3} \left[ \frac{\partial(h_\beta a_\beta)}{\partial\alpha} - \frac{\partial(h_\alpha a_\alpha)}{\partial\beta} \right]. \end{aligned}$$

Substituting the last equation into (16.40), we have that

$$\hat{\boldsymbol{\gamma}} \cdot \nabla \times \mathbf{a} = \frac{h_\gamma}{h^3} \left[ \frac{\partial(h_\beta a_\beta)}{\partial\alpha} - \frac{\partial(h_\alpha a_\alpha)}{\partial\beta} \right].$$

[26] The appearance of both $\mathbf{a}$ and $A$ in (16.40) is unfortunate but coincidental, as is the appearance of both $\hat{\boldsymbol{\gamma}}$ and $\Gamma$. The capital and minuscule symbols here represent unrelated quantities.

Likewise,

$$\hat{\boldsymbol{\alpha}} \cdot \nabla \times \mathbf{a} = \frac{h_\alpha}{h^3}\left[\frac{\partial(h_\gamma a_\gamma)}{\partial\beta} - \frac{\partial(h_\beta a_\beta)}{\partial\gamma}\right],$$
$$\hat{\boldsymbol{\beta}} \cdot \nabla \times \mathbf{a} = \frac{h_\beta}{h^3}\left[\frac{\partial(h_\alpha a_\alpha)}{\partial\gamma} - \frac{\partial(h_\gamma a_\gamma)}{\partial\alpha}\right].$$

But one can split any vector $\mathbf{v}$ into locally rectangular components as $\mathbf{v} = \hat{\boldsymbol{\alpha}}(\hat{\boldsymbol{\alpha}} \cdot \mathbf{v}) + \hat{\boldsymbol{\beta}}(\hat{\boldsymbol{\beta}} \cdot \mathbf{v}) + \hat{\boldsymbol{\gamma}}(\hat{\boldsymbol{\gamma}} \cdot \mathbf{v})$, so

$$\begin{aligned}
\nabla \times \mathbf{a} &= \hat{\boldsymbol{\alpha}}(\hat{\boldsymbol{\alpha}} \cdot \nabla \times \mathbf{a}) + \hat{\boldsymbol{\beta}}(\hat{\boldsymbol{\beta}} \cdot \nabla \times \mathbf{a}) + \hat{\boldsymbol{\gamma}}(\hat{\boldsymbol{\gamma}} \cdot \nabla \times \mathbf{a}) \\
&= \frac{\hat{\boldsymbol{\alpha}} h_\alpha}{h^3}\left[\frac{\partial(h_\gamma a_\gamma)}{\partial\beta} - \frac{\partial(h_\beta a_\beta)}{\partial\gamma}\right] + \frac{\hat{\boldsymbol{\beta}} h_\beta}{h^3}\left[\frac{\partial(h_\alpha a_\alpha)}{\partial\gamma} - \frac{\partial(h_\gamma a_\gamma)}{\partial\alpha}\right] \\
&\qquad + \frac{\hat{\boldsymbol{\gamma}} h_\gamma}{h^3}\left[\frac{\partial(h_\beta a_\beta)}{\partial\alpha} - \frac{\partial(h_\alpha a_\alpha)}{\partial\beta}\right] \\
&= \frac{1}{h^3}\begin{vmatrix} \hat{\boldsymbol{\alpha}} h_\alpha & \hat{\boldsymbol{\beta}} h_\beta & \hat{\boldsymbol{\gamma}} h_\gamma \\ \partial/\partial\alpha & \partial/\partial\beta & \partial/\partial\gamma \\ h_\alpha a_\alpha & h_\beta a_\beta & h_\gamma a_\gamma \end{vmatrix};
\end{aligned}$$

or, in Einstein notation,[27]

$$\nabla \times \mathbf{a} = \frac{\epsilon_{\tilde{\imath}\tilde{\jmath}\tilde{k}} \hat{\mathbf{\imath}} h_{\tilde{\imath}}\, \partial(h_{\tilde{k}} a_{\tilde{k}})}{h^3\, \partial\tilde{\jmath}}. \tag{16.41}$$

Compared to the formulas (16.39) and (16.41) for divergence and curl, the corresponding gradient formula is trivial. It is

$$\nabla\psi = \frac{\hat{\mathbf{\imath}}\, \partial\psi}{h_{\tilde{\imath}}\, \partial\tilde{\imath}}. \tag{16.42}$$

One can generate most of the vector-derivative formulas of Tables 16.4 and 16.5 by means of this subsection's (16.39), (16.41) and (16.42). One can generate additional vector-derivative formulas for special coordinate systems like the parabolic systems of § 15.7 by means of the same equations.

[27] What a marvel mathematical notation is! If you can read (16.41) and understand the message it conveys, then let us pause a moment to appreciate a few of the many concepts the notation implicitly encapsulates. There are the vector, the unit vector, the field, the derivative, the integral, circulation, parity, rotational invariance, nonrectangular coordinates, three-dimensional geometry, the dummy variable and so on—each of which concepts itself yet encapsulates several further ideas—not to mention multiplication and division which themselves are not trivial. It is doubtful that one could explain it even tersely to the uninitiated in fewer than fifty pages, and yet to the initiated one can express it all in half a line.

## 16.10 Vector infinitesimals

To integrate a field over a contour or surface is a typical maneuver of vector calculus. One might integrate in any of the forms

$$\begin{array}{cccc} \displaystyle\int_C \psi\,d\ell & \displaystyle\int_C \mathbf{a}\,d\ell & \displaystyle\int_S \psi\,ds & \displaystyle\int_S \mathbf{a}\,ds \\ \displaystyle\int_C \psi\,d\boldsymbol{\ell} & \displaystyle\int_C \mathbf{a}\cdot d\boldsymbol{\ell} & \displaystyle\int_S \psi\,d\mathbf{s} & \displaystyle\int_S \mathbf{a}\cdot d\mathbf{s} \\ & \displaystyle\int_C \mathbf{a}\times d\boldsymbol{\ell} & & \displaystyle\int_S \mathbf{a}\times d\mathbf{s} \end{array}$$

among others. Where the integration is over a contour, a pair of functions $\alpha(\gamma)$ and $\beta(\gamma)$ typically can serve to specify the contour. Where over a surface, a single function $\gamma(\alpha;\beta)$ can serve. Given such functions and a field integral to compute, one wants an expression for the integrand's infinitesimal $d\boldsymbol{\ell}$ or $d\mathbf{s}$ in terms respectively of the contour functions $\alpha(\gamma)$ and $\beta(\gamma)$ or of the surface function $\gamma(\alpha;\beta)$.

The contour infinitesimal is evidently

$$d\boldsymbol{\ell} = \left(\hat{\boldsymbol{\gamma}}h_\gamma + \hat{\boldsymbol{\alpha}}\frac{h_\alpha\,d\alpha}{d\gamma} + \hat{\boldsymbol{\beta}}\frac{h_\beta\,d\beta}{d\gamma}\right)\,d\gamma, \tag{16.43}$$

consisting of a step in the $\hat{\boldsymbol{\gamma}}$ direction plus the corresponding steps in the orthogonal $\hat{\boldsymbol{\alpha}}$ and $\hat{\boldsymbol{\beta}}$ directions. This is easy once you see how to do it. Harder is the surface infinitesimal, but one can nevertheless correctly construct it as the cross product

$$\begin{aligned} d\mathbf{s} &= \left[\left(\hat{\boldsymbol{\alpha}}h_\alpha + \hat{\boldsymbol{\gamma}}\frac{h_\gamma\,\partial\gamma}{\partial\alpha}\right)\,d\alpha\right] \times \left[\left(\hat{\boldsymbol{\beta}}h_\beta + \hat{\boldsymbol{\gamma}}\frac{h_\gamma\,\partial\gamma}{\partial\beta}\right)\,d\beta\right] \\ &= \left(\hat{\boldsymbol{\gamma}}\frac{1}{h_\gamma} - \hat{\boldsymbol{\alpha}}\frac{\partial\gamma}{h_\alpha\,\partial\alpha} - \hat{\boldsymbol{\beta}}\frac{\partial\gamma}{h_\beta\,\partial\beta}\right)h^3\,d\alpha\,d\beta \end{aligned} \tag{16.44}$$

of two vectors that lie on the surface, one vector normal to $\hat{\boldsymbol{\beta}}$ and the other to $\hat{\boldsymbol{\alpha}}$, edges not of a rectangular patch of the surface but of a patch whose projection onto the $\alpha$-$\beta$ plane is an $(h_\alpha\,d\alpha)$-by-$(h_\beta\,d\beta)$ rectangle.

So, that's it. Those are the essentials of the three-dimensional geometrical vector—of its analysis and of its calculus. The geometrical vector of Chs. 15 and 16 and the matrix of Chs. 11 through 14 have in common that they represent well-developed ways of marshaling several quantities together to a common purpose: three quantities in the specialized case of the

geometrical vector; $n$ quantities in the generalized case of the matrix. Matrices and vectors have admittedly not been easy for us to treat but after a slow start, it must be said, they have proven unexpectedly interesting. In applications, they are exceedingly significant. Matrices and vectors vastly expand the domain of physical phenomena a scientist or engineer can model. Mathematically, one cannot well manage without them.

The time nevertheless has come to change the subject. Turning the page, we will begin from the start of the next chapter to introduce a series of advanced topics that pick up where Ch. 9 has left off, entering first upon the broad topic of the Fourier transform.

# Part III

# Transforms and special functions

# Chapter 17

# The Fourier series

It might fairly be said that, among advanced mathematical techniques, none is so useful, and few so appealing, as the one Lord Kelvin has acclaimed "a great mathematical poem."[1] It is the Fourier transform, which this chapter and the next will develop. This first of the two chapters brings the Fourier transform in its primitive guise as the *Fourier series.*

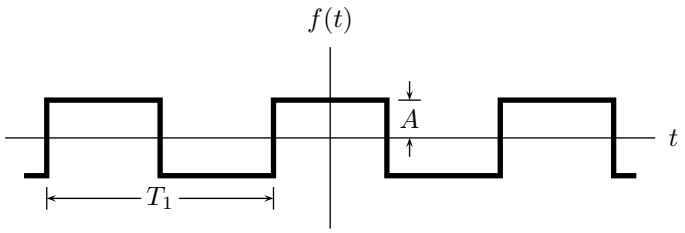
The Fourier series is an analog of the Taylor series of Ch. 8 but meant for *repeating waveforms,* functions $f(t)$ of which

$$f(t) = f(t + nT_1),\ \Im(T_1) = 0,\ T_1 \neq 0,\ \text{for all } n \in \mathbb{Z}, \tag{17.1}$$

where $T_1$ is the waveform's characteristic *period.* Examples include the *square wave* of Fig. 17.1. A Fourier series expands such a repeating waveform as a superposition of complex exponentials or, alternately, if the waveform is real, of sinusoids.

[1] [35, Ch. 17]

Figure 17.1: A square wave.



Suppose that you wanted to approximate the square wave of Fig. 17.1 by a single sinusoid. You might try the sinusoid at the top of Fig. 17.2—which is not very convincing, maybe, but if you added to the sinusoid another, suitably scaled sinusoid of thrice the frequency then you would obtain the somewhat better fitting curve in the figure's middle. The curve at the figure's bottom would yet result after you had added in four more sinusoids respectively of five, seven, nine and eleven times the primary frequency. Algebraically,

$$\begin{aligned} f(t) &= \frac{8A}{2\pi}\left[\cos\frac{(2\pi)t}{T_1} - \frac{1}{3}\cos\frac{3(2\pi)t}{T_1}\right. \\ &\qquad \left. + \frac{1}{5}\cos\frac{5(2\pi)t}{T_1} - \frac{1}{7}\cos\frac{7(2\pi)t}{T_1} + \cdots\right]. \end{aligned} \tag{17.2}$$

How faithfully (17.2) really represents the repeating waveform and why its coefficients happen to be $1, -\frac{1}{3}, \frac{1}{5}, -\frac{1}{7}, \ldots$ are among the questions this chapter will try to answer; but, visually at least, it looks as though superimposing sinusoids worked.

The chapter begins in preliminaries, starting with a discussion of Parseval's principle.

## 17.1 Parseval's principle

*Parseval's principle* is that *a step in every direction is no step at all.* In the Argand plane (Fig. 2.5), stipulated that

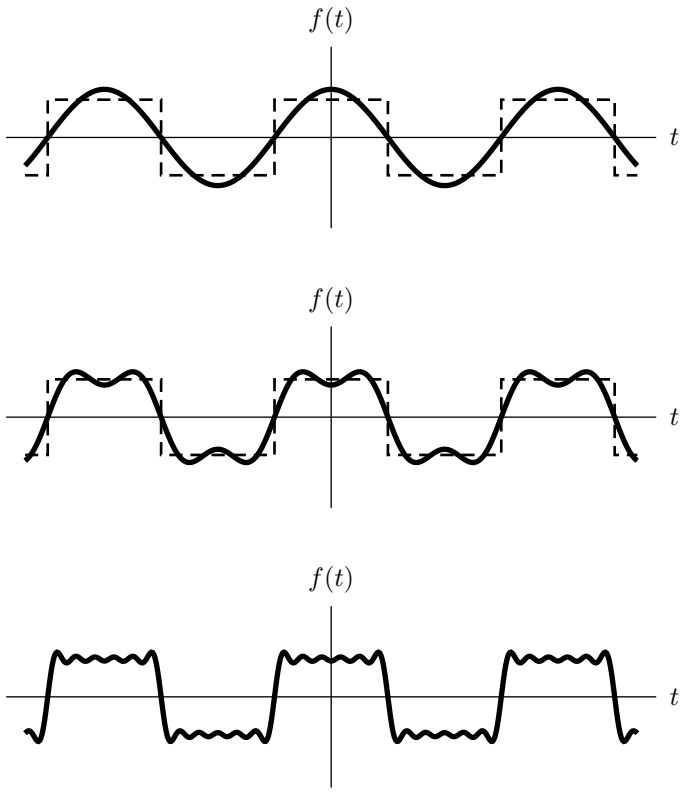
$$\begin{aligned} \Delta\omega\, T_1 &= 2\pi, \\ \Im(\Delta\omega) &= 0, \\ \Im(t_o) &= 0, \\ \Im(T_1) &= 0, \\ T_1 &\neq 0, \end{aligned} \tag{17.3}$$

and also that[2]

$$\begin{aligned} j, n, N &\in \mathbb{Z}, \\ n &\neq 0, \\ |n| &< N, \\ 2 &\leq N, \end{aligned} \tag{17.4}$$

[2]That $2 \leq N$ is a redundant requirement, since (17.4)'s other lines imply it, but it doesn't hurt to state it anyway.

Figure 17.2: Superpositions of one, two and six sinusoids to approximate the square wave of Fig. 17.1.



the principle is expressed algebraically as that[3]

$$\int_{t_o-T_1/2}^{t_o+T_1/2} e^{in\,\Delta\omega\,\tau}\,d\tau = 0 \tag{17.5}$$

or alternately in discrete form as that

$$\sum_{j=0}^{N-1} e^{i2\pi nj/N} = 0. \tag{17.6}$$

Because the product $\Delta\omega\,T_1 = 2\pi$ relates $\Delta\omega$ to $T_1$, the symbols $\Delta\omega$ and $T_1$ together represent in (17.3) and (17.5) not two but only one independent parameter. If $T_1$ bears physical units then these typically will be units of time (seconds, for instance), whereupon $\Delta\omega$ will bear the corresponding units of angular frequency (such as radians per second). The frame offset $t_o$ and the dummy variable $\tau$ naturally must have the same dimensions[4] $T_1$ has and normally will bear the same units. This matter is discussed further in § 17.2.

To prove (17.5) symbolically is easy: one merely carries out the indicated integration. To prove (17.6) symbolically is not much harder: one replaces the complex exponential $e^{i2\pi nj/N}$ by $\lim_{\epsilon\to 0^+} e^{(i-\epsilon)2\pi nj/N}$ and then uses (2.34) to evaluate the summation. Notwithstanding, we can do better, for an alternate, more edifying, physically more insightful explanation of the two equations is possible as follows. Because $n$ is a nonzero integer, (17.5) and (17.6) represent sums of steps in every direction—that is, steps in every phase—in the Argand plane (more precisely, eqn. 17.6 represents a sum over a discrete but balanced, uniformly spaced selection of phases). An appeal to symmetry forbids such sums from favoring any one phase $n\,\Delta\omega\,\tau$ or $2\pi nj/N$ over any other. This being the case, how could the sums of (17.5) and (17.6) ever come to any totals other than zero? The plain answer is that they can come to no other totals. A step in every direction is indeed no step at all. This is why (17.5) and (17.6) are so.[5]

We have actually already met Parseval's principle, informally, in § 9.6.2.

[3] An expression like $t_o \pm T_1/2$ means $t_o \pm (T_1/2)$, here and elsewhere in the book.

[4] The term *dimension* in this context refers to the kind of physical unit. A quantity like $T_1$ for example, measurable in seconds or years (but not, say, in kilograms or dollars), has dimensions of time. An automobile's speed having dimensions of length divided by time can be expressed in miles per hour as well as in meters per second but not directly, say, in volts per centimeter; and so on.

[5] The writer unfortunately knows of no conventionally established name for Parseval's principle. The name *Parseval's principle* seems as apt as any and this is the name this book will use.

One can translate Parseval's principle from the Argand realm to the analogous realm of geometrical vectors, if needed, in the obvious way.

## 17.2 Time, space and frequency

A *frequency* is the inverse of an associated period of time, expressing the useful concept of the rate at which a cycle repeats. For example, an internal-combustion engine whose crankshaft revolves once every 20 milliseconds—which is to say, once every 1/3000 of a minute—runs thereby at a frequency of 3000 revolutions per minute (RPM). Frequency however comes in two styles: cyclic frequency (as in the engine's example), conventionally represented by letters like $\nu$ and $f$; and angular frequency, by letters like $\omega$ and $k$. If $T$, $\nu$ and $\omega$ are letters taken to stand respectively for a period of time, the associated cyclic frequency and the associated angular frequency, then by definition

$$\begin{aligned} \nu T &= 1, \\ \omega T &= 2\pi, \\ \omega &= 2\pi\nu. \end{aligned} \tag{17.7}$$

The period $T$ will have dimensions of time like seconds. The cyclic frequency $\nu$ will have dimensions of inverse time like hertz (cycles per second).[6] The angular frequency $\omega$ will have dimensions of inverse time like radians per second.

The applied mathematician should make himself aware, and thereafter keep in mind, that the cycle per second and the radian per second do not differ dimensionally from one another. Both are technically units of $[\text{second}]^{-1}$, whereas the words "cycle" and "radian" in the contexts of the phrases "cycle per second" and "radian per second" are verbal cues that, in and of themselves, play no actual part in the mathematics. This is not because the

A pedagogical knot seems to tangle Marc-Antoine Parseval's various namesakes. Because Parseval's principle can be extracted as a special case from Parseval's theorem (eqn. 18.34 in the next chapter), the literature sometimes indiscriminately applies the name "Parseval's theorem" to both. This is fine as far as it goes, but the knot arrives when one needs Parseval's principle to derive the Fourier series, which one needs to derive the Fourier transform, which one needs in turn to derive Parseval's theorem, at least as this book develops them. The way to untie the knot is to give Parseval's principle its own name and to let it stand as an independent result.

[6] Notice incidentally, contrary to the improper verbal usage one sometimes hears, that there is no such thing as a "hert." Rather, "Hertz" is somebody's name. The uncapitalized form "hertz" thus is singular as well as plural.

cycle and the radian were ephemeral but rather because the second is unfundamental. The second is an arbitrary unit of measure. The cycle and the radian are definite, discrete, inherently countable things; and, where things are counted, it is ultimately up to the mathematician to interpret the count (consider for instance that nine baseball hats may imply nine baseball players and one baseball team, but that there is nothing in the number nine itself to tell us so). To distinguish angular frequencies from cyclic frequencies, it remains to the mathematician to lend factors of $2\pi$ where needed.

The word "frequency" without a qualifying adjective is usually taken to mean cyclic frequency unless the surrounding context implies otherwise.

Frequencies exist in space as well as in time:

$$k\lambda = 2\pi. \tag{17.8}$$

Here, $\lambda$ is a *wavelength* measured in meters or other units of length. The *wavenumber*[7] $k$ is an angular spatial frequency measured in units like radians per meter. (Oddly, no conventional symbol for cyclic spatial frequency seems to be current. The literature just uses $k/2\pi$ which, in light of the potential for confusion between $\nu$ and $\omega$ in the temporal domain, is probably for the best.)

Where a wave propagates the propagation speed

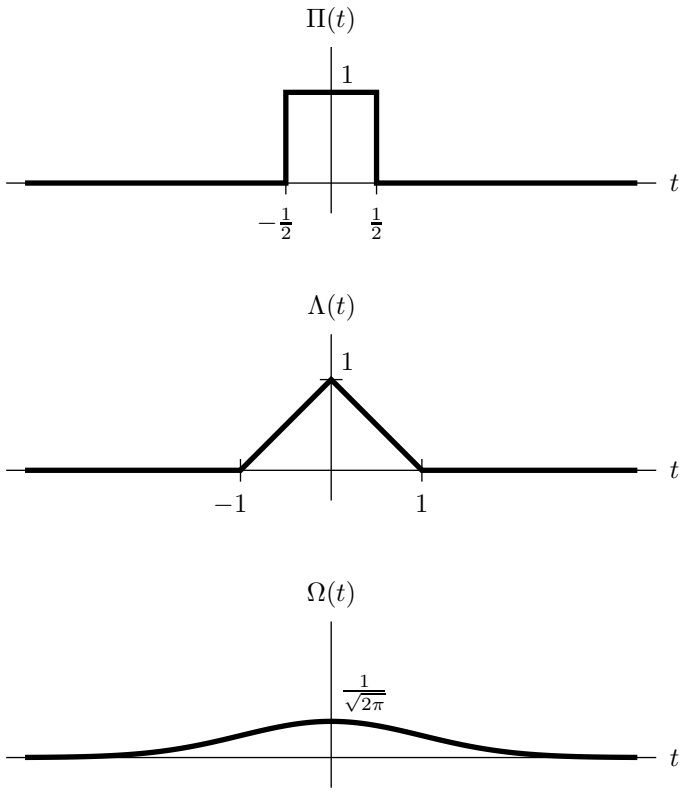
$$v = \frac{\lambda}{T} = \frac{\omega}{k} \tag{17.9}$$

relates periods and frequencies in space and time.

Now, we must admit that we fibbed when we said that $T$ had to have dimensions of time. Physically, that is the usual interpretation, but mathematically $T$ (and $T_1$, $t$, $t_o$, $\tau$, etc.) can bear any units and indeed are not required to bear units at all, as § 17.1 has observed. The only mathematical requirement is that the product $\omega T = 2\pi$ (or $\Delta\omega\, T_1 = 2\pi$ or the like, as appropriate) be dimensionless. However, when $T$ has dimensions of length rather than time it is conventional—indeed, it is practically mandatory if one wishes to be understood—to change $\lambda \leftarrow T$ and $k \leftarrow \omega$ as this section has done, though the essential Fourier mathematics is the same regardless of $T$'s dimensions (if any) or of whether alternate symbols like $\lambda$ and $k$ are used.

[7]The wavenumber $k$ is no integer, notwithstanding that the letter $k$ tends to represent integers in other contexts.

Figure 17.3: The square, triangular and Gaussian pulses.




## 17.3 The square, triangular and Gaussian pulses

The Dirac delta of § 7.7 and of Fig. 7.10 is useful for the unit area it covers among other reasons, but for some purposes its curve is too sharp. One occasionally finds it expedient to substitute either the *square* or the *triangular pulse* of Fig. 17.3,

$$\begin{aligned}\Pi(t) &\equiv \begin{cases} 1 & \text{if } |t| \le 1/2, \\ 0 & \text{otherwise;} \end{cases} \\ \Lambda(t) &\equiv \begin{cases} 1 - |t| & \text{if } |t| \le 1, \\ 0 & \text{otherwise;} \end{cases}\end{aligned} \tag{17.10}$$

for the Dirac delta, both of which pulses evidently share Dirac's property

that

$$\begin{aligned}
\int_{-\infty}^{\infty} \frac{1}{T}\,\delta\left(\frac{\tau - t_o}{T}\right)\,d\tau &= 1, \\
\int_{-\infty}^{\infty} \frac{1}{T}\Pi\left(\frac{\tau - t_o}{T}\right)\,d\tau &= 1, \\
\int_{-\infty}^{\infty} \frac{1}{T}\Lambda\left(\frac{\tau - t_o}{T}\right)\,d\tau &= 1,
\end{aligned} \tag{17.11}$$

for any real $T > 0$ and real $t_o$. In the limit,

$$\begin{aligned}
\lim_{T\to 0^+} \frac{1}{T}\Pi\left(\frac{t - t_o}{T}\right) &= \delta(t - t_o), \\
\lim_{T\to 0^+} \frac{1}{T}\Lambda\left(\frac{t - t_o}{T}\right) &= \delta(t - t_o),
\end{aligned} \tag{17.12}$$

constituting at least two possible implementations of the Dirac delta in case such an implementation were needed. Looking ahead, if we may further abuse the Greek capitals to let them represent pulses whose shapes they accidentally resemble, then a third, subtler implementation—more complicated to handle but analytic (§ 8.4) and therefore preferable for some purposes—is the *Gaussian pulse*

$$\begin{aligned}
\lim_{T\to 0^+} \frac{1}{T}\Omega\left(\frac{t - t_o}{T}\right) &= \delta(t - t_o), \\
\Omega(t) &\equiv \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{t^2}{2}\right),
\end{aligned} \tag{17.13}$$

the mathematics of which § 18.5 and Chs. 19 and 20 will unfold.

## 17.4 Expanding repeating waveforms in Fourier series

The Fourier series represents a repeating waveform (17.1) as a superposition of sinusoids. More precisely, inasmuch as Euler's formula (5.18) makes a sinusoid a sum of two complex exponentials, the Fourier series supposes that a repeating waveform were a superposition

$$f(t) = \sum_{j=-\infty}^{\infty} a_j e^{ij\,\Delta\omega\,t} \tag{17.14}$$

of many complex exponentials, in which (17.3) is obeyed yet in which neither the several Fourier coefficients $a_j$ nor the waveform $f(t)$ itself need be real. Whether one can properly represent every repeating waveform as a superposition (17.14) of complex exponentials is a question §§ 17.4.4 and 17.6 will address later; but, at least to the extent to which one can properly represent such a waveform, we will now assert that one can recover any or all of the waveform's Fourier coefficients $a_j$ by choosing an arbitrary frame offset $t_o$ ($t_o = 0$ is a typical choice) and then integrating

$$a_j = \frac{1}{T_1} \int_{t_o - T_1/2}^{t_o + T_1/2} e^{-ij\,\Delta\omega\,\tau} f(\tau)\,d\tau. \tag{17.15}$$

### 17.4.1 Derivation of the Fourier-coefficient formula

But why should (17.15) work? How is it to recover a Fourier coefficient $a_j$? The answer is that it recovers a Fourier coefficient $a_j$ by isolating it, and that it isolates it by shifting frequencies and integrating.

Equation (17.14) has proposed to express a repeating waveform as a series of complex exponentials, each exponential of the form $a_j e^{ij\,\Delta\omega\,t}$ in which $a_j$ is a weight to be determined. Unfortunately, (17.14) can hardly be very useful until the several $a_j$ actually are determined, whereas how to determine $a_j$ from (17.14) for a given value of $j$ is not immediately obvious.

The trouble with using (17.14) to determine the several coefficients $a_j$ is that it includes all the terms of the series and, hence, all the coefficients $a_j$ at once. To determine $a_j$ for a given value of $j$, one should like to suppress the entire series except the single element $a_j e^{ij\,\Delta\omega\,t}$, isolating this one element for analysis. Fortunately, Parseval's principle (17.5) gives us a way to do this, as we shall soon see.

Now, to prove (17.15) we mean to use (17.15), a seemingly questionable act. Nothing prevents us however from taking only the right side of (17.15)—not as an equation but as a mere expression—and doing some algebra with it to see where the algebra leads, for if the algebra should lead to the *left* side of (17.15) then we should have proven the equation. Accordingly, changing dummy variables $\tau \leftarrow t$ and $\ell \leftarrow j$ in (17.14) and then substituting into (17.15)'s right side the resulting expression for $f(\tau)$, we have by suc-

cessive steps that[8]

$$\begin{aligned}
&\frac{1}{T_1}\int_{t_o-T_1/2}^{t_o+T_1/2} e^{-ij\,\Delta\omega\,\tau} f(\tau)\,d\tau \\
&\quad= \frac{1}{T_1}\int_{t_o-T_1/2}^{t_o+T_1/2} e^{-ij\,\Delta\omega\,\tau} \sum_{\ell=-\infty}^{\infty} a_\ell e^{i\ell\,\Delta\omega\,\tau}\,d\tau \\
&\quad= \frac{1}{T_1}\sum_{\ell=-\infty}^{\infty} a_\ell \int_{t_o-T_1/2}^{t_o+T_1/2} e^{i(\ell-j)\,\Delta\omega\,\tau}\,d\tau \\
&\quad= \frac{a_j}{T_1}\int_{t_o-T_1/2}^{t_o+T_1/2} e^{i(j-j)\,\Delta\omega\,\tau}\,d\tau \\
&\quad= \frac{a_j}{T_1}\int_{t_o-T_1/2}^{t_o+T_1/2} d\tau = a_j,
\end{aligned}$$

in which Parseval's principle (17.5) has killed all but the $\ell = j$ term in the summation. Thus is (17.15) formally proved.

Though the foregoing formally completes the proof, the idea behind the formality remains more interesting than the formality itself, for one would like to know not only the fact that (17.15) is true but also the thought which leads one to propose the equation in the first place. The thought is as follows. Assuming that (17.14) indeed can represent the waveform $f(t)$ properly, one observes that the transforming factor $e^{-ij\,\Delta\omega\,\tau}$ of (17.15) serves to shift the waveform's $j$th component $a_j e^{ij\,\Delta\omega\,t}$—whose angular frequency evidently is $\omega = j\,\Delta\omega$—down to a frequency of zero, incidentally shifting the waveform's several other components to various nonzero frequencies as well. Significantly, the transforming factor leaves each shifted frequency to be a whole multiple of the waveform's fundamental frequency $\Delta\omega$. By Parseval's principle, (17.15)'s integral then kills all the thus frequency-shifted components except the zero-shifted one *by integrating the components over complete cycles,* passing only the zero-shifted component which, once shifted, has no cycle. Such is the thought which has given rise to the equation.

[8]It is unfortunately conventional to footnote steps like these with some formal remarks on convergence and the swapping of summational/integrodifferential operators. Refer to §§ 7.3.4 and 7.3.5.

### 17.4.2 The square wave

According to (17.15), the Fourier coefficients of Fig. 17.1's square wave are, if $t_o = T_1/4$ is chosen and by successive steps,

$$\begin{aligned} a_j &= \frac{1}{T_1}\int_{-T_1/4}^{3T_1/4} e^{-ij\,\Delta\omega\,\tau} f(\tau)\,d\tau \\ &= \frac{A}{T_1}\left[\int_{-T_1/4}^{T_1/4} - \int_{T_1/4}^{3T_1/4}\right] e^{-ij\,\Delta\omega\,\tau}\,d\tau \\ &= \frac{iA}{2\pi j} e^{-ij\,\Delta\omega\,\tau}\left[\Big|_{-T_1/4}^{T_1/4} - \Big|_{T_1/4}^{3T_1/4}\right]. \end{aligned}$$

But

$$\begin{aligned} e^{-ij\,\Delta\omega\,\tau}\big|_{\tau=-T_1/4} = e^{-ij\,\Delta\omega\,\tau}\big|_{\tau=3T_1/4} &= i^j, \\ e^{-ij\,\Delta\omega\,\tau}\big|_{\tau=T_1/4} &= (-i)^j, \end{aligned}$$

so

$$\begin{aligned} & e^{-ij\,\Delta\omega\,\tau}\left[\Big|_{-T_1/4}^{T_1/4} - \Big|_{T_1/4}^{3T_1/4}\right] \\ &= [(-i)^j - i^j] - [i^j - (-i)^j] = 2[(-i)^j - i^j] \\ &= \ldots, -i4, 0, i4, 0, -i4, 0, i4, \ldots \text{ for } j = \ldots, -3, -2, -1, 0, 1, 2, 3, \ldots \end{aligned}$$

Therefore,

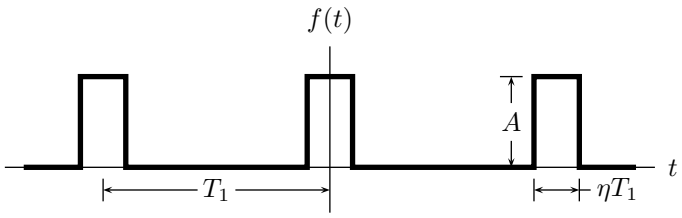
$$\begin{aligned} a_j &= \left[(-i)^j - i^j\right]\frac{i2A}{2\pi j} \\ &= \begin{cases} (-)^{(j-1)/2} 4A/2\pi j & \text{for odd } j, \\ 0 & \text{for even } j \end{cases} \end{aligned} \tag{17.16}$$

are the square wave's Fourier coefficients which, when the coefficients are applied to (17.14) and when (5.18) is invoked, indeed yield the specific series of sinusoids (17.2) and Fig. 17.2 have proposed.

### 17.4.3 The rectangular pulse train

The square wave of § 17.4.2 is an important, canonical case and (17.2) is arguably worth memorizing. After the square wave however the variety of

Figure 17.4: A rectangular pulse train.




possible repeating waveforms has no end. Whenever an unfamiliar repeating waveform arises, one can calculate its Fourier coefficients (17.15) on the spot by the straightforward routine of § 17.4.2. There seems little point therefore in trying to tabulate waveforms here.

One variant on the square wave nonetheless is interesting enough to attract special attention. This variant is the *pulse train* of Fig. 17.4,

$$f(t) = A \sum_{j=-\infty}^{\infty} \Pi\left(\frac{t - jT_1}{\eta T_1}\right); \tag{17.17}$$

where $\Pi(\cdot)$ is the square pulse of (17.10); the symbol $A$ represents the pulse's full height rather than the half-height of Fig. 17.1; and the dimensionless factor $0 \leq \eta \leq 1$ is the train's *duty cycle,* the fraction of each cycle its pulse is as it were on duty. By the routine of § 17.4.2,

$$\begin{aligned} a_j &= \frac{1}{T_1} \int_{-T_1/2}^{T_1/2} e^{-ij\,\Delta\omega\,\tau} f(\tau)\,d\tau \\ &= \frac{A}{T_1} \int_{-\eta T_1/2}^{\eta T_1/2} e^{-ij\,\Delta\omega\,\tau}\,d\tau \\ &= \frac{iA}{2\pi j} e^{-ij\,\Delta\omega\,\tau}\Big|_{-\eta T_1/2}^{\eta T_1/2} = \frac{2A}{2\pi j} \sin\frac{2\pi\eta j}{2} \end{aligned}$$

for $j \neq 0$. On the other hand,

$$a_0 = \frac{1}{T_1} \int_{-T_1/2}^{T_1/2} f(\tau)\,d\tau = \frac{A}{T_1} \int_{-\eta T_1/2}^{\eta T_1/2} d\tau = \eta A$$

is the waveform's mean value. Altogether for the pulse train,

$$a_j = \begin{cases} \dfrac{2A}{2\pi j} \sin \dfrac{2\pi\eta j}{2} & \text{if } j \neq 0, \\ \eta A & \text{if } j = 0 \end{cases} \tag{17.18}$$

(though eqn. 17.26 will improve the notation later).

An especially interesting special case occurs when the duty cycle grows very short. Since $\lim_{\eta\to 0^+} \sin(2\pi\eta j/2) = 2\pi\eta j/2$ according to (8.32), it follows from (17.18) that

$$\lim_{\eta\to 0^+} a_j = \eta A, \tag{17.19}$$

the same for every index $j$. As the duty cycle $\eta$ tends to vanish the pulse tends to disappear and the Fourier coefficients along with it; but we can compensate for vanishing duty if we wish by increasing the pulse's amplitude $A$ proportionally, maintaining the product

$$\eta T_1 A = 1 \tag{17.20}$$

of the pulse's width $\eta T_1$ and its height $A$, thus preserving unit area[9] under the pulse. In the limit $\eta \to 0^+$, the pulse then by definition becomes the Dirac delta of Fig. 7.10, and the pulse train by construction becomes the *Dirac delta pulse train* of Fig. 17.5. Enforcing (17.20) on (17.19) yields the Dirac delta pulse train's Fourier coefficients

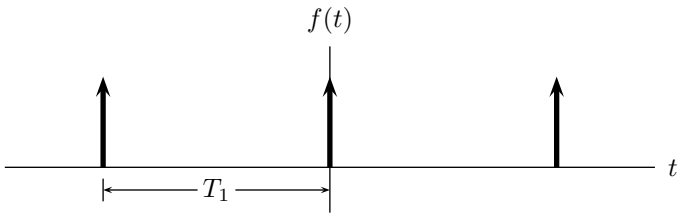
$$a_j = \frac{1}{T_1}. \tag{17.21}$$

### 17.4.4 Linearity and sufficiency

The Fourier series is evidently linear according to the rules of § 7.3.3. That is, if the Fourier coefficients of $f_1(t)$ are $a_{j1}$ and the Fourier coefficients of $f_2(t)$ are $a_{j2}$, and if the two waveforms $f_1(t)$ and $f_2(t)$ share the same fundamental period $T_1$, then the Fourier coefficients of $f(t) = f_1(t) + f_2(t)$ are $a_j = a_{j1} + a_{j2}$. Likewise, the Fourier coefficients of $\alpha f(t)$ are $\alpha a_j$ and

[9] In light of the discussion of time, space and frequency in § 17.2, we should clarify that we do not here mean a physical area measurable in square meters or the like. We merely mean the dimensionless product of the width (probably measured in units of time like seconds) and the height (correspondingly probably measured in units of frequency like inverse seconds) of the rectangle a single pulse encloses in Fig. 17.4. Though it is not a physical area the rectangle one sketches on paper to represent it, as in the figure, of course does have an area. The word *area* here is meant in the latter sense.

Figure 17.5: A Dirac delta pulse train.




the Fourier coefficients of the null waveform $f_{\text{null}}(t) \equiv 0$ are themselves null, thus satisfying the conditions of linearity.

All this however supposes that the Fourier series actually works.[10] Though Fig. 17.2 is suggestive, the figure alone hardly serves to demonstrate that every repeating waveform were representable as a Fourier series. To try to consider every repeating waveform at once would be too much to try at first in any case, so let us start from a more limited question: does there exist any continuous, repeating waveform[11] $f(t) \neq 0$ of period $T_1$ whose Fourier coefficients $a_j = 0$ are identically zero?

If the waveform $f(t)$ in question is continuous then nothing prevents us from discretizing (17.15) as

$$\begin{aligned} a_j &= \lim_{M\to\infty} \frac{1}{T_1} \sum_{\ell=-M}^{M} e^{(-ij\,\Delta\omega)(t_o+\ell\,\Delta\tau_M)} f(t_o + \ell\,\Delta\tau_M)\,\Delta\tau_M, \\ \Delta\tau_M &\equiv \frac{T_1}{2M+1}, \end{aligned}$$

and further discretizing the waveform itself as

$$f(t) = \lim_{M\to\infty} \sum_{p=-\infty}^{\infty} f(t_o + p\,\Delta\tau_M)\,\Pi\left[\frac{t - (t_o + p\,\Delta\tau_M)}{\Delta\tau_M}\right],$$

in which $\Pi[\cdot]$ is the square pulse of (17.10). Substituting the discretized

[10] The remainder of this dense subsection can be regarded as optional reading.

[11] As elsewhere in the book, the notation $f(t) \neq 0$ here forbids only the all-zero waveform. It does not forbid waveforms like $f(t) = A \sin \omega t$ that happen to take a zero value at certain values of $t$.

waveform into the discretized formula for $a_j$, we have that

$$\begin{aligned} a_j &= \lim_{M\to\infty} \frac{\Delta\tau_M}{T_1} \sum_{\ell=-M}^{M} \sum_{p=-\infty}^{\infty} e^{(-ij\,\Delta\omega)(t_o+\ell\,\Delta\tau_M)} f(t_o + p\,\Delta\tau_M)\Pi(\ell - p) \\ &= \lim_{M\to\infty} \frac{\Delta\tau_M}{T_1} \sum_{\ell=-M}^{M} e^{(-ij\,\Delta\omega)(t_o+\ell\,\Delta\tau_M)} f(t_o + \ell\,\Delta\tau_M). \end{aligned}$$

If we define the $(2M+1)$-element vectors and $(2M+1)\times(2M+1)$ matrix

$$\begin{aligned} [\mathbf{f}_M]_\ell &\equiv f(t_o + \ell\,\Delta\tau_M), \\ [\mathbf{a}_M]_j &\equiv a_j, \\ [C_M]_{j\ell} &\equiv \frac{\Delta\tau_M}{T_1} e^{(-ij\,\Delta\omega)(t_o+\ell\,\Delta\tau_M)}, \\ -M \le (j,\ell) &\le M, \end{aligned}$$

then matrix notation renders the last equation as

$$\lim_{M\to\infty} \mathbf{a}_M = \lim_{M\to\infty} C_M \mathbf{f}_M,$$

whereby

$$\lim_{M\to\infty} \mathbf{f}_M = \lim_{M\to\infty} C_M^{-1} \mathbf{a}_M,$$

assuming that $C_M$ is invertible.

But is $C_M$ invertible? This seems a hard question to answer until we realize that the rows of $C_M$ consist of sampled complex exponentials which repeat over the interval $T_1$ and thus stand subject to Parseval's principle (17.6). Realizing this, we can do better than merely to state that $C_M$ is invertible: we can write down its actual inverse,

$$[C_M^{-1}]_{\ell j} = \frac{T_1}{(2M+1)\,\Delta\tau_M} e^{(+ij\,\Delta\omega)(t_o+\ell\,\Delta\tau_M)},$$

such that[12] $C_M C_M^{-1} = I_{-M}^{M}$ and thus per (13.2) also that $C_M^{-1} C_M = I_{-M}^{M}$. So, the answer to our question is that, yes, $C_M$ is invertible.

Because $C_M$ is invertible, § 14.2 has it that neither $\mathbf{f}_M$ nor $\mathbf{a}_M$ can be null unless both are. In the limit $M \to \infty$, this implies that no continuous,

[12] Equation (11.30) has defined the notation $I_{-M}^{M}$, representing a $(2M+1)$-dimensional identity matrix whose string of ones extends along its main diagonal from $j = \ell = -M$ through $j = \ell = M$.

repeating waveform $f(t) \neq 0$ exists whose Fourier coefficients $a_j = 0$ are identically zero.

Now consider a continuous, repeating waveform $F(t)$ and its Fourier series $f(t)$. Let $\Delta F(t) \equiv F(t) - f(t)$ be the part of $F(t)$ unrepresentable as a Fourier series, continuous because both $F(t)$ and $f(t)$ are continuous. Being continuous and unrepresentable as a Fourier series, $\Delta F(t)$ has null Fourier coefficients; but as the last paragraph has concluded this can only be so if $\Delta F(t) = 0$. Hence, $\Delta F(t) = 0$ indeed, which implies[13] that $f(t) = F(t)$. In other words, *every continuous, repeating waveform is representable as a Fourier series.*

And what of discontinuous waveforms? Well, the square wave of Figs. 17.1 and 17.2 this chapter has posed as its principal example is a repeating waveform but, of course, not a continuous one. A truly discontinuous waveform would admittedly invalidate the discretization above of $f(t)$, but see: nothing prevents us from approximating the square wave's discontinuity by an arbitrarily steep slope, whereupon this subsection's conclusion again applies.[14]

The better, more subtle, more complete answer to the question though is that a discontinuity incurs Gibbs' phenomenon, which § 17.6 will derive.

### 17.4.5 The trigonometric form

It is usually best, or at least neatest and cleanest, and moreover more evocative, to calculate Fourier coefficients and express Fourier series in terms of complex exponentials as (17.14) and (17.15) do. Occasionally, though, when the repeating waveform $f(t)$ is real, one prefers to work in sines and cosines rather than in complex exponentials. One writes (17.14) by Euler's formula (5.12) as

$$f(t) = a_0 + \sum_{j=1}^{\infty} [(a_j + a_{-j}) \cos j\,\Delta\omega\, t + i(a_j - a_{-j}) \sin j\,\Delta\omega\, t]\,.$$

[13] Chapter 8's footnote 6 has argued in a similar style, earlier in the book.

[14] Where this subsection's conclusion cannot be made to apply is where unreasonable waveforms like $A \sin[B/\sin \omega t]$ come into play. We will leave to the professional mathematician the classification of such unreasonable waveforms, the investigation of the waveforms' Fourier series and the provision of greater rigor generally.

Then, superimposing coefficients in (17.15),

$$\begin{aligned} a_0 &= \frac{1}{T_1}\int_{t_o-T_1/2}^{t_o+T_1/2} f(\tau)\,d\tau, \\ b_j \equiv\ (a_j + a_{-j}) &= \frac{2}{T_1}\int_{t_o-T_1/2}^{t_o+T_1/2} \cos(j\,\Delta\omega\,\tau) f(\tau)\,d\tau, \\ c_j \equiv i(a_j - a_{-j}) &= \frac{2}{T_1}\int_{t_o-T_1/2}^{t_o+T_1/2} \sin(j\,\Delta\omega\,\tau) f(\tau)\,d\tau, \end{aligned} \tag{17.22}$$

which give the Fourier series the trigonometric form

$$f(t) = a_0 + \sum_{j=1}^{\infty} \left(b_j \cos j\,\Delta\omega\,t + c_j \sin j\,\Delta\omega\,t\right). \tag{17.23}$$

The complex conjugate of (17.15) is

$$a_j^* = \frac{1}{T_1}\int_{t_o-T_1/2}^{t_o+T_1/2} e^{+ij\,\Delta\omega\,\tau} f^*(\tau)\,d\tau.$$

If the waveform happens to be real then $f^*(t) = f(t)$, which in light of the last equation and (17.15) implies that

$$a_{-j} = a_j^* \text{ if } \Im[f(t)] = 0. \tag{17.24}$$

Combining (17.22) and (17.24), we have that

$$\left.\begin{aligned} b_j &= \quad 2\Re(a_j) \\ c_j &= -2\Im(a_j) \end{aligned}\right\} \text{ if } \Im[f(t)] = 0. \tag{17.25}$$

## 17.5 The sine-argument function

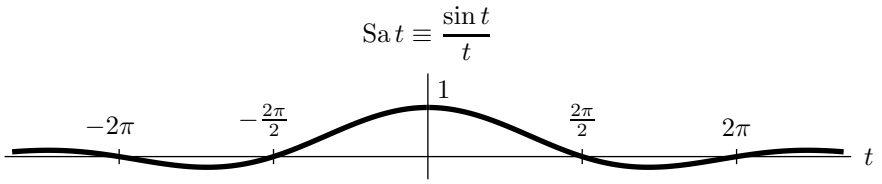
Equation (17.18) gives the pulse train of Fig. 17.4 its Fourier coefficients, but a better notation for (17.18) is

$$a_j = \eta A\,\mathrm{Sa}\,\frac{2\pi\eta j}{2}, \tag{17.26}$$

where

$$\mathrm{Sa}\,z \equiv \frac{\sin z}{z} \tag{17.27}$$

Figure 17.6: The sine-argument function.




is the *sine-argument function,*[15] plotted in Fig. 17.6. The function's Taylor series is

$$\operatorname{Sa} z = \sum_{j=0}^{\infty} \left[ \prod_{m=1}^{j} \frac{-z^2}{(2m)(2m+1)} \right], \tag{17.28}$$

the Taylor series of $\sin z$ from Table 8.1, divided by $z$.

This section introduces the sine-argument function and some of its properties, plus also the related sine integral.[16]

### 17.5.1 Derivative and integral

The sine-argument function's derivative is computed from the definition (17.27) and the derivative product rule (4.26) to be

$$\frac{d}{dz} \operatorname{Sa} z = \frac{\cos z - \operatorname{Sa} z}{z}. \tag{17.29}$$

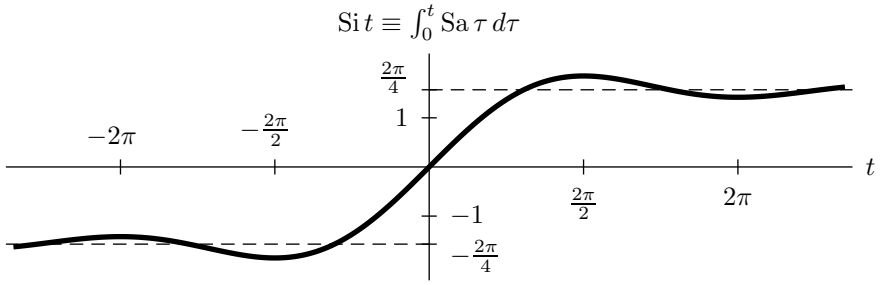
[15] Many (including the author himself in other contexts) call it the *sinc function,* denoting it sinc(·) and pronouncing it as "sink." Unfortunately, some [48, § 4.3][13, § 2.2][18] use the sinc(·) notation for another function,

$$\operatorname{sinc}_{\text{alternate}} z \equiv \operatorname{Sa} \frac{2\pi z}{2} = \frac{\sin(2\pi z/2)}{2\pi z/2}.$$

The unambiguous Sa(·) suits this particular book better, anyway, so this is the notation we will use.

[16] Readers interested in Gibbs' phenomenon, § 17.6, will read the present section because Gibbs depends on its results. Among other readers however some, less interested in special functions than in basic Fourier theory, may find this section unprofitably tedious. They can skip ahead to the start of the next chapter without great loss.

Figure 17.7: The sine integral.




The function's integral is expressed as a Taylor series after integrating the function's own Taylor series (17.28) term by term to obtain the form

$$\operatorname{Si} z \equiv \int_0^z \operatorname{Sa} \tau \, d\tau = \sum_{j=0}^{\infty} \left[ \frac{z}{2j+1} \prod_{m=1}^{j} \frac{-z^2}{(2m)(2m+1)} \right], \tag{17.30}$$

plotted in Fig. 17.7. Convention gives this integrated function its own name and notation: it calls it the *sine integral*[17,18] and denotes it by $\operatorname{Si}(\cdot)$.

### 17.5.2 Properties of the sine-argument function

Sine-argument properties include the following.

- The sine-argument function is real over the real domain. That is, if $\Im(t) = 0$ then $\Im(\operatorname{Sa} t) = 0$.
- The zeros of $\operatorname{Sa} z$ occur at $z = n\pi$, $n \neq 0$, $n \in \mathbb{Z}$.
- It is that $|\operatorname{Sa} t| < 1$ over the real domain $\Im(t) = 0$ except at the global maximum $t = 0$, where
$$\operatorname{Sa} 0 = 1. \tag{17.31}$$
- Over the real domain $\Im(t) = 0$, the function $\operatorname{Sa} t$ alternates between distinct, positive and negative lobes. Specifically, $(-)^n \operatorname{Sa}(\pm t) > 0$ over $n\pi < t < (n+1)\pi$ for each $n \geq 0$, $n \in \mathbb{Z}$.

[17] [43, § 3.3]

[18] The name "sine-argument" incidentally seems to have been back-constructed from the name "sine integral."

- Each of the sine-argument's lobes has but a single peak. That is, over the real domain $\Im(t) = 0$, the derivative $(d/dt)\operatorname{Sa} t = 0$ is zero at only a single value of $t$ on each lobe.

- The sine-argument function and its derivative converge toward

$$\begin{aligned} \lim_{t\to\pm\infty} \operatorname{Sa} t &= 0, \\ \lim_{t\to\pm\infty} \frac{d}{dt} \operatorname{Sa} t &= 0. \end{aligned} \tag{17.32}$$

Some of these properties are obvious in light of the sine-argument function's definition (17.27). Among the less obvious properties, that $|\operatorname{Sa} t| < 1$ says merely that $|\sin t| < |t|$ for nonzero $t$; which must be true since $t$, interpreted as an angle—which is to say, as a curved distance about a unit circle—can hardly be shorter than $\sin t$, interpreted as the corresponding direct shortcut to the axis (see Fig. 3.1). For $t = 0$, (8.32) obtains—or, if you prefer, (17.28).

That each of the sine-argument function's lobes should have but a single peak seems right in view of Fig. 17.6 but is nontrivial to prove. To assert that each lobe has but a single peak is to assert that $(d/dt)\operatorname{Sa} t = 0$ exactly once in each lobe; or, equivalently—after setting (17.29)'s left side to zero, multiplying by $z^2/\cos z$ and changing $t \leftarrow z$—it is to assert that

$$\tan t = t$$

exactly once in each interval

$$n\pi \le t < (n+1)\pi, \ n \ge 0,$$

for $t \ge 0$; and similarly for $t \le 0$. But according to Table 5.2

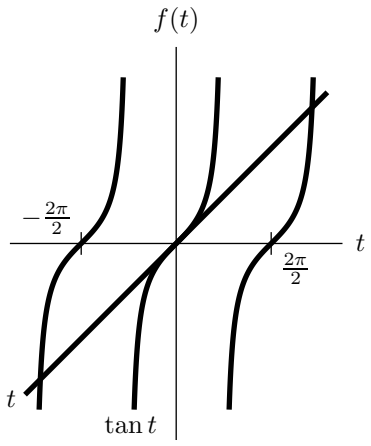
$$\frac{d}{dt}\tan t = \frac{1}{\cos^2 t} \ge 1,$$

whereas $dt/dt = 1$, implying that $\tan t$ is everywhere at least as steep as $t$ is—and, well, the most concise way to finish the argument is to draw a picture of it, as in Fig. 17.8, where the curves evidently cannot but intersect exactly once in each interval.

### 17.5.3 Properties of the sine integral

Properties of the sine integral $\operatorname{Si} t$ of (17.30) include the following.

Figure 17.8: The points at which $t$ intersects $\tan t$.




- Over the real domain $\Im(t) = 0$, the sine integral $\operatorname{Si} t$ is positive for positive $t$, negative for negative $t$ and, of course, zero for $t = 0$.

- The local extrema of $\operatorname{Si} t$ over the real domain $\Im(t) = 0$ occur at the zeros of $\operatorname{Sa} t$.

- The global maximum and minimum of $\operatorname{Si} t$ over the real domain $\Im(t) = 0$ occur respectively at the first positive and negative zeros of $\operatorname{Sa} t$, which are $t = \pm\pi$.

- The sine integral converges toward

$$\lim_{t \to \pm\infty} \operatorname{Si} t = \pm\frac{2\pi}{4}. \tag{17.33}$$

That the sine integral should reach its local extrema at the sine-argument's zeros ought to be obvious to the extent to which the concept of integration is understood. To explain the other properties it helps first to have expressed

the sine integral in the form

$$\begin{aligned}
\operatorname{Si} t &= S_n + \int_{n\pi}^{t} \operatorname{Sa}\tau \, d\tau, \\
S_n &\equiv \sum_{j=0}^{n-1} U_j, \\
U_j &\equiv \int_{j\pi}^{(j+1)\pi} \operatorname{Sa}\tau \, d\tau, \\
n\pi &\leq t < (n+1)\pi, \\
0 &\leq n, \quad (j,n) \in \mathbb{Z},
\end{aligned}$$

where each partial integral $U_j$ integrates over a single lobe of the sine-argument. The several $U_j$ alternate in sign but, because each lobe majorizes the next (§ 8.10.2)—that is, because,[19] in the integrand, $|\operatorname{Sa}\tau| \geq |\operatorname{Sa}\tau + \pi|$ for all $\tau \geq 0$—the magnitude of the area under each lobe exceeds that under the next, such that

$$\begin{aligned}
0 \leq (-)^j \int_{j\pi}^{t} \operatorname{Sa}\tau \, d\tau < (-)^j U_j < (-)^{j-1} U_{j-1}, \\
j\pi \leq t < (j+1)\pi, \\
0 \leq j, \ j \in \mathbb{Z}
\end{aligned}$$

(except that the $U_{j-1}$ term of the inequality does not apply when $j = 0$, since there is no $U_{-1}$) and thus that

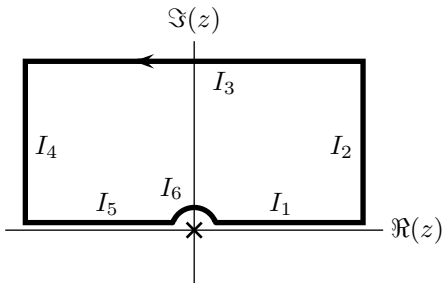
$$0 = S_0 < S_{2m} < S_{2m+2} < S_\infty < S_{2m+3} < S_{2m+1} < S_1$$
$$\text{for all } m > 0, \ m \in \mathbb{Z}.$$

The foregoing applies only when $t \geq 0$ but naturally one can reason similarly for $t \leq 0$, concluding that the integral's global maximum and minimum over the real domain occur respectively at the sine-argument function's first positive and negative zeros, $t = \pm\pi$; and further concluding that the integral is positive for all positive $t$ and negative for all negative $t$.

Equation (17.33) wants some cleverness to calculate and will be the subject of the next subsection.

[19] More rigorously, to give the reason perfectly unambiguously, one could fuss here for a third of a page or so over signs, edges and the like. To do so is left as an exercise to those who aspire to the pure mathematics profession.

Figure 17.9: A complex contour about which to integrate $e^{iz}/i2z$.




### 17.5.4 The sine integral's limit by complex contour

Equation (17.33) has proposed that the sine integral converges toward a value of $2\pi/4$, but why? The integral's Taylor series (17.30) is impractical to compute for large $t$ and is useless for $t \to \infty$, so it cannot answer the question. To evaluate the integral in the infinite limit, we shall have to think of something cleverer.

Noticing per (5.19) that

$$\mathrm{Sa}\, z = \frac{e^{+iz} - e^{-iz}}{i2z},$$

rather than trying to integrate the sine-argument function all at once let us first try to integrate just one of its two complex terms, leaving the other term aside to handle later, for the moment computing only

$$I_1 \equiv \int_0^\infty \frac{e^{iz}\,dz}{i2z}.$$

To compute the integral $I_1$, we will apply the closed-contour technique of § 9.5, choosing a contour in the Argand plane that incorporates $I_1$ but shuts out the integrand's pole at $z = 0$.

Many contours are possible and one is unlikely to find an amenable contour on the first attempt, but perhaps after several false tries we discover and choose the contour of Fig. 17.9. The integral about the inner semicircle of this contour is

$$I_6 = \int_{C_6} \frac{e^{iz}\,dz}{i2z} = \lim_{\rho \to 0^+} \int_{2\pi/2}^{0} \frac{e^{iz}(i\rho e^{i\phi}\,d\phi)}{i2(\rho e^{i\phi})} = \int_{2\pi/2}^{0} \frac{e^{i0}\,d\phi}{2} = -\frac{2\pi}{4}.$$

The integral across the contour's top segment is

$$I_3 = \int_{C_3} \frac{e^{iz}\,dz}{i2z} = \lim_{a\to\infty} \int_a^{-a} \frac{e^{i(x+ia)}\,dx}{i2z} = \lim_{a\to\infty} \int_{-a}^{a} \frac{-e^{ix}e^{-a}\,dx}{i2z},$$

from which, according to the continuous triangle inequality (9.15),

$$|I_3| \le \lim_{a\to\infty} \int_{-a}^{a} \left| \frac{-e^{ix}e^{-a}\,dx}{i2z} \right| = \lim_{a\to\infty} \int_{-a}^{a} \frac{e^{-a}\,dx}{2\,|z|};$$

which, since $0 < a \le |z|$ across the segment, we can weaken to read

$$|I_3| \le \lim_{a\to\infty} \int_{-a}^{a} \frac{e^{-a}\,dx}{2a} = \lim_{a\to\infty} e^{-a} = 0,$$

only possible if

$$I_3 = 0.$$

The integral up the contour's right segment is

$$I_2 = \int_{C_2} \frac{e^{iz}\,dz}{i2z} = \lim_{a\to\infty} \int_0^{a} \frac{e^{i(a+iy)}\,dy}{2z} = \lim_{a\to\infty} \int_0^{a} \frac{e^{ia}e^{-y}\,dy}{2z},$$

from which, according to the continuous triangle inequality,

$$|I_2| \le \lim_{a\to\infty} \int_0^{a} \left| \frac{e^{ia}e^{-y}\,dy}{2z} \right| = \lim_{a\to\infty} \int_0^{a} \frac{e^{-y}\,dy}{2\,|z|};$$

which, since $0 < a \le |z|$ across the segment, we can weaken to read

$$|I_2| \le \lim_{a\to\infty} \int_0^{a} \frac{e^{-y}\,dy}{2a} = \lim_{a\to\infty} \frac{1}{2a} = 0,$$

only possible if

$$I_2 = 0.$$

The integral down the contour's left segment is

$$I_4 = 0$$

for like reason. Because the contour encloses no pole,

$$\oint \frac{e^{iz}\,dz}{i2z} = I_1 + I_2 + I_3 + I_4 + I_5 + I_6 = 0,$$

which in light of the foregoing calculations implies that

$$I_1 + I_5 = \frac{2\pi}{4}.$$

Now,

$$I_1 = \int_{C_1} \frac{e^{iz}\,dz}{i2z} = \int_0^\infty \frac{e^{ix}\,dx}{i2x}$$

is the integral we wanted to compute in the first place, but what is that $I_5$? Answer:

$$I_5 = \int_{C_5} \frac{e^{iz\,dz}}{i2z} = \int_{-\infty}^0 \frac{e^{ix}\,dx}{i2x};$$

or, changing $-x \leftarrow x$,

$$I_5 = \int_0^\infty \frac{-e^{-ix}\,dx}{i2x},$$

which fortuitously happens to integrate the heretofore neglected term of the sine-argument function we started with. Thus,

$$\lim_{t\to\infty} \mathrm{Si}\,t = \int_0^\infty \mathrm{Sa}\,x\,dx = \int_0^\infty \frac{e^{+ix} - e^{-ix}}{i2x}\,dx = I_1 + I_5 = \frac{2\pi}{4},$$

which was to be computed.[20]

## 17.6 Gibbs' phenomenon

Section 17.4.4 has shown how the Fourier series suffices to represent a continuous, repeating waveform. Paradoxically, the chapter's examples have been of discontinuous waveforms like the square wave. At least in Fig. 17.2 the Fourier series seems to work for such discontinuous waveforms, though

[20] Integration by closed contour is a subtle technique, is it not? What a finesse this subsection's calculation has been! The author rather strongly sympathizes with the reader who still somehow cannot quite believe that contour integration actually works, but in the case of the sine integral another, quite independent method to evaluate the integral is known and it finds the same number $2\pi/4$. The interested reader can extract this other method from Gibbs' calculation in § 17.6, which refers a sine integral to the known amplitude of a square wave.

We said that it was fortuitous that $I_5$, which we did not know how to eliminate, turned out to be something we needed anyway; but is it really merely fortuitous, once one has grasped the technique? An integration of $-e^{-iz}/i2z$ is precisely the sort of thing an experienced applied mathematician would expect to fall out as a byproduct of the contour integration of $e^{iz}/i2z$. The trick is to discover the contour from which it actually does fall out, the discovery being a process of informed trial and error.

we have never exactly demonstrated that it should work for them, or how. So, what does all this imply?

In one sense, it does not imply much of anything. One can represent a discontinuity by a relatively sharp continuity—as for instance one can represent the Dirac delta of Fig. 7.10 by the triangular pulse of Fig. 17.3, with its sloped edges, if $T$ in (17.12) is sufficiently small—and, considered in this light, the Fourier series works. Mathematically however one is more likely to approximate a Fourier series by truncating it after some finite number $N$ of terms; and, indeed, so-called[21] "low-pass" physical systems that naturally suppress high frequencies[22] are common, in which case to truncate the series is more or less the right thing to do. Yet, a significant thing happens when one truncates the Fourier series. *At a discontinuity, the Fourier series oscillates and overshoots.*[23]

Henry Wilbraham investigated this phenomenon as early as 1848. J. Willard Gibbs explored its engineering implications in 1899.[24] Let us along with them refer to the square wave of Fig. 17.2 on page 489. As further Fourier components are added the Fourier waveform better approximates the square wave, but, as we said, it oscillates about and overshoots—it "rings about" in the electrical engineer's vernacular—the square wave's discontinuities (the verb "to ring" here recalls the ringing of a bell or steel beam). This oscillation and overshot turn out to be irreducible, and moreover they can have significant physical effects.

Changing $t - T_1/4 \leftarrow t$ in (17.2) to delay the square wave by a quarter cycle yields

$$f(t) = \frac{8A}{2\pi} \sum_{j=0}^{\infty} \frac{1}{2j+1} \sin\left[\frac{(2j+1)(2\pi)t}{T_1}\right],$$

which we can, if we like, write as

$$f(t) = \lim_{N\to\infty} \frac{8A}{2\pi} \sum_{j=0}^{N-1} \frac{1}{2j+1} \sin\left[\frac{(2j+1)(2\pi)t}{T_1}\right].$$

Again changing

$$\Delta v \leftarrow \frac{2(2\pi)t}{T_1}$$

[21] So called because they pass low frequencies while suppressing high ones, though systems encountered in practice admittedly typically suffer a middle frequency domain through which frequencies are only partly suppressed.

[22] [35, § 15.2]

[23] [38]

[24] [67][25]

makes this

$$f\left[\frac{T_1}{2(2\pi)}\,\Delta v\right] = \lim_{N\to\infty}\frac{4A}{2\pi}\sum_{j=0}^{N-1}\operatorname{Sa}\left[\left(j+\frac{1}{2}\right)\,\Delta v\right]\,\Delta v.$$

Stipulating that $\Delta v$ be infinitesimal,

$$0 < \Delta v \ll 1,$$

(which in light of the definition of $\Delta v$ is to stipulate that $0 < t \ll T_1$) such that $dv \equiv \Delta v$ and, therefore, that the summation become an integration; and further defining

$$u \equiv N\,\Delta v;$$

we have that

$$\lim_{N\to\infty} f\left[\frac{T_1}{2(2\pi)N}u\right] = \frac{4A}{2\pi}\int_0^u \operatorname{Sa} v\,dv = \frac{4A}{2\pi}\operatorname{Si} u. \tag{17.34}$$

Equation (17.33) gives us that $\lim_{u\to\infty}\operatorname{Si} u = 2\pi/4$, so (17.34) as it should has it that $f(t) \approx A$ when[25] $t \gtrapprox 0$. When $t \approx 0$ however it gives the waveform locally the sine integral's shape of Fig. 17.7.

Though unexpected the effect can and does actually arise in physical systems. When it does, the maximum value of $f(t)$ is of interest to mechanical and electrical engineers among others because, if an element in an engineered system will overshoot its designed position, the engineer wants to allow safely for the overshot. According to § 17.5.3, the sine integral $\operatorname{Si} u$ reaches its maximum at

$$u = \frac{2\pi}{2},$$

where according to (17.30)

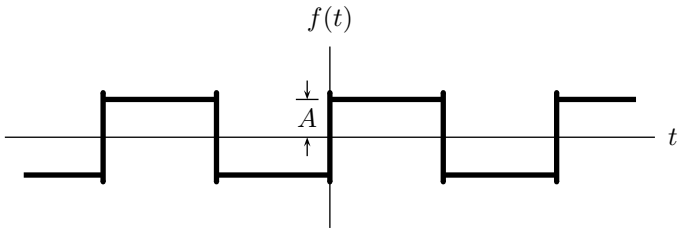
$$f_{\max} = \frac{4A}{2\pi}\operatorname{Si}\frac{2\pi}{2} = \frac{4A}{2\pi}\sum_{j=0}^{\infty}\left[\frac{2\pi/2}{2j+1}\prod_{m=1}^{j}\frac{-(2\pi/2)^2}{(2m)(2m+1)}\right] \approx (0\text{x}1.2\text{DD}2)A.$$

This overshot, peaking momentarily at $(0\text{x}1.2\text{DD}2)A$, and the associated sine-integral ringing constitute *Gibbs' phenomenon,* as Fig. 17.10 depicts.

We have said that Gibbs' phenomenon is irreducible, and indeed strictly this is so: a true discontinuity, if it is to obey Fourier, must overshoot according to Gibbs. Admittedly as earlier alluded, one can sometimes substantially evade Gibbs by softening a discontinuity's edge, giving it a steep

[25] Here is an exotic symbol: $\gtrapprox$. It means what it appears to mean, that $t > 0$ and $t \not\approx 0$.

Figure 17.10: Gibbs' phenomenon.




but not vertical slope and maybe rounding its corners a little;[26] or, alternately, by rolling the Fourier series off gradually rather than truncating it exactly at $N$ terms. Engineers may do one or the other, or both, explicitly or implicitly, which is why the full Gibbs is not always observed in engineered systems. Nature may do likewise. Neither however is the point. The point is that sharp discontinuities do not behave in the manner one might naïvely have expected, yet that one can still analyze them profitably, adapting this section's subtle technique as the circumstance might demand. A good engineer or other applied mathematician will make himself aware of Gibbs' phenomenon and of the mathematics behind it for this reason.

[26] If the applied mathematician is especially exacting he might represent a discontinuity by the probability integral of [not yet written] or maybe (if slightly less exacting) as an arctangent, and indeed there are times at which he might do so. However, such extra-fine mathematical craftsmanship is unnecessary to this section's purpose.

# Chapter 18

# The Fourier and Laplace transforms

The Fourier series of Ch. 17 though quite useful applies solely to waveforms that repeat. An effort to extend the Fourier series to the broader domain of nonrepeating waveforms leads to the *Fourier transform,* this chapter's chief subject. [This chapter is yet only a rough draft.]

## 18.1 The Fourier transform

This section derives and presents the Fourier transform, extending the Fourier series.

### 18.1.1 Fourier's equation

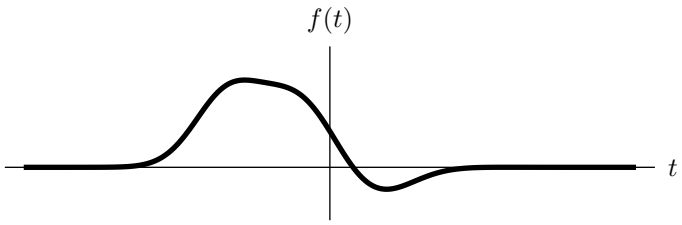
Consider the nonrepeating waveform or *pulse* of Fig. 18.1. Because the pulse does not repeat it has no Fourier series, yet one can however give it something very like a Fourier series in the following way. First, convert the pulse $f(t)$ into the pulse train

$$g(t) \equiv \sum_{n=-\infty}^{\infty} f(t - nT_1),$$

which naturally does repeat.[1] Second, by (17.15), calculate the Fourier coefficients of this pulse train $g(t)$. Third, use these coefficients in the Fourier

[1] One could divert rigorously from this point to consider formal requirements against $f(t)$ but it suffices that $f(t)$ be sufficiently limited in extent that $g(t)$ exist for all $\Re(T_1) > 0$, $\Im(T_1) = 0$. Formally, such a condition would forbid a function like $f(t) = A\cos\omega_o t$,

Figure 18.1: A pulse.




series (17.14) to reconstruct

$$g(t) = \sum_{j=-\infty}^{\infty} \left\{ \left[ \frac{1}{T_1} \int_{-T_1/2}^{T_1/2} e^{-ij\,\Delta\omega\,\tau} g(\tau)\,d\tau \right] e^{ij\,\Delta\omega\,t} \right\}.$$

Fourth, observing that $\lim_{T_1\to\infty} g(t) = f(t)$, recover from the train the original pulse

$$f(t) = \lim_{T_1\to\infty} \sum_{j=-\infty}^{\infty} \left\{ \left[ \frac{1}{T_1} \int_{-T_1/2}^{T_1/2} e^{-ij\,\Delta\omega\,\tau} f(\tau)\,d\tau \right] e^{ij\,\Delta\omega\,t} \right\};$$

or, observing per (17.3) that $\Delta\omega\,T_1 = 2\pi$ and reordering factors,

$$f(t) = \lim_{\Delta\omega\to 0^+} \frac{1}{\sqrt{2\pi}} \sum_{j=-\infty}^{\infty} e^{ij\,\Delta\omega\,t} \left[ \frac{1}{\sqrt{2\pi}} \int_{-2\pi/2\Delta\omega}^{2\pi/2\Delta\omega} e^{-ij\,\Delta\omega\,\tau} f(\tau)\,d\tau \right] \Delta\omega.$$

Fifth, defining the symbol $\omega \equiv j\,\Delta\omega$ observe that the summation is really an integration in the limit, such that

$$f(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{i\omega t} \left[ \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-i\omega\tau} f(\tau)\,d\tau \right] d\omega. \tag{18.1}$$

This is *Fourier's equation,* a remarkable, highly significant result.

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but one can evade this formality, among other ways, by defining the function as $f(t) = \lim_{T_2\to\infty} \Pi(t/T_2) A\cos\omega_o t$, where $\Pi(t)$ is the rectangular pulse of (17.10).

We will leave to the professionals further consideration of formal requirements.

### 18.1.2 The transform and inverse transform

The reader may agree that Fourier's equation (18.1) is curious, but in what way is it remarkable? To answer, let us observe that the quantity in (18.1)'s square braces,

$$F(\omega) = \mathcal{F}\left\{f(t)\right\} \equiv \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-i\omega\tau} f(\tau)\, d\tau, \tag{18.2}$$

is a function not of $t$ but rather of $\omega$. We conventionally give this function the capitalized symbol $F(\omega)$ and name it the *Fourier transform* of $f(t)$, introducing also the useful notation $\mathcal{F}\{\cdot\}$ (where the script letter $\mathcal{F}$ stands for "Fourier" and is only coincidentally, unfortunately, the same letter here as $f$ and $F$) as a short form to represent the transformation (18.2) serves to define. Substituting (18.2) into (18.1) and changing $\eta \leftarrow \omega$ as the dummy variable of integration, we have that

$$f(t) = \mathcal{F}^{-1}\left\{F(\omega)\right\} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{i\eta t} F(\eta)\, d\eta. \tag{18.3}$$

This last is the *inverse Fourier transform* of the function $F(\omega)$.

The Fourier transform (18.2) serves as a continuous measure of a function's frequency content. To understand why this should be so, consider that (18.3) constructs a function $f(t)$ of an infinity of infinitesimally graded complex exponentials and that (18.2) provides the weights $F(\omega)$ for the construction. Indeed, the Fourier transform's complementary equations (18.3) and (18.2) are but continuous versions of the earlier complementary equations (17.14) and (17.15) of the discrete Fourier series. The transform finds even wider application than does the series.[2]

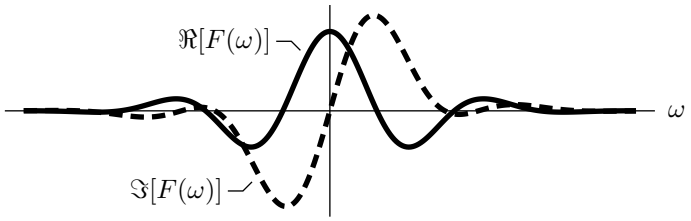
Figure 18.2 plots the Fourier transform of the pulse of Fig. 18.1.

### 18.1.3 The complementary variables of transformation

If $t$ represents time then $\omega$ represents angular frequency as § 17.2 has explained. In this case the function $f(t)$ is said to operate in the *time domain* and the corresponding transformed function $F(\omega)$, in the *frequency domain.*

[2]Regrettably, several alternate definitions and usages of the Fourier series are broadly current in the writer's country alone. Alternate definitions [48][13] handle the factors of $1/\sqrt{2\pi}$ differently. Alternate usages [19] change $-i \leftarrow i$ in certain circumstances. The essential Fourier mathematics however remains the same in any case. The reader can adapt the book's presentation to the Fourier definition and usage his colleagues prefer at need.

Figure 18.2: The Fourier transform of the pulse of Fig. 18.1.




The mutually independent variables $\omega$ and $t$ are then the *complementary variables of transformation.*

Formally, one can use any two letters in place of $t$ and $\omega$; and indeed one need not even use two different letters, for it is sometimes easier just to write

$$\begin{aligned} F(v) = \mathcal{F}\{f(v)\} &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-iv\theta} f(\theta)\, d\theta, \\ f(v) = \mathcal{F}^{-1}\{F(v)\} &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{iv\theta} F(\theta)\, d\theta, \\ \mathcal{F} &\equiv \mathcal{F}_{vv}, \end{aligned} \tag{18.4}$$

in which the $\theta$ is in itself no variable of transformation but only a dummy variable. To emphasize the distinction between the untransformed and transformed (respectively typically time and frequency) domains, however, one can instead write

$$\begin{aligned} F(\omega) = \mathcal{F}\{f(t)\} &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-i\omega t} f(t)\, dt, \\ f(t) = \mathcal{F}^{-1}\{F(\omega)\} &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{i\omega t} F(\omega)\, d\omega, \\ \mathcal{F} &\equiv \mathcal{F}_{\omega t}, \end{aligned} \tag{18.5}$$

where (18.5) is just (18.2) and (18.3) together with appropriate changes of dummy variable. Notice here the usage of the symbol $\mathcal{F}$, incidentally. As clarity demands, one can elaborate the $\mathcal{F}$—here or wherever else it appears—as $\mathcal{F}_{vv}$, $\mathcal{F}_{\omega t}$ or the like to identify the complementary variables of transformation explicitly. The unadorned symbol $\mathcal{F}$ however usually acquits itself clearly enough in context (refer to § A.2).

Whichever letter or letters might be used for the independent variable, the functions

$$f(v) \stackrel{\mathcal{F}}{\to} F(v) \tag{18.6}$$

constitute a *Fourier transform pair.*

### 18.1.4 An example

As a Fourier example, consider the triangular pulse $\Lambda(t)$ of (17.10). Its Fourier transform according to (18.4) is

$$\begin{aligned}
\mathcal{F}\left\{\Lambda(v)\right\} &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-iv\theta} \Lambda(\theta)\, d\theta \\
&= \frac{1}{\sqrt{2\pi}} \left\{ \int_{-1}^{0} e^{-iv\theta}(1+\theta)\, d\theta + \int_{0}^{1} e^{-iv\theta}(1-\theta)\, d\theta \right\}.
\end{aligned}$$

According to Table 9.1 (though it is easy enough to figure it out without recourse to the table), $\theta e^{-iv\theta} = [d/d\theta][e^{-iv\theta}(1+iv\theta)/v^2]$; so, continuing,

$$\begin{aligned}
\mathcal{F}\left\{\Lambda(v)\right\} &= \frac{1}{v^2\sqrt{2\pi}} \left\{ \left[ e^{-iv\theta}[1+(iv)(1+\theta)] \right]_{-1}^{0} \right. \\
&\qquad \left. + \left[ e^{-iv\theta}[-1+(iv)(1-\theta)] \right]_{0}^{1} \right\} \\
&= \frac{\mathrm{Sa}^2(v/2)}{\sqrt{2\pi}},
\end{aligned}$$

where $\mathrm{Sa}(\cdot)$ is the sine-argument function of (17.27). Thus we find the Fourier transform pair

$$\Lambda(v) \stackrel{\mathcal{F}}{\to} \frac{\mathrm{Sa}^2(v/2)}{\sqrt{2\pi}}. \tag{18.7}$$

One can compute other Fourier transforms in like manner, such as that

$$\Pi(v) \stackrel{\mathcal{F}}{\to} \frac{\mathrm{Sa}(v/2)}{\sqrt{2\pi}}. \tag{18.8}$$

and yet further transforms by the duality rule and the other properties of § 18.2.

## 18.2 Properties of the Fourier transform

The Fourier transform obeys an algebra of its own, exhibiting several broadly useful properties one might grasp to wield the transform effectively. This section derives and lists the properties.

### 18.2.1 Duality

Changing $-v \leftarrow v$ makes (18.4)'s second line to read

$$f(-v) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-iv\theta} F(\theta)\, d\theta.$$

However, according to (18.4)'s first line, this says neither more nor less than that

$$F(v) \stackrel{\mathcal{F}}{\to} f(-v), \tag{18.9}$$

which is that the transform of the transform is the original function with the independent variable reversed, an interesting and useful property. It is entertaining, and moreover enlightening, to combine (18.6) and (18.9) to form the endless transform progression

$$\cdots \stackrel{\mathcal{F}}{\to} f(v) \stackrel{\mathcal{F}}{\to} F(v) \stackrel{\mathcal{F}}{\to} f(-v) \stackrel{\mathcal{F}}{\to} F(-v) \stackrel{\mathcal{F}}{\to} f(v) \stackrel{\mathcal{F}}{\to} \cdots \tag{18.10}$$

Equation (18.9), or alternately (18.10), expresses the Fourier transform's *duality* rule.

The Fourier transform evinces duality in another guise too, *compositional duality,* expressed abstractly as

$$\begin{aligned} g[v, f(h_g(v))] &\stackrel{\mathcal{F}}{\to} G[v, F(h_G(v))], \\ G[v, f(h_G(v))] &\stackrel{\mathcal{F}}{\to} g[-v, F(-h_g(-v))]. \end{aligned} \tag{18.11}$$

This is best introduced by example. Consider the Fourier pair $\Lambda(v) \stackrel{\mathcal{F}}{\to} \mathrm{Sa}^2[v/2]/\sqrt{2\pi}$ mentioned in § 18.1.4, plus the Fourier identity $f(v-a) \stackrel{\mathcal{F}}{\to} e^{-iav}F(v)$ which we have not yet met but will in § 18.2.4 below. Identifying $f(v) = \Lambda(v)$ and $F(v) = \mathrm{Sa}^2[v/2]/\sqrt{2\pi}$, the identity extends the pair to $\Lambda(v-a) \stackrel{\mathcal{F}}{\to} e^{-iav}\,\mathrm{Sa}^2[v/2]/\sqrt{2\pi}$. On the other hand, recognizing $h_g(v) = v - a$, $g[v,(\cdot)] = (\cdot)$, $h_G(v) = v$, and $G[v,(\cdot)] = (e^{-iav})(\cdot)$, eqn. (18.11) converts the identity to its compositional dual $e^{-iav}f(v) \stackrel{\mathcal{F}}{\to} F(v+a)$, which in turn extends the pair to $e^{-iav}\Lambda(v) \stackrel{\mathcal{F}}{\to} \mathrm{Sa}^2[(v+a)/2]/\sqrt{2\pi}$. Note incidentally that the direct dual of the original pair per (18.10) is the pair $\mathrm{Sa}^2[v/2]/\sqrt{2\pi} \stackrel{\mathcal{F}}{\to} \Lambda(-v)$ which, since it happens that $\Lambda(-v) = \Lambda(v)$, is just the pair $\mathrm{Sa}^2[v/2]/\sqrt{2\pi} \stackrel{\mathcal{F}}{\to} \Lambda(v)$; but that we need neither the identity nor (18.11) to determine this.

So, assuming that (18.11) is correct, it does seem useful; but is it correct? To show that it is, take the direct dual on $v$ of (18.11)'s first line to get the

formal pair

$$G[v, F(h_G(v))] \xrightarrow{\mathcal{F}} g[-v, f(h_g(-v))],$$

then change the symbols $\phi \leftarrow f$ and $\Phi \leftarrow F$ to express the same formal pair as

$$G[v, \Phi(h_G(v))] \xrightarrow{\mathcal{F}} g[-v, \phi(h_g(-v))]. \tag{18.12}$$

Now, this as we said is merely a formal pair, which is to say that it represents no functions in particular but presents a pattern to which functions can be fitted. Therefore, $\phi([\cdot])$ might represent any function so long as $\Phi([\cdot])$ were let to represent the same function's Fourier transform on $[\cdot]$, as[3]

$$\phi([\cdot]) \xrightarrow{\mathcal{F}} \Phi([\cdot]).$$

Suppose some particular function $f([\cdot])$ whose Fourier transform on $[\cdot]$ is $F([\cdot])$, for the two of which there must exist—by direct duality thrice on $[\cdot]$—the Fourier pair

$$F(-[\cdot]) \xrightarrow{\mathcal{F}} f([\cdot]).$$

Let us define

$$\Phi([\cdot]) \equiv f([\cdot]),$$

whose inverse Fourier transform on $[\cdot]$, in view of the foregoing, cannot but be

$$\phi([\cdot]) \equiv F(-[\cdot]);$$

then observe that substituting these two, complementary definitions together into the formal pair (18.12) yields (18.11)'s second line, completing the proof.

Once the proof is understood, (18.11) is readily extended to

$$\begin{aligned} g[v, f_1(h_{g1}(v)), f_2(h_{g2}(v))] &\xrightarrow{\mathcal{F}} G[v, F_1(h_{G1}(v)), F_2(h_{G2}(v))], \\ G[v, f_1(h_{G1}(v)), f_2(h_{G2}(v))] &\xrightarrow{\mathcal{F}} g[-v, F_1(-h_{g1}(-v)), F_2(-h_{g2}(-v))]; \end{aligned} \tag{18.13}$$

[3]To be symbolically precise, the $\mathcal{F}$ here is $\mathcal{F}_{[\cdot][\cdot]}$, such that

$$\phi([\cdot]) \xrightarrow{\mathcal{F}_{[\cdot][\cdot]}} \Phi([\cdot]),$$
$$F(-[\cdot]) \xrightarrow{\mathcal{F}_{[\cdot][\cdot]}} f([\cdot]);$$

whereas the $\mathcal{F}$ in the formal pairs was $\mathcal{F}_{vv}$, such that

$$g[v, f(h_g(v))] \xrightarrow{\mathcal{F}_{vv}} G[v, F(h_G(v))],$$
$$G[v, f(h_G(v))] \xrightarrow{\mathcal{F}_{vv}} g[-v, F(-h_g(-v))].$$

Refer to § 18.1.3.

Table 18.1: Fourier duality rules. (Observe that the compositional rules, the table's several rules involving $g$, transform only properties valid for all $f[v]$.)

$$\begin{aligned}
f(v) &\overset{\mathcal{F}}{\to} F(v) \\
F(v) &\overset{\mathcal{F}}{\to} f(-v) \\
f(-v) &\overset{\mathcal{F}}{\to} F(-v) \\
F(-v) &\overset{\mathcal{F}}{\to} f(v)
\end{aligned}$$

$$\begin{aligned}
g[v, f(h_g(v))] &\overset{\mathcal{F}}{\to} G[v, F(h_G(v))] \\
G[v, f(h_G(v))] &\overset{\mathcal{F}}{\to} g[-v, F(-h_g(-v))]
\end{aligned}$$

$$\begin{aligned}
g[v, f_1(h_{g1}(v)), f_2(h_{g2}(v))] &\overset{\mathcal{F}}{\to} G[v, F_1(h_{G1}(v)), F_2(h_{G2}(v))] \\
G[v, f_1(h_{G1}(v)), f_2(h_{G2}(v))] &\overset{\mathcal{F}}{\to} g[-v, F_1(-h_{g1}(-v)), F_2(-h_{g2}(-v))]
\end{aligned}$$

$$\begin{aligned}
g[v, f_k(h_{gk}(v))] &\overset{\mathcal{F}}{\to} G[v, F_k(h_{Gk}(v))] \\
G[v, f_k(h_{Gk}(v))] &\overset{\mathcal{F}}{\to} g[-v, F_k(-h_{gk}(-v))]
\end{aligned}$$

and indeed generalized to

$$\begin{aligned}
g[v, f_k(h_{gk}(v))] &\overset{\mathcal{F}}{\to} G[v, F_k(h_{Gk}(v))], \\
G[v, f_k(h_{Gk}(v))] &\overset{\mathcal{F}}{\to} g[-v, F_k(-h_{gk}(-v))],
\end{aligned} \tag{18.14}$$

in which $g[v, f_k(h_{gk}(v))]$ means $g[v, f_1(h_{g1}(v)), f_2(h_{g2}(v)), f_3(h_{g3}(v)), \ldots]$.

Table 18.1 summarizes.

### 18.2.2 Real and imaginary parts

The Fourier transform of a function's conjugate according to (18.4) is

$$\mathcal{F}\{f^*(v)\} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-iv\theta} f^*(\theta)\, d\theta = \left[ \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{iv^*\theta} f(\theta)\, d\theta \right]^*,$$

in which we have taken advantage of the fact that the dummy variable $\theta = \theta^*$ happens to be real. This implies by (18.4) and (18.10) that

$$\mathcal{F}\{f^*(v)\} = \mathcal{F}^{-*}\{f(v^*)\} = \mathcal{F}^*\{f(-v^*)\}, \tag{18.15}$$

where the symbology $\mathcal{F}^{-*}\{\cdot\} \equiv [\mathcal{F}^{-1}\{\cdot\}]^*$ is used and $\mathcal{F}^{-1}\{g(w)\} = \mathcal{F}\{\mathcal{F}^{-1}\{\mathcal{F}^{-1}\{g(w)\}\}\} = \mathcal{F}\{g(-w)\}$. In the arrow notation, it implies that[4]

$$f^*(v) \stackrel{\mathcal{F}}{\to} F^*(-v^*). \tag{18.16}$$

If we express the real and imaginary parts of $f(v)$ in the style of (2.58) as

$$\begin{aligned}\Re[f(v)] &= \frac{f(v) + f^*(v)}{2},\\ \Im[f(v)] &= \frac{f(v) - f^*(v)}{i2},\end{aligned}$$

then the Fourier transforms of these parts according to (18.16) are[5]

$$\begin{aligned}\Re[f(v)] &\stackrel{\mathcal{F}}{\to} \frac{F(v) + F^*(-v^*)}{2},\\ \Im[f(v)] &\stackrel{\mathcal{F}}{\to} \frac{F(v) - F^*(-v^*)}{i2}.\end{aligned} \tag{18.17}$$

For real $v$ and an $f(v)$ which itself is real for all real $v$, the latter line becomes

$$0 \stackrel{\mathcal{F}}{\to} \frac{F(v) - F^*(-v)}{i2} \quad \text{if } \Im(v) = 0 \text{ and, for all such } v,\ \Im[f(v)] = 0,$$

[4]From past experience with complex conjugation, an applied mathematician might naturally have expected of (18.16) that $f^*(v) \stackrel{\mathcal{F}}{\to} F^*(v)$, but this natural expectation would actually have been incorrect. Readers whom this troubles might consider that, unlike most of the book's mathematics before Ch. 17, eqns. (17.14) and (17.15)—and thus ultimately also the Fourier transform's definition (18.4)—have arbitrarily chosen a particular sign for the $i$ in the phasing factor $e^{-ij\,\Delta\omega\,\tau}$ or $e^{-iv\theta}$, which phasing factor the Fourier integration bakes into the transformed function $F(v)$, so to speak. The Fourier transform as such therefore does not meet § 2.12.2's condition for (2.64) to hold. Fortunately, (18.16) does hold.

Viewed from another angle, it must be so, because Fourier transforms real functions into complex ones. See Figs. 18.1 and 18.2.

[5]The precisely orderly reader might note that a forward reference to Table 18.3 is here implied, but the property referred to, Fourier superposition $A_1 f_1(v) + A_2 f_2(v) \stackrel{\mathcal{F}}{\to} A_1 F_1(v) + A_2 F_2(v)$, which does not depend on this subsection's results anyway, is so trivial to prove that we will not bother about the precise ordering in this case.

Table 18.2: Real and imaginary parts of the Fourier transform.

$$\begin{aligned} f^*(v) &\stackrel{\mathcal{F}}{\to} F^*(-v^*) \\ \Re[f(v)] &\stackrel{\mathcal{F}}{\to} \frac{F(v) + F^*(-v^*)}{2} \\ \Im[f(v)] &\stackrel{\mathcal{F}}{\to} \frac{F(v) - F^*(-v^*)}{i2} \end{aligned}$$

If $\Im(v) = 0$ and, for all such $v$, $\Im[f(v)] = 0$, then

$$F(v) = F^*(-v).$$

whereby

$$F(v) = F^*(-v) \text{ if } \Im(v) = 0 \text{ and, for all such } v,\ \Im[f(v)] = 0. \tag{18.18}$$

Interpreted, (18.18) says for real $v$ and $f(v)$ that the plot of $\Re[F(v)]$ is symmetric about the vertical axis whereas the plot of $\Im[F(v)]$ is symmetric about the origin, as Fig. 18.2 has illustrated.

Table 18.2 summarizes.

### 18.2.3 The Fourier transform of the Dirac delta

Section 18.3 will compute several Fourier transform pairs but § 18.2.4 will need one particular pair and its dual sooner, so let us pause to compute these now. Applying (18.4) to the Dirac delta (7.12) and invoking its sifting property (7.14), we find curiously that

$$\delta(v) \stackrel{\mathcal{F}}{\to} \frac{1}{\sqrt{2\pi}}, \tag{18.19}$$

the dual of which according to (18.10) is

$$1 \stackrel{\mathcal{F}}{\to} \left(\sqrt{2\pi}\right) \delta(v) \tag{18.20}$$

inasmuch as $\delta(-v) = \delta(v)$. (The duality rule proves its worth in eqn. 18.20, incidentally. Had we tried to calculate the Fourier transform of 1—that is, of $f[v] \equiv 1$—directly according to eqn. 18.4 we would have found $1 \stackrel{\mathcal{F}}{\to} [1/\sqrt{2\pi}] \int_{-\infty}^{\infty} e^{-iv\theta}\, d\theta$, an impossible integral to evaluate.)

Table 18.3: Fourier properties involving shifting, scaling, differentiation and integration.

$$\begin{aligned}
f(v-a) &\overset{\mathcal{F}}{\to} e^{-iav}F(v) \\
e^{iav}f(v) &\overset{\mathcal{F}}{\to} F(v-a) \\
Af(\alpha v) &\overset{\mathcal{F}}{\to} \frac{A}{|\alpha|}F\left(\frac{v}{\alpha}\right) \quad \text{if } \Im(\alpha)=0,\ \Re(\alpha)\neq 0 \\
A_1 f_1(v) + A_2 f_2(v) &\overset{\mathcal{F}}{\to} A_1 F_1(v) + A_2 F_2(v) \\
\frac{d}{dv}f(v) &\overset{\mathcal{F}}{\to} ivF(v) \\
ivf(v) &\overset{\mathcal{F}}{\to} -\frac{d}{dv}F(v) \\
\int_{-\infty}^{v} f(\tau)\,d\tau &\overset{\mathcal{F}}{\to} \frac{F(v)}{iv} + \frac{2\pi}{2}F(0)\delta(v)
\end{aligned}$$

### 18.2.4 Shifting, scaling and differentiation

Table 18.3 lists several Fourier properties involving shifting, scaling and differentiation, plus an expression of the Fourier transform's linearity.

The table's first property is proved by applying (18.4) to $f(v-a)$ then changing $\xi \leftarrow \theta - a$. The table's second property is proved by applying (18.4) to $e^{iav}f(v)$; or, alternately, is proved through (18.11) as the composition dual of the table's first property. The table's third property is proved by applying (18.4) to $Af(\alpha v)$ then changing $\xi \leftarrow \alpha\theta$. The table's fourth property is proved trivially.

The table's fifth and sixth properties begin from the derivative of the inverse Fourier transform; that is, of (18.4)'s second line. This derivative is

$$\begin{aligned}
\frac{d}{dv}f(v) &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} i\theta e^{iv\theta}F(\theta)\,d\theta \\
&= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{iv\theta}[i\theta F(\theta)]\,d\theta \\
&= \mathcal{F}^{-1}\{ivF(v)\},
\end{aligned}$$

which implies

$$\mathcal{F}\left\{\frac{d}{dv}f(v)\right\} = ivF(v),$$

the table's fifth property. The sixth and last property is the compositional dual (18.11) of the fifth.

Besides the identities this section derives, Table 18.3 also includes (18.45), which § 18.4 will prove.

### 18.2.5 Convolution and correlation

The concept of *convolution* emerges from mechanical engineering (or from its subdisciplines electrical and chemical engineering), in which the response of a linear system to an impulse $\delta(t)$ is some characteristic *transfer function* $h(t)$. Since the system is linear, it follows that its response to an arbitrary input $f(t)$ is

$$g(t) \equiv \int_{-\infty}^{\infty} h(t-\tau) f(\tau)\, d\tau;$$

or, changing $t/2 + \tau \leftarrow \tau$ to improve the equation's symmetry,

$$g(t) \equiv \int_{-\infty}^{\infty} h\left(\frac{t}{2} - \tau\right) f\left(\frac{t}{2} + \tau\right) d\tau. \tag{18.21}$$

This integral defines[6] convolution of the two functions $f(t)$ and $h(t)$.

Changing $v \leftarrow t$ and $\psi \leftarrow \tau$ in (18.21) to comport with the notation found elsewhere in this section then applying (18.4) yields[7]

$$\begin{aligned}
&\mathcal{F}\left\{\int_{-\infty}^{\infty} h\left(\frac{v}{2} - \psi\right) f\left(\frac{v}{2} + \psi\right) d\psi\right\} \\
&= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-iv\theta} \int_{-\infty}^{\infty} h\left(\frac{\theta}{2} - \psi\right) f\left(\frac{\theta}{2} + \psi\right) d\psi\, d\theta \\
&= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-iv\theta} h\left(\frac{\theta}{2} - \psi\right) f\left(\frac{\theta}{2} + \psi\right) d\theta\, d\psi.
\end{aligned}$$

Now changing $\phi \leftarrow \theta/2 + \psi$,

$$\begin{aligned}
&\mathcal{F}\left\{\int_{-\infty}^{\infty} h\left(\frac{v}{2} - \psi\right) f\left(\frac{v}{2} + \psi\right) d\psi\right\} \\
&= \frac{2}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-iv(2\phi - 2\psi)} h(\phi - 2\psi) f(\phi)\, d\phi\, d\psi \\
&= \frac{2}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-iv\phi} f(\phi) \int_{-\infty}^{\infty} e^{-iv(\phi - 2\psi)} h(\phi - 2\psi)\, d\psi\, d\phi.
\end{aligned}$$

[6] [?, § 2.2]

[7] See Ch. 17's footnote 8.

Again changing $\chi \leftarrow \phi - 2\psi$,

$$\begin{aligned}
&\mathcal{F}\left\{\int_{-\infty}^{\infty} h\left(\frac{v}{2}-\psi\right) f\left(\frac{v}{2}+\psi\right)\, d\psi\right\} \\
&= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-iv\phi} f(\phi) \int_{-\infty}^{\infty} e^{-iv\chi} h(\chi)\, d\chi\, d\phi \\
&= \left[\sqrt{2\pi}\right]\left[\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-iv\chi} h(\chi)\, d\chi\right]\left[\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-iv\phi} f(\phi)\, d\phi\right] \\
&= \left(\sqrt{2\pi}\right) H(v)F(v).
\end{aligned}$$

That is,

$$\int_{-\infty}^{\infty} h\left(\frac{v}{2}-\psi\right) f\left(\frac{v}{2}+\psi\right)\, d\psi \stackrel{\mathcal{F}}{\to} \left(\sqrt{2\pi}\right) H(v)F(v). \tag{18.22}$$

The compositional dual (18.11) of (18.22) is

$$h(v)f(v) \stackrel{\mathcal{F}}{\to} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} H\left(\frac{v}{2}-\psi\right) F\left(\frac{v}{2}+\psi\right)\, d\psi, \tag{18.23}$$

in which we have changed $-\psi \leftarrow \psi$ as the dummy variable of integration. Whether by (18.22) or by (18.23), convolution in the one domain evidently transforms to multiplication in the other.

Closely related to the convolutional integral (18.21) is the integral

$$g(t) \equiv \int_{-\infty}^{\infty} h\left(\tau - \frac{t}{2}\right) f\left(\tau + \frac{t}{2}\right)\, d\tau, \tag{18.24}$$

whose transform and dual transform are computed as in the last paragraph to be

$$\begin{gathered}
\int_{-\infty}^{\infty} h\left(\psi-\frac{v}{2}\right) f\left(\psi+\frac{v}{2}\right)\, d\psi \stackrel{\mathcal{F}}{\to} \left(\sqrt{2\pi}\right) H(-v)F(v), \\
h(-v)f(v) \stackrel{\mathcal{F}}{\to} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} H\left(\psi-\frac{v}{2}\right) F\left(\psi+\frac{v}{2}\right)\, d\psi.
\end{gathered} \tag{18.25}$$

Furthermore, according to (18.16), $h^*(v) \stackrel{\mathcal{F}}{\to} H^*(-v^*)$, so

$$\begin{gathered}
\int_{-\infty}^{\infty} h^*\left(\psi-\frac{v}{2}\right) f\left(\psi+\frac{v}{2}\right)\, d\psi \stackrel{\mathcal{F}}{\to} \left(\sqrt{2\pi}\right) H^*(v^*)F(v), \\
h^*(v^*)f(v) \stackrel{\mathcal{F}}{\to} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} H^*\left(\psi-\frac{v}{2}\right) F\left(\psi+\frac{v}{2}\right)\, d\psi,
\end{gathered} \tag{18.26}$$

in which the second line is the compositional dual of the first with, as before, the dummy variable $-\psi \leftarrow \psi$ changed; and indeed one can do the same to the transform (18.22) of the convolutional integral, obtaining

$$\begin{aligned}\int_{-\infty}^{\infty} h^*\left(\frac{v}{2}-\psi\right) f\left(\frac{v}{2}+\psi\right)\,d\psi &\overset{\mathcal{F}}{\to} \left(\sqrt{2\pi}\right) H^*(-v^*)F(v),\\ h^*(v)f(v) &\overset{\mathcal{F}}{\to} \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} H^*\left(\frac{v}{2}-\psi\right) F\left(\frac{v}{2}+\psi\right)\,d\psi.\end{aligned} \tag{18.27}$$

Unlike the operation the integral (18.21) expresses, known as convolution, the operation the integral (18.24) expresses has no special name as far as the writer is aware. However, the operation its variant

$$g(t) \equiv \int_{-\infty}^{\infty} h^*\left(\tau-\frac{t}{2}\right) f\left(\tau+\frac{t}{2}\right)\,d\tau \tag{18.28}$$

expresses does have a name. It is called *correlation,* being a measure of the degree to which one function tracks another with an offset in the independent variable. Reviewing this subsection, we see that in (18.26) we have already determined the transform and dual transform of the correlational integral (18.28). Convolution and correlation arise often enough in applications to enjoy their own, peculiar notations[8]

$$h(t) * f(t) \equiv \int_{-\infty}^{\infty} h\left(\frac{t}{2}-\tau\right) f\left(\frac{t}{2}+\tau\right)\,d\tau \tag{18.29}$$

for convolution and

$$R_{fh}(t) \equiv \int_{-\infty}^{\infty} h^*\left(\tau-\frac{t}{2}\right) f\left(\tau+\frac{t}{2}\right)\,d\tau \tag{18.30}$$

for correlation. Nothing prevents one from correlating a function with itself, incidentally. The *autocorrelation*

$$R_{ff}(t) = \int_{-\infty}^{\infty} f^*\left(\tau-\frac{t}{2}\right) f\left(\tau+\frac{t}{2}\right)\,d\tau \tag{18.31}$$

proves useful at times.[9] For convolution, the commutative and associative properties that

$$\begin{aligned}f(t) * h(t) &= h(t) * f(t),\\ f(t) * [g(t) * h(t)] &= [f(t) * g(t)] * h(t),\end{aligned} \tag{18.32}$$

[8] [35, § 19.4]
[9] [**?**, § 1.6A]

are useful, too, where the former may be demonstrated by changing $-\tau \leftarrow \tau$ in (18.29) and, through Fourier transformation, both may be demonstrated as $f(v) * [g(v) * h(v)] \overset{\mathcal{F}}{\to} (\sqrt{2\pi})F(v)[(\sqrt{2\pi})G(v)H(v)] = (\sqrt{2\pi})[(\sqrt{2\pi})F(v)G(v)]H(v) \overset{\mathcal{F}^{-1}}{\to} [f(v) * g(v)] * h(v)$ and similarly for the commutative property.

Tables 18.4 and 18.5 summarize.

Before closing the section, we should take note of one entry of 18.5 in particular, $R_{fh}(v) \overset{\mathcal{F}}{\to} (\sqrt{2\pi})H^*(v^*)F(v)$. This same entry is also found in Table 18.4 in other notation—indeed it is just the first line of (18.26)—but when written in the correlation's peculiar notation it draws attention to a peculiar result. Scaled by $1/\sqrt{2\pi}$, the *autocorrelation* and its Fourier transform are evidently

$$\frac{1}{\sqrt{2\pi}} R_{ff}(v) \overset{\mathcal{F}}{\to} F^*(v^*)F(v).$$

For[10] real $v$,

$$\frac{1}{\sqrt{2\pi}} R_{ff}(v) \overset{\mathcal{F}}{\to} |F(v)|^2 \quad \text{if } \Im(v) = 0. \tag{18.33}$$

### 18.2.6 Parseval's theorem

By successive steps,

$$\begin{aligned}
\int_{-\infty}^{\infty} h^*(v)f(v)\,dv &= \int_{-\infty}^{\infty} h^*(v) \left[ \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{iv\theta} F(\theta)\,d\theta \right] dv \\
&= \int_{-\infty}^{\infty} \left[ \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{iv\theta} h^*(v)\,dv \right] F(\theta)\,d\theta \\
&= \int_{-\infty}^{\infty} \left[ \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-iv\theta} h(v)\,dv \right]^* F(\theta)\,d\theta. \\
&= \int_{-\infty}^{\infty} H^*(\theta)F(\theta)\,d\theta,
\end{aligned}$$

in which we have used (18.4), interchanged the integrations and assumed that the dummy variables $v$ and $\theta$ of integration remain real. Changing $v \leftarrow \theta$ on the right, we have that

$$\int_{-\infty}^{\infty} h^*(v)f(v)\,dv = \int_{-\infty}^{\infty} H^*(v)F(v)\,dv. \tag{18.34}$$

[10] Electrical engineers call the quantity $|F(v)|^2$ on (18.33)'s right the *energy spectral density* of $f(v)$. [**?**, § 1.6B]

Table 18.4: Convolution and correlation, and their Fourier properties.

$$
\begin{aligned}
\int_{-\infty}^{\infty} h\left(\frac{v}{2}-\psi\right) f\left(\frac{v}{2}+\psi\right)\, d\psi &\stackrel{\mathcal{F}}{\to} \left(\sqrt{2\pi}\right) H(v)F(v) \\
h(v)f(v) &\stackrel{\mathcal{F}}{\to} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} H\left(\frac{v}{2}-\psi\right) \\
&\qquad \times F\left(\frac{v}{2}+\psi\right)\, d\psi \\
\int_{-\infty}^{\infty} h\left(\psi-\frac{v}{2}\right) f\left(\psi+\frac{v}{2}\right)\, d\psi &\stackrel{\mathcal{F}}{\to} \left(\sqrt{2\pi}\right) H(-v)F(v) \\
h(-v)f(v) &\stackrel{\mathcal{F}}{\to} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} H\left(\psi-\frac{v}{2}\right) \\
&\qquad \times F\left(\psi+\frac{v}{2}\right)\, d\psi \\
\int_{-\infty}^{\infty} h^*\left(\frac{v}{2}-\psi\right) f\left(\frac{v}{2}+\psi\right)\, d\psi &\stackrel{\mathcal{F}}{\to} \left(\sqrt{2\pi}\right) H^*(-v^*)F(v) \\
h^*(v)f(v) &\stackrel{\mathcal{F}}{\to} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} H^*\left(\frac{v}{2}-\psi\right) \\
&\qquad \times F\left(\frac{v}{2}+\psi\right)\, d\psi \\
\int_{-\infty}^{\infty} h^*\left(\psi-\frac{v}{2}\right) f\left(\psi+\frac{v}{2}\right)\, d\psi &\stackrel{\mathcal{F}}{\to} \left(\sqrt{2\pi}\right) H^*(v^*)F(v) \\
h^*(v^*)f(v) &\stackrel{\mathcal{F}}{\to} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} H^*\left(\psi-\frac{v}{2}\right) \\
&\qquad \times F\left(\psi+\frac{v}{2}\right)\, d\psi
\end{aligned}
$$

Table 18.5: Convolution and correlation in their peculiar notation. (Note that the $*$ which appears in the table as $h[t] * f[t]$ differs in meaning from the $^*$ in $h^*[v^*]$.)

$$\begin{aligned}
f(t) * h(t) = h(t) * f(t) &\equiv \int_{-\infty}^{\infty} h\left(\frac{t}{2} - \tau\right) f\left(\frac{t}{2} + \tau\right)\, d\tau \\
R_{fh}(t) &\equiv \int_{-\infty}^{\infty} h^*\left(\tau - \frac{t}{2}\right) f\left(\tau + \frac{t}{2}\right)\, d\tau \\
h(v) * f(v) &\stackrel{\mathcal{F}}{\to} \left(\sqrt{2\pi}\right) H(v)F(v) \\
h(v)f(v) &\stackrel{\mathcal{F}}{\to} \frac{1}{\sqrt{2\pi}}[H(v) * F(v)] \\
R_{fh}(v) &\stackrel{\mathcal{F}}{\to} \left(\sqrt{2\pi}\right) H^*(v^*)F(v) \\
h^*(v^*)f(v) &\stackrel{\mathcal{F}}{\to} \frac{1}{\sqrt{2\pi}} R_{FH}(v) \\
f(t) * [g(t) * h(t)] &= [f(t) * g(t)] * h(t)
\end{aligned}$$

This is *Parseval's theorem.*[11]

Especially interesting is the special case $h(t) = f(t)$, when

$$\int_{-\infty}^{\infty} |f(v)|^2\, dv = \int_{-\infty}^{\infty} |F(v)|^2\, dv. \tag{18.35}$$

When this is written as

$$\int_{-\infty}^{\infty} |f(t)|^2\, dt = \int_{-\infty}^{\infty} |F(\omega)|^2\, d\omega,$$

and $t$, $|f(t)|^2$, $\omega$ and $|F(\omega)|^2$ respectively have physical dimensions of time, energy per unit time, angular frequency and energy per unit angular frequency, then the theorem conveys the important physical insight that energy transmitted at various times can equally well be regarded as energy transmitted at various frequencies. This works for space and spatial frequencies, too: see § 17.2. For real $f(v)$, one can write (18.35) as

$$\int_{-\infty}^{\infty} f^2(v)\, dv = \int_{-\infty}^{\infty} \Re^2[F(v)]\, dv + \int_{-\infty}^{\infty} \Im^2[F(v)]\, dv \tag{18.36}$$

[11] [13, § 2-2][?, § 1.6B]

which expresses the principle of *quadrature,* conveying the additional physical insight that a single frequency can carry energy in not one but each of two distinct, independent channels; namely, a *real-phased, in-phase* or $I$ channel and an *imaginary-phased, quadrature-phase* or $Q$ channel.[12] Practical digital electronic communications systems, wired or wireless, often do precisely this—effectively transmitting two, independent streams of information at once, without conflict, in the selfsame band.

### 18.2.7 Oddness and evenness

Odd functions have odd transforms. Even functions have even transforms. Symbolically,

- if $f(-v) = -f(v)$ for all $v$, then $F(-v) = -F(v)$;
- if $f(-v) = f(v)$ for all $v$, then $F(-v) = F(v)$.

In the odd case, this is seen by expressing $F(-v)$ per (18.4) as

$$F(-v) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-i(-v)\theta} f(\theta)\, d\theta,$$

then changing the dummy variable $-\theta \leftarrow \theta$ to get

$$\begin{aligned} F(-v) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-i(-v)(-\theta)} f(-\theta)\, d\theta \\ &= -\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-iv\theta} f(\theta)\, d\theta \\ &= -F(v). \end{aligned}$$

The even case is analyzed likewise. See § 8.12.

## 18.3 The Fourier transforms of selected functions

We have already computed the Fourier transforms of $\Pi(v)$, $\Lambda(v)$, $\delta(v)$ and the constant 1 in (18.8), (18.7), (18.19) and (18.20), respectively. The duals (18.10) of the first two of these are evidently

$$\begin{aligned} \mathrm{Sa}\left(\frac{v}{2}\right) &\overset{\mathcal{F}}{\to} \left(\sqrt{2\pi}\right) \Pi(-v), \\ \mathrm{Sa}^2\left(\frac{v}{2}\right) &\overset{\mathcal{F}}{\to} \left(\sqrt{2\pi}\right) \Lambda(-v); \end{aligned}$$

[12] [13, § 5-1]

or, since $\Pi(-v) = \Pi(v)$ and $\Lambda(-v) = \Lambda(v)$,

$$\begin{aligned} \mathrm{Sa}\left(\frac{v}{2}\right) &\overset{\mathcal{F}}{\to} \left(\sqrt{2\pi}\right)\Pi(v), \\ \mathrm{Sa}^2\left(\frac{v}{2}\right) &\overset{\mathcal{F}}{\to} \left(\sqrt{2\pi}\right)\Lambda(v), \end{aligned}$$

which by the scaling property of Table 18.3 imply that[13]

$$\begin{aligned} \mathrm{Sa}(v) &\overset{\mathcal{F}}{\to} \frac{\sqrt{2\pi}}{2}\Pi\left(\frac{v}{2}\right), \\ \mathrm{Sa}^2(v) &\overset{\mathcal{F}}{\to} \frac{\sqrt{2\pi}}{2}\Lambda\left(\frac{v}{2}\right). \end{aligned} \tag{18.37}$$

Applying the Fourier transform's definition (18.4) to $u(v)e^{-av}$, where $u(v)$ is the Heaviside unit step (7.11), yields

$$\mathcal{F}\left\{u(v)e^{-av}\right\} = \frac{1}{\sqrt{2\pi}}\int_0^\infty e^{-(a+iv)\theta}\,d\theta = \frac{1}{\sqrt{2\pi}}\left[\frac{e^{-(a+iv)\theta}}{-(a+iv)}\right]_0^\infty,$$

revealing the transform pair

$$u(v)e^{-av} \overset{\mathcal{F}}{\to} \frac{1}{\left(\sqrt{2\pi}\right)(a+iv)}, \quad \Re(a) > 0. \tag{18.38}$$

Interesting is the limit $a \to 0^+$ in (18.38),

$$u(v) \overset{\mathcal{F}}{\to} \frac{1}{\left(\sqrt{2\pi}\right)(iv)} + C\delta(v),$$

where the necessary term $C\delta(v)$, with scale $C$ to be determined, merely admits that we do not yet know how to evaluate (18.38) when both $a$ and $v$ vanish at once. What we do know from § 18.2.7 is that odd functions have odd transforms and that (as one can readily see in Fig. 7.9) one can convert $u(v)$ to an odd function by the simple expedient of subtracting $1/2$ from it. Since $1/2 \overset{\mathcal{F}}{\to} (\sqrt{2\pi}/2)\delta(v)$ according to (18.20), we have then that

$$u(v) - \frac{1}{2} \overset{\mathcal{F}}{\to} \frac{1}{\left(\sqrt{2\pi}\right)(iv)} + \left(C - \frac{\sqrt{2\pi}}{2}\right)\delta(v),$$

[13] In electronic communications systems, including radio, the first line of (18.37) implies significantly that, to spread energy evenly over an available "baseband" but to let no energy leak outside that band, one should transmit sine-argument-shaped pulses as in Fig. 17.6.

which to make its right side odd demands that $C = \sqrt{2\pi}/2$. The transform pair

$$u(v) \stackrel{\mathcal{F}}{\to} \frac{1}{\left(\sqrt{2\pi}\right) iv} + \frac{\sqrt{2\pi}}{2}\delta(v) \tag{18.39}$$

results. On the other hand, according to Table 9.1,

$$e^{-av}v^n = \frac{d}{dv}\sum_{k=0}^{n} \frac{-e^{-av}v^k}{(n!/k!)a^{n-k+1}},$$

so

$$\begin{aligned}\mathcal{F}\left\{u(v)e^{-av}v^n\right\} &= \frac{1}{\sqrt{2\pi}}\int_0^\infty e^{-(a+iv)\theta}\theta^n\,d\theta \\ &= \frac{1}{\sqrt{2\pi}}\sum_{k=0}^{n}\frac{-e^{-(a+iv)\theta}\theta^k}{(n!/k!)(a+iv)^{n-k+1}}\bigg|_0^\infty .\end{aligned}$$

Since all but the $k = 0$ term vanish, the last equation implies the transform pair

$$u(v)e^{-av}v^n \stackrel{\mathcal{F}}{\to} \frac{1}{\sqrt{2\pi}n!(a+iv)^{n+1}}, \quad \Re(a) > 0,\ n \in \mathbb{Z},\ n \geq 0. \tag{18.40}$$

The Fourier transforms of $\sin av$ and $\cos av$ are interesting and important, and can be computed straightforwardly from the pairs

$$\begin{aligned} e^{iav} &\stackrel{\mathcal{F}}{\to} \left(\sqrt{2\pi}\right)\delta(v-a), \\ e^{-iav} &\stackrel{\mathcal{F}}{\to} \left(\sqrt{2\pi}\right)\delta(v+a), \end{aligned} \tag{18.41}$$

which result by applying to (18.20) Table 18.3's property that $e^{iav}f(v) \stackrel{\mathcal{F}}{\to} F(v-a)$. Composing per Table 5.1 the trigonometrics from their complex parts, we have that

$$\begin{aligned} \sin av &\stackrel{\mathcal{F}}{\to} \frac{\sqrt{2\pi}}{j2}\left[\delta(v-a) - \delta(v+a)\right], \\ \cos av &\stackrel{\mathcal{F}}{\to} \frac{\sqrt{2\pi}}{2}\left[\delta(v-a) + \delta(v+a)\right]. \end{aligned} \tag{18.42}$$

Curiously, the Fourier transform of the Dirac delta pulse train of Fig. 17.5 turns out to be another Dirac delta pulse train. The reason is that the Dirac delta pulse train's Fourier series according to (17.21) and (17.14) is

$$\sum_{j=-\infty}^{\infty}\delta(v - jT_1) = \sum_{j=-\infty}^{\infty}\frac{e^{ij(2\pi/T_1)v}}{T_1},$$

the transform of which according to (18.41) is

$$\sum_{j=-\infty}^{\infty} \delta(v - jT_1) \stackrel{\mathcal{F}}{\to} \frac{\sqrt{2\pi}}{T_1} \sum_{j=-\infty}^{\infty} \delta\left(v - j\frac{2\pi}{T_1}\right). \tag{18.43}$$

Apparently, the further the pulses of the original train, the closer the pulses of the transformed train, and vice versa; yet, even when transformed, the train remains a train of Dirac deltas. Letting $T_1 = \sqrt{2\pi}$ in (18.43) we find the pair

$$\sum_{j=-\infty}^{\infty} \delta\left(v - j\sqrt{2\pi}\right) \stackrel{\mathcal{F}}{\to} \sum_{j=-\infty}^{\infty} \delta\left(v - j\sqrt{2\pi}\right), \tag{18.44}$$

discovering a pulse train whose Fourier transform is itself.

Table 18.6 summarizes. Besides gathering transform pairs from this and earlier sections, the table also covers the Gaussian pulse of § 18.5.

## 18.4 The Fourier transform of the integration operation

Though it includes the Fourier transform of the differentiation operation, Table 18.3 omits the complementary identity

$$\int_{-\infty}^{v} f(\tau)\, d\tau \stackrel{\mathcal{F}}{\to} \frac{F(v)}{iv} + \frac{2\pi}{2} F(0)\delta(v), \tag{18.45}$$

the Fourier transform of the integration operation, for when we compiled the table we lacked the needed theory. We have the theory now.[14]

To develop (18.45), we begin by observing that one can express the integration in question in either of the equivalent forms

$$\int_{-\infty}^{v} f(\tau)\, d\tau = \int_{-\infty}^{\infty} u\left(\frac{v}{2} - \tau\right) f\left(\frac{v}{2} + \tau\right)\, d\tau.$$

Invoking an identity of Table 18.4 on the rightward form, then substituting the leftward form, yields the Fourier pair

$$\begin{aligned} \int_{-\infty}^{v} f(\tau)\, d\tau &\stackrel{\mathcal{F}}{\to} \left(\sqrt{2\pi}\right) H(v)F(v), \\ H(v) &\equiv \mathcal{F}\{u(v)\}. \end{aligned}$$

[14] [?, Prob. 5.33]

Table 18.6: Fourier transform pairs.

$$
\begin{aligned}
1 &\overset{\mathcal{F}}{\to} \left(\sqrt{2\pi}\right)\delta(v) \\
u(v) &\overset{\mathcal{F}}{\to} \frac{1}{\left(\sqrt{2\pi}\right)iv} + \frac{\sqrt{2\pi}}{2}\delta(v) \\
\delta(v) &\overset{\mathcal{F}}{\to} \frac{1}{\sqrt{2\pi}} \\
\Pi(v) &\overset{\mathcal{F}}{\to} \frac{\mathrm{Sa}(v/2)}{\sqrt{2\pi}} \\
\Lambda(v) &\overset{\mathcal{F}}{\to} \frac{\mathrm{Sa}^2(v/2)}{\sqrt{2\pi}} \\
u(v)e^{-av} &\overset{\mathcal{F}}{\to} \frac{1}{\left(\sqrt{2\pi}\right)(a+iv)}, \quad \Re(a) > 0 \\
u(v)e^{-av}v^n &\overset{\mathcal{F}}{\to} \frac{1}{\left(\sqrt{2\pi}\right)n!(a+iv)^{n+1}}, \\
&\qquad \Re(a) > 0,\ n \in \mathbb{Z},\ n \geq 0 \\
e^{iav} &\overset{\mathcal{F}}{\to} \left(\sqrt{2\pi}\right)\delta(v-a) \\
\sin av &\overset{\mathcal{F}}{\to} \frac{\sqrt{2\pi}}{j2}\left[\delta(v-a) - \delta(v+a)\right] \\
\cos av &\overset{\mathcal{F}}{\to} \frac{\sqrt{2\pi}}{2}\left[\delta(v-a) + \delta(v+a)\right] \\
\mathrm{Sa}(v) &\overset{\mathcal{F}}{\to} \frac{\sqrt{2\pi}}{2}\Pi\left(\frac{v}{2}\right) \\
\mathrm{Sa}^2(v) &\overset{\mathcal{F}}{\to} \frac{\sqrt{2\pi}}{2}\Lambda\left(\frac{v}{2}\right) \\
\sum_{j=-\infty}^{\infty}\delta(v - jT_1) &\overset{\mathcal{F}}{\to} \frac{\sqrt{2\pi}}{T_1}\sum_{j=-\infty}^{\infty}\delta\left(v - j\frac{2\pi}{T_1}\right) \\
\sum_{j=-\infty}^{\infty}\delta\left(v - j\sqrt{2\pi}\right) &\overset{\mathcal{F}}{\to} \sum_{j=-\infty}^{\infty}\delta\left(v - j\sqrt{2\pi}\right) \\
\Omega(v) &\overset{\mathcal{F}}{\to} \Omega(v)
\end{aligned}
$$

But according to Table 18.6, $\mathcal{F}\{u(v)\} = 1/[(\sqrt{2\pi})iv] + (\sqrt{2\pi}/2)\delta(v)$, so

$$\int_{-\infty}^{v} f(\tau)\,d\tau \stackrel{\mathcal{F}}{\to} \frac{F(v)}{iv} + \frac{2\pi}{2}F(v)\delta(v),$$

of which sifting the rightmost term produces (18.45).

## 18.5 The Gaussian pulse

While studying the derivative in Chs. 4 and 5, we asked among other questions whether any function could be its own derivative. We found that a sinusoid could be its own derivative after a fashion—differentiation shifted its curve leftward but did not alter its shape—but that the only nontrivial function to be exactly its own derivative was the natural exponential $f(z) = Ae^z$. We later found the same natural exponential to fill several significant mathematical roles—largely, whether directly or indirectly, because it was indeed its own derivative.

Studying the Fourier transform, the question now arises again: can any function be its own transform? Well, we have already found in § 18.3 that the Dirac delta pulse train can be; but this train unlike the natural exponential is abrupt and ungraceful, perhaps not the sort of function one had in mind. One should like an analytic function, and preferably not a train but a single pulse.

In Chs. 19 and 20, during the study of special functions and probability, we shall encounter a most curious function, the *Gaussian pulse,* also known as the *bell curve* among other names. We will defer discussion of the Gaussian pulse's provenance to the coming chapters but, for now, we can just copy here the pulse's definition from (20.15) as

$$\Omega(t) \equiv \frac{\exp\left(-t^2/2\right)}{\sqrt{2\pi}}, \tag{18.46}$$

plotted on pages 562 below and 493 above, respectively in Figs. 20.1 and 17.3. The Fourier transform of the Gaussian pulse is even trickier to compute than were the transforms of § 18.3, but known techniques to compute it include the following.[15] From the definition (18.4) of the Fourier transform,

$$\mathcal{F}\{\Omega(v)\} = \frac{1}{2\pi}\int_{-\infty}^{\infty} \exp\left(-\frac{\theta^2}{2} - iv\theta\right)\,d\theta.$$

[15] An alternate technique is outlined in [?, Prob. 5.43].

Completing the square (§ 2.2),[16]

$$\begin{aligned}\mathcal{F}\{\Omega(v)\} &= \frac{\exp\left(-v^2/2\right)}{2\pi}\int_{-\infty}^{\infty}\exp\left(-\frac{\theta^2}{2}-iv\theta+\frac{v^2}{2}\right)\,d\theta \\ &= \frac{\exp\left(-v^2/2\right)}{2\pi}\int_{-\infty}^{\infty}\exp\left[-\frac{(\theta+iv)^2}{2}\right]\,d\theta.\end{aligned}$$

Changing $\xi \leftarrow \theta + iv$,

$$\mathcal{F}\{\Omega(v)\} = \frac{\exp\left(-v^2/2\right)}{2\pi}\int_{-\infty+iv}^{\infty+iv}\exp\left(-\frac{\xi^2}{2}\right)\,d\xi.$$

Had we not studied complex contour integration in § 9.5 we should find such an integral hard to integrate in closed form. However, since happily we have studied it, observing that the integrand $\exp(\xi^2)$ is an entire function (§ 8.6) of $\xi$—that is, that it is everywhere analytic—we recognize that one can trace the path of integration from $-\infty+i\theta$ to $\infty+i\theta$ along any contour one likes. Let us trace it along the real Argand axis from $-\infty$ to $\infty$, leaving only the two, short complex segments at the ends which (as is easy enough to see, and the formal proof is left as an exercise to the interested reader[17]) lie so far away that—for this integrand—they integrate to nothing. So tracing leaves us with

$$\mathcal{F}\{\Omega(v)\} = \frac{\exp\left(-v^2/2\right)}{2\pi}\int_{-\infty}^{\infty}\exp\left(-\frac{\xi^2}{2}\right)\,d\xi. \tag{18.47}$$

How to proceed from here is not immediately obvious. None of the techniques of Ch. 9 seems especially suitable to evaluate

$$I \equiv \int_{-\infty}^{\infty}\exp\left(-\frac{\xi^2}{2}\right)\,d\xi,$$

though if a search for a suitable contour of integration failed one might fall back on the Taylor-series technique of § 9.9. Fortunately, mathematicians have been searching hundreds of years for clever techniques to evaluate just such integrals and, when occasionally they should discover such a technique and reveal it to us, why, we record it in books like this, not to forget.

[16] [?]

[17] The short complex segments at the ends might integrate to something were the real part of $\xi^2$ negative, but the real part happens to be positive—indeed, most extremely positive—over the domains of the segments in question.

Here is the technique.[18] The equations

$$I = \int_{-\infty}^{\infty} \exp\left(-\frac{x^2}{2}\right)\,dx,$$
$$I = \int_{-\infty}^{\infty} \exp\left(-\frac{y^2}{2}\right)\,dy,$$

express the same integral $I$ in two different ways, the only difference being in the choice of letter for the dummy variable. What if we multiply the two? Then

$$\begin{aligned} I^2 &= \int_{-\infty}^{\infty} \exp\left(-\frac{x^2}{2}\right)\,dx \int_{-\infty}^{\infty} \exp\left(-\frac{y^2}{2}\right)\,dy \\ &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \exp\left(-\frac{x^2+y^2}{2}\right)\,dx\,dy. \end{aligned}$$

One, geometrical way to interpret this $I^2$ is as a double integration over a plane in which $(x, y)$ are rectangular coordinates. If we interpret thus, nothing then prevents us from double-integrating by the cylindrical coordinates $(\rho; \phi)$, instead, as

$$\begin{aligned} I^2 &= \int_{-2\pi/2}^{2\pi/2}\int_{0}^{\infty} \exp\left(-\frac{\rho^2}{2}\right)\rho\,d\rho\,d\phi \\ &= 2\pi\left[\int_{0}^{\infty} \exp\left(-\frac{\rho^2}{2}\right)\rho\,d\rho\right]. \end{aligned}$$

At a casual glance, the last integral in square brackets does not look much different from the integral with which we started, but see: it is not only that the lower limit of integration and the letter of the dummy variable have changed, but that an extra factor of the dummy variable has appeared—that the integrand ends not with $d\rho$ but with $\rho\,d\rho$. Once we have realized this, the integral's solution by antiderivative (§ 9.1) becomes suddenly easy to guess:

$$I^2 = 2\pi\left[-\exp\left(-\frac{\rho^2}{2}\right)\right]_0^{\infty} = 2\pi.$$

So evidently,

$$I = \sqrt{2\pi},$$

which means that

$$\int_{-\infty}^{\infty} \exp\left(-\frac{\xi^2}{2}\right)\,d\xi = \sqrt{2\pi} \tag{18.48}$$

[18] [19, § I:40-4]

as was to be calculated.

Finally substituting (18.48) into (18.47), we have that

$$\mathcal{F}\{\Omega(v)\} = \frac{\exp\left(-v^2/2\right)}{\sqrt{2\pi}}.$$

which in view of (18.46) reveals the remarkable transform pair

$$\Omega(v) \stackrel{\mathcal{F}}{\to} \Omega(v), \tag{18.49}$$

The Gaussian pulse transforms to itself. Old Fourier, who can twist and knot other curves with ease, seems powerless to bend Gauss' mighty curve.

It is worth observing incidentally in light of (18.46) and (18.48) that

$$\int_{-\infty}^{\infty} \Omega(t)\,dt = 1, \tag{18.50}$$

the same as for $\Pi(t)$, $\Lambda(t)$ and indeed $\delta(t)$. Section 17.3 and its (17.13) have recommended the shape of the Gaussian pulse, in the tall, narrow limit, to implement the Dirac delta $\delta(t)$. This section lends a bit more force to the recommendation, for not only is the Gaussian pulse analytic (unlike the Dirac delta) but it also behaves uncommonly well under Fourier transformation (like the Dirac delta), thus rendering the Dirac delta susceptible to an analytic limiting process which transforms amenably. Too, the Gaussian pulse is about as tightly localized as a nontrivial, uncontrived analytic function can be.[19] The passion of one of the author's mentors in extolling the Gaussian pulse as "absolutely a beautiful function" seems well supported by the practical mathematical virtues exhibited by the function itself.

The Gaussian pulse resembles the natural exponential in its general versatility. Indeed, though the book has required several chapters through this Ch. 18 to develop the fairly deep mathematics underlying the Gaussian pulse and supporting its basic application, now that we have the Gaussian pulse in hand we shall find that it ably fills all sorts of roles—not least the principal role of Ch. 20 to come.

## 18.6 The Laplace transform

Equation (18.5), defining the Fourier transform in the $\mathcal{F}_{\omega t}$ notation, transforms pulses like those of Figs. 18.1 and 17.3 straightforwardly but stumbles

[19] Consider that $\Omega(t) \approx 0\text{x}0.6621, 0\text{x}0.3\text{DF}2, 0\text{x}0.0\text{DD}2, 0\text{x}0.0122, 0\text{x}0.0009, 0\text{x}0.0000$ at $t = 0, \pm1, \pm2, \pm3, \pm4, \pm5$; and that $\Omega(\pm8) < 2^{-0\text{x}2\text{F}}$. Away from its middle region $|t| \lesssim 1$, the Gaussian pulse evidently vanishes rather convincingly.

on time-unlimited functions like $f(t) = \cos \omega_o t$ or even the ridiculously simple $f(t) = 1$. Only by the indirect techniques of §§ 18.2 and 18.3 have we been able to transform such functions. Such indirect techniques are valid and even interesting, but nonetheless can prompt the tasteful mathematician to wonder whether a simpler alternative to the Fourier transform were not possible.

At the sometimes acceptable cost of omitting one of the Fourier integral's two tails,[20] the *Laplace transform*

$$F(s) = \mathcal{L}\{f(t)\} \equiv \int_{0^-}^{\infty} e^{-st} f(t)\, dt \tag{18.51}$$

offers such an alternative. Here, $s = j\omega$ is the transform variable and, when $s$ is purely imaginary, the Laplace transform is very like the Fourier; but Laplace's advantage lies in that it encourages the use of a complex $s$, usually with a negative real part, which in (18.51)'s integrand suppresses even the tail the transform does not omit, thus effectively converting even a time-unlimited function to an integrable pulse—and Laplace does so without resort to indirect techniques.[21]

As we said, the Laplace transform's definition (18.51)—quite unlike the Fourier transform's definition (18.5)—tends to transform simple functions straightforwardly. For instance, the pair

$$1 \stackrel{\mathcal{L}}{\to} \frac{1}{s}.$$

(in which one can elaborate the symbol $\mathcal{L}$ as $\mathcal{L}_{st}$ if desired) comes by direct application of the definition. The first several of the Laplace properties of Table 18.7 likewise come by direct application. The differentiation property comes by

$$\mathcal{L}\left\{\frac{d}{dt} f(t)\right\} = \int_{0^-}^{\infty} e^{-st} \frac{d}{dt} f(t)\, dt = \int_{0^-}^{\infty} e^{-st}\, d\left[f(t)\right]$$

and thence by integration by parts (§ 9.3); whereafter the higher-order differentiation property comes by repeated application of the differentiation property. The integration property merely reverses the integration property on the function $g(t) \equiv \int_{0^-}^{\infty} f(\tau)\, d\tau$, for which $dg/dt = f(t)$ and $g(0^-) = 0$. The ramping property comes by differentiating and negating (18.51) as

$$-\frac{d}{ds} F(s) = -\frac{d}{ds} \int_{0^-}^{\infty} e^{-st} f(t)\, dt = \int_{0^-}^{\infty} e^{-st} [tf(t)]\, dt = \mathcal{L}\{tf(t)\};$$

[20] There has been invented a version of the Laplace transform which omits no tail [**?**, Ch. 3]. This book does not treat it.

[21] [**?**, Ch. 3][48, Ch. 7][35, Ch. 19]

whereafter again the higher-order property comes by repeated application. The convolution property comes as it did in § 18.2.5 except that here we take advantage the presence of Heaviside's unit step to manipulate the limits of integration, beginning

$$\begin{aligned}&\mathcal{L}\left\{\int_{-\infty}^{\infty} u\left(\frac{t}{2}-\psi\right) h\left(\frac{t}{2}-\psi\right) u\left(\frac{t}{2}+\psi\right) f\left(\frac{t}{2}+\psi\right)\,d\psi\right\}\\ &\quad= \int_{-\infty}^{\infty} e^{-st}\int_{-\infty}^{\infty} u\left(\frac{t}{2}-\psi\right) h\left(\frac{t}{2}-\psi\right) u\left(\frac{t}{2}+\psi\right) f\left(\frac{t}{2}+\psi\right)\,d\psi\,dt,\end{aligned}$$

wherein evidently $u(t/2-\psi)u(t/2+\psi)=0$ for all $t<0$, regardless of the value of $\psi$. As in § 18.2.5, here also we change $\phi \leftarrow t/2+\psi$ and $\chi \leftarrow \phi-2\psi$, eventually reaching the form

$$\begin{aligned}&\mathcal{L}\left\{\int_{-\infty}^{\infty} u\left(\frac{t}{2}-\psi\right) h\left(\frac{t}{2}-\psi\right) u\left(\frac{t}{2}+\psi\right) f\left(\frac{t}{2}+\psi\right)\,d\psi\right\}\\ &\quad= \left[\int_{-\infty}^{\infty} e^{-s\chi}u(\chi)h(\chi)\,d\chi\right]\left[\int_{-\infty}^{\infty} e^{-s\phi}u(\phi)f(\phi)\,d\phi\right],\end{aligned}$$

after which once more we take advantage of Heaviside, this time to curtail each integration to begin at $0^-$ rather than at $-\infty$, thus completing the convolution property's proof.

As Table 18.7 lists Laplace properties, so Table 18.8 lists Laplace transform pairs. As the former table's, most too of the latter table's entries come by direct application of the Laplace transform's definition (18.51) (though to reach the sine and cosine entries one should first split the sine and cosine functions per Table 5.1 into their complex exponential components). The pair $t \overset{\mathcal{L}}{\to} 1/s^2$ comes by application of the property that $tf(t) \overset{\mathcal{L}}{\to} -(d/ds)F(s)$ to the pair $1 \overset{\mathcal{L}}{\to} 1/s$, and the pair $t^n \overset{\mathcal{L}}{\to} n!/s^{n+1}$ comes by repeated application of the same property. The pairs transforming $e^{-at}t$ and $e^{-at}t^n$ come similarly.

In the application of either table, $a$ may be, and $s$ usually is, complex, but $\alpha$ and $t$ are normally real.

## 18.7 Solving differential equations by Laplace

The Laplace transform is curious, but admittedly one often finds in practice that the more straightforward—though harder to analyze—Fourier transform is a better tool for frequency-domain analysis, among other reasons because Fourier brings an inverse transformation formula (18.5) whereas

Table 18.7: Properties of the Laplace transform.

$$
\begin{aligned}
u(t-t_o)f(t-t_o) &\overset{\mathcal{L}}{\to} e^{-st_o}F(s) \\
e^{-at}f(t) &\overset{\mathcal{L}}{\to} F(s+a) \\
Af(\alpha t) &\overset{\mathcal{L}}{\to} \frac{A}{\alpha}F\left(\frac{s}{\alpha}\right) \quad \text{if } \Im(\alpha)=0,\ \Re(\alpha)>0 \\
A_1f_1(t)+A_2f_2(t) &\overset{\mathcal{L}}{\to} A_1F_1(t)+A_2F_2(t) \\
\frac{d}{dt}f(t) &\overset{\mathcal{L}}{\to} sF(s)-f(0^-) \\
\frac{d^n}{dt^n}f(t) &\overset{\mathcal{L}}{\to} s^nF(s)-\sum_{k=0}^{n-1}\left\{s^k\left[\frac{d^{n-1-k}}{dt^{n-1-k}}f(t)\right]_{t=0^-}\right\} \\
\int_{0^-}^{\infty} f(\tau)\,d\tau &\overset{\mathcal{L}}{\to} \frac{F(s)}{s} \\
tf(t) &\overset{\mathcal{L}}{\to} -\frac{d}{ds}F(s) \\
t^nf(t) &\overset{\mathcal{L}}{\to} (-)^n\frac{d^n}{ds^n}F(s) \\
[u(t)h(t)]*[u(t)f(t)] &\overset{\mathcal{L}}{\to} H(s)F(s)
\end{aligned}
$$

Table 18.8: Laplace transform pairs.

$$
\begin{aligned}
\delta(t) &\overset{\mathcal{L}}{\to} 1 \\
1 &\overset{\mathcal{L}}{\to} \frac{1}{s} & e^{-at} &\overset{\mathcal{L}}{\to} \frac{1}{s+a} \\
t &\overset{\mathcal{L}}{\to} \frac{1}{s^2} & e^{-at}t &\overset{\mathcal{L}}{\to} \frac{1}{(s+a)^2} \\
t^n &\overset{\mathcal{L}}{\to} \frac{n!}{s^{n+1}} & e^{-at}t^n &\overset{\mathcal{L}}{\to} \frac{n!}{(s+a)^{n+1}} \\
\sin\omega_o t &\overset{\mathcal{L}}{\to} \frac{\omega_o}{s^2+\omega_o^2} & e^{-at}\sin\omega_o t &\overset{\mathcal{L}}{\to} \frac{\omega_o}{(s+a)^2+\omega_o^2} \\
\cos\omega_o t &\overset{\mathcal{L}}{\to} \frac{s}{s^2+\omega_o^2} & e^{-at}\cos\omega_o t &\overset{\mathcal{L}}{\to} \frac{s+a}{(s+a)^2+\omega_o^2}
\end{aligned}
$$

Laplace does not.[22] This depends on the application. However, by the way, another use use for the Laplace transform happens to arise. The latter use emerges from the Laplace property of Table 18.7 that $(d/dt)f(t) \stackrel{\mathcal{L}}{\to} sF(s) - f(0^-)$, according to which, evidently, *differentiation in the time (untransformed) domain corresponds to multiplication by the transform variable $s$ in the frequency (transformed) domain.*

Now, one might say the same of the Fourier transform, for it has a differentiation property, too, $(d/dv)f(v) \stackrel{\mathcal{F}}{\to} ivF(s)$, which looks rather alike. The difference however lies in Laplace's extra Laplace term $-f(0^-)$ which, significantly, represents the untransformed function's initial condition.

To see the significance, consider for example the linear differential equation[23,24]

$$\begin{aligned} \frac{d^2}{dt^2}f(t) + 4\frac{d}{dt}f(t) + 3f(t) &= e^{-2t}, \\ f(t)\big|_{t=0^-} &= 1, \\ \frac{d}{dt}f(t)\bigg|_{t=0^-} &= 2. \end{aligned}$$

Applying the properties of Table 18.7 and transforms of Table 18.8, term by term, yields the transformed equation

$$\begin{aligned} &\left\{ s^2F(s) - s\left[f(t)\right]_{t=0^-} - \left[\frac{d}{dt}f(t)\right]_{t=0^-} \right\} \\ &\qquad + 4\left\{ sF(s) - \left[f(t)\right]_{t=0^-} \right\} + 3F(s) = \frac{1}{s+2}. \end{aligned}$$

That is,

$$(s^2 + 4s + 3)F(s) - (s+4)\left[f(t)\right]_{t=0^-} - \left[\frac{d}{dt}f(t)\right]_{t=0^-} = \frac{1}{s+2}.$$

[22] Actually, formally, Laplace does bring an uncouth contour-integrating inverse transformation formula in footnote 25, but we'll not use it.

[23] [35, Example 19.31]

[24] It is rather enlightening to study the same differential equation, written in the *state-space* style [48, Ch. 8]

$$\frac{d}{dt}\mathbf{f}(t) = \begin{bmatrix} 0 & 1 \\ -3 & -4 \end{bmatrix}\mathbf{f}(t) + \begin{bmatrix} 0 \\ e^{-2t} \end{bmatrix}, \quad \mathbf{f}(0) = \begin{bmatrix} 1 \\ 2 \end{bmatrix},$$

where $\mathbf{f}(t) \equiv [1 \;\; d/dt]^T f(t)$. The effort required to assimilate the notation rewards the student with significant insight into the manner in which initial conditions—here symbolized $\mathbf{f}(0)$—determine a system's subsequent evolution.

Applying the known initial conditions,

$$(s^2 + 4s + 3)F(s) - (s + 4)[1] - [2] = \frac{1}{s + 2}.$$

Combining like terms,

$$(s^2 + 4s + 3)F(s) - (s + 6) = \frac{1}{s + 2}.$$

Multiplying by $s + 2$ and rearranging,

$$(s + 2)(s^2 + 4s + 3)F(s) = s^2 + 8s + 13.$$

Isolating the heretofore unknown frequency-domain function $F(s)$,

$$F(s) = \frac{s^2 + 8s + 0\text{xD}}{(s + 2)(s^2 + 4s + 3)}.$$

Factoring the denominator,

$$F(s) = \frac{s^2 + 8s + 0\text{xD}}{(s + 1)(s + 2)(s + 3)}.$$

Expanding in partial fractions (this step being the key to the whole technique: see § 9.6),

$$F(s) = \frac{3}{s + 1} - \frac{1}{s + 2} - \frac{1}{s + 3}.$$

Though we lack an inverse transformation formula, it seems that we do not need one because—having split the frequency-domain equation into such simple terms—we can just look up the inverse transformation in Table 18.8, term by term. The time-domain solution

$$f(t) = 3e^{-t} - e^{-2t} - e^{-3t}$$

results to the differential equation with which we started.[25]

This section's Laplace technique neatly solves many linear differential equations.

[25]The careful reader might object that we have never proved that the Laplace transform cannot map distinct time-domain functions atop one another in the frequency domain; that is, that we have never shown the Laplace transform to be invertible. The objection has merit. Consider for instance the time-domain function $f_2(t) = u(t)[3e^{-t} - e^{-2t} - e^{-3t}]$, whose Laplace transform does not differ from that of $f(t)$.

However, even the careful reader will admit that the suggested $f_2(t)$ differs from $f(t)$ only over $t < 0$, a domain Laplace ignores. What one thus ought to ask is whether the Laplace transform can map time-domain functions, the functions being *distinct for* $t \geq 0$,

## 18.8 Initial and final values by Laplace

The method of § 18.7 though effective is sometimes too much work, when all one wants to know are the initial and/or final values of a function $f(t)$, when one is not interested in the details between. The Laplace transform's *initial-* and *final-value theorems,*

$$\begin{aligned} f(0^+) &= \lim_{s\to\infty} sF(s), \\ \lim_{t\to\infty} f(t) &= \lim_{s\to 0} sF(s), \end{aligned} \tag{18.52}$$

meet this want. (Note that these are not transform pairs as in Tables 18.7 and 18.8 but actual equations.)

One derives the initial-value theorem by the successive steps

$$\begin{aligned} \lim_{s\to\infty} \mathcal{L}\left\{\frac{d}{dt}f(t)\right\} &= \lim_{s\to\infty} sF(s) - f(0^-), \\ \lim_{s\to\infty} \int_{0^-}^{\infty} e^{-st}\frac{d}{dt}f(t)\,dt &= \lim_{s\to\infty} sF(s) - f(0^-), \\ \int_{0^-}^{0^+} \frac{d}{dt}f(t)\,dt &= \lim_{s\to\infty} sF(s) - f(0^-), \\ f(0^+) - f(0^-) &= \lim_{s\to\infty} sF(s) - f(0^-), \end{aligned}$$

which invoke the time-differentiation property of Table 18.7 and the last of

atop one another in the frequency domain.

In one sense it may be unnecessary to answer even the latter question, for one can check the correctness, and probably also the sufficiency, of any solution Laplace might offer to a particular differential equation by the expedient of substituting the solution back into the equation. However, one can answer the latter question formally nonetheless by changing $s \leftarrow i\omega$ in (18.1) and observing the peculiar, contour-integrating inverse of the Laplace transform, $f(t) = (1/i2\pi)\int_{-i\infty}^{i\infty} e^{st}F(s)\,ds$, which results [48, eqn. 7.2]. To consider the choice of contours of integration and otherwise to polish the answer is left as an exercise to the interested reader; here it is noted only that, to cause the limits of integration involved to behave nicely, one might insist as a precondition to answering the question something like that $f(t) = 0$ for all $t < 0$, the precondition being met by any $f(t) = u(t)g(t)$ (in which $u[t]$ is formally defined for the present purpose such that $u[0] = 1$).

which implies (18.52)'s first line. For the final value, one begins

$$\begin{aligned}
\lim_{s\to 0} \mathcal{L}\left\{\frac{d}{dt}f(t)\right\} &= \lim_{s\to 0} sF(s) - f(0^-), \\
\lim_{s\to 0} \int_{0^-}^{\infty} e^{-st}\frac{d}{dt}f(t)\,dt &= \lim_{s\to 0} sF(s) - f(0^-), \\
\int_{0^-}^{\infty} \frac{d}{dt}f(t)\,dt &= \lim_{s\to 0} sF(s) - f(0^-), \\
\lim_{t\to\infty} f(t) - f(0^-) &= \lim_{s\to 0} sF(s) - f(0^-),
\end{aligned}$$

and (18.52)'s second line follows immediately.[26]

## 18.9 The spatial Fourier transform

In the study of wave mechanics, physicists and engineers sometimes elaborate this chapter's kernel $e^{iv\theta}$ or $e^{i\omega t}$, or by whichever pair of letters is let to represent the complementary variables of transformation, into the more general, spatiotemporal phase factor[27] $e^{i(\pm\omega t\mp\mathbf{k}\cdot\mathbf{r})}$; where $\mathbf{k}$ and $\mathbf{r}$ are three-dimensional geometrical vectors and $\mathbf{r}$ in particular represents a position in space. To review the general interpretation and use of such a factor lies beyond the chapter's scope but the factor's very form,

$$e^{i(\mp\omega t\pm\mathbf{k}\cdot\mathbf{r})} = e^{i(\mp\omega t\pm k_x x\pm k_y y\pm k_z z)},$$

suggests Fourier transformation with respect not merely to time but also to space. There results the *spatial Fourier transform*

$$\begin{aligned}
F(\mathbf{k}) &= \frac{1}{(2\pi)^{3/2}}\int_V e^{+i\mathbf{k}\cdot\mathbf{r}} f(\mathbf{r})\,d\mathbf{r}, \\
f(\mathbf{r}) &= \frac{1}{(2\pi)^{3/2}}\int_V e^{-i\mathbf{k}\cdot\mathbf{r}} F(\mathbf{k})\,d\mathbf{k},
\end{aligned} \tag{18.53}$$

analogous to (18.5) but cubing the $1/\sqrt{2\pi}$ scale factor for the triple integration and reversing the sign of the kernel's exponent. The transform variable $\mathbf{k}$, analogous to $\omega$, is a *spatial frequency,* also for other reasons called a *propagation vector.*

[26] [48, § 7.5]

[27] The choice of sign here is a matter of convention, which differs by discipline. This book tends to reflect its author's preference for $e^{i(-\omega t+\mathbf{k}\cdot\mathbf{r})}$, convenient in electrical modeling but slightly less convenient in quantum-mechanical work.

Nothing prevents one from extending (18.53) to four dimensions, including a fourth integration to convert time $t$ to temporal frequency $\omega$ while also converting position $\mathbf{r}$ to spatial frequency $\mathbf{k}$. On the other hand, one can restrict it to two dimensions or even one. Thus, various plausibly useful Fourier transforms include

$$\begin{aligned}
F(\omega) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-i\omega t} f(t)\, dt, \\
F(k_z) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{+ik_z z} f(z)\, dz, \\
F(\mathbf{k}_\rho) &= \frac{1}{2\pi} \int_S e^{+i\mathbf{k}_\rho \cdot \boldsymbol{\rho}} f(\boldsymbol{\rho})\, d\boldsymbol{\rho}, \\
F(\mathbf{k}) &= \frac{1}{(2\pi)^{3/2}} \int_V e^{+i\mathbf{k}\cdot\mathbf{r}} f(\mathbf{r})\, d\mathbf{r}, \\
F(\mathbf{k},\omega) &= \frac{1}{(2\pi)^2} \int_V \int_{-\infty}^{\infty} e^{i(-\omega t + \mathbf{k}\cdot\mathbf{r})} f(\mathbf{r},t)\, dt\, d\mathbf{r},
\end{aligned}$$

among others.

# Chapter 19

# Introduction to special functions

[This chapter is a rough, partial draft.]

No topic more stirs the pure mathematician's imagination than that of number theory, so briefly addressed by this book's § 6.1. So it is said. However, an applied mathematician is pleased to follow a lesser muse, and the needful topic that most provokes his mathematical curiosity may be that of special functions.

What is a *special function?* Trouble arises at once, before the first mathematical symbol strikes the page, for it is not easy to discover a precise definition of the term. N.N. Lebedev and Larry C. Andrews, authors respectively in Russian and English of two of the better post-World War II books on the topic,[1] seem to decline to say what a special function *is,* merely suggesting in their respective prefaces some things it *does,* diving thence fluidly into the mathematics before anyone should notice that those authors have motivated but never quite delineated their topic (actually, it's not a bad approach, and reading Lebedev or Andrews one does soon perceive the shape of the thing). Abramowitz and Stegun, whose celebrated handbook[2] though not expressly a book on special functions is largely about them, are even silenter on the question. The sober dictionary[3] on the author's desk correctly defines such mathematical terms as "eigenvalue" and "Fourier transform" but seems unacquainted with "special function." What are we to make of

[1] The books respectively are [43] and [2], the latter originally in English, the former available at the time of this writing in Richard A. Silverman's excellent, inexpensive English translation.

[2] [1]

[3] [**?**]

this?

Here is what. A *special function* is an analytic function (§ 8.4)—likely of a single and at most of a few complex scalar variables, harder to analyze and evaluate than the *elementary functions* of Chs. 2 through 5, defined in a suitably canonical form—that serves to evaluate an integral, to solve an integral equation,[4] or to solve a differential equation elementary functions alone cannot evaluate or solve. Such a definition approximates at least the aspect and use of the special functions this book means to treat. The definition will do to go on with.

The fascination of special functions to the scientist and engineer lies in how gracefully they analyze otherwise intractable physical models; in how reluctantly they yield their mathematical secrets; in how readily they conform to unexpected applications; in how often they connect seemingly unrelated phenomena; and in that, the more intrepidly one explores their realm, the more disquietly one feels that one had barely penetrated the realm's frontier. The topic of special functions seems inexhaustible. We surely will not begin to exhaust the topic in this book; yet, even so, useful results will start to flow from it almost at once.

## 19.1 The Gaussian pulse and its moments

We have already met

$$\Omega(x) = \frac{\exp\left(-x^2/2\right)}{\sqrt{2\pi}},$$

as (20.15).

[4] It need not interest you but in case it does, an *integral equation* is an equation like

$$\int_{-\infty}^{\infty} g(z,w) f(w)\, dw = h(z),$$

in which the unknown is not a variable but a function $f(w)$ that operates on a dummy variable of integration. Actually, we have already met integral equations in disguise, in discretized form, in matrix notation (Chs. 11 through 14) resembling

$$G\mathbf{f} = \mathbf{h},$$

which means no more than it seems to mean; so maybe integral equations are not so strange as they look. The integral equation is just the matrix equation with the discrete vectors $\mathbf{f}$ and $\mathbf{h}$ replaced by their continuous versions $\Delta w\, f(j\,\Delta w)$ and $\Delta z\, h(i\,\Delta z)$ (the $i$ representing not the imaginary unit here but just an index, as in Ch. 11).

# Chapter 20

# Probability

[This chapter is still quite a rough draft.]

Of all mathematical fields of study, none may be so counterintuitive yet, paradoxically, so widely applied as that of probability, whether as *probability* in the technical term's conventional, restricted meaning or as probability in its expanded or inverted guise as *statistics*.[1] Man's mind, untrained, seems somehow to rebel against the concept. Sustained reflection on the concept however gradually reveals to the mind a fascinating mathematical landscape.

As calculus is the mathematics of change, so *probability* is the mathematics of uncertainty. If I tell you that my thumb is three inches long, I likely do not mean that it is exactly three inches. I mean that it is about three inches. Quantitatively, I might report the length as $3.0 \pm 0.1$ inches, thus indicating not only the length but the degree of uncertainty in the length. Probability in its guise as *statistics* is the mathematics which produces, analyzes and interprets such quantities.

More obviously statistical is a report that the average, say, 25-year-old American male is $70 \pm 3$ inches tall, inferred from actual measurements of some number $N > 1$ of 25-year-old American males. Deep mathematics underlie such a report, for the report implies among other things that a little over two-thirds—$(1/\sqrt{2\pi}) \int_{-1}^{1} \exp(-\tau^2/2)\, d\tau \approx$ 0x0.AEC5, to be precise—of a typical, randomly chosen sample of 25-year-old American males ought

[1] The nomenclature is slightly unfortunate. Were statistics called "inferred probability" or "probabilistic estimation" the name would suggest something like the right taxonomy. Actually, the nomenclature is fine once you know what it means, but on first encounter it provokes otherwise unnecessary footnotes like this one.

Statistics (singular noun) the expanded mathematical discipline—as opposed to the statistics (plural noun) mean and standard deviation of § 20.2—as such lies mostly beyond this book's scope, but the chapter will have at least a little to say about it in § 20.6.

to be found to have heights between 67 and 73 inches.

Probability is also met in games of chance and in models of systems which—from the model's point of view—logically resemble games of chance, and in this setting probability is not statistics. The reason it is not is that its mathematics in this case is based not on observation but on a teleological assumption of some kind, often an assumption of symmetry such as that no face of a die or card from a deck ought to turn up more often than another. Entertaining so plausible an assumption, if you should draw three cards at random from a standard 52-card deck (let us use decimal notation rather than hexadecimal in this paragraph, since neither you nor I have ever heard of a 0x34-card deck), the deck comprising four cards each of thirteen ranks, then there would be some finite probability—which is $(3/51)(2/50) = 1/425$—that the three cards you draw would share the same rank (why?). If I should however shuffle the deck, draw three cards off the top, and look at the three cards without showing them to you, all before inviting you to draw three, then the probability that your three would share the same rank were again 1/425 (why?). On the other hand, if before you drew I let you peek at my three hidden cards, and you saw that they were ace, queen and ten, this knowledge alone must slightly lower your estimation of the probability that your three would subsequently share the same rank as one another to $(40/49)(3/48)(2/47) + (9/49)(2/48)(1/47) \approx 1/428$ (again, why?).

That the probability should be 1/425 suggests that one would draw three of the same rank once in 425 tries. However, were I to shuffle 425 decks and you to draw three cards from each, then you *might* draw three of the same rank from two, three or four decks, or from none at all, though very unlikely from twenty decks—so what does a probability of 1/425 really mean? The answer is that it means something like this: were I to shuffle 425 *million* decks then you would draw three of the same rank from very nearly 1.0 million decks—almost certainly not from as many as 1.1 million nor as few as 0.9 million. It means that the ratio of the number of three-of-the-same-rank events to the number of trials must converge exactly upon 1/425 as the number of trials tends toward infinity.

See also § 4.2.

Other than by this brief introduction, the book you are reading is not well placed to offer a gentle tutorial in probabilistic thought.[2] What it does offer, in the form of the present chapter, is the discovery and derivation of the essential mathematical functions of probability theory (including in § 20.5 the derivation of one critical result undergraduate statistics textbooks

[2]The late R.W. Hamming's fine book [27] ably fills such a role.

usually state but, understandably, omit to prove), plus a brief investigation of these functions' principal properties and typical use.

## 20.1 Definitions and basic concepts

A *probability* is the chance that a trial of some kind will result in some specified event of interest. Conceptually,

$$P_{\text{event}} \equiv \lim_{N\to\infty} \frac{N_{\text{event}}}{N},$$

where $N$ and $N_{\text{event}}$ are the numbers respectively of trials and events. A *probability density function* (PDF) or *distribution* is a function $f(x)$ defined such that

$$\begin{aligned} P_{ba} &= \int_a^b f(x)\,dx, \\ 1 &= \int_{-\infty}^{\infty} f(x)\,dx, \\ 0 &\le f(x), \quad \Im[f(x)] = 0. \end{aligned} \tag{20.1}$$

where the event of interest is that the *random variable* $x$ fall[3] in the interval[4] $a < x < b$ and $P_{ba}$ is the probability of this event. The corresponding *cumulative distribution function* (CDF) is

$$F(x) \equiv \int_{-\infty}^{x} f(\tau)\,d\tau, \tag{20.2}$$

where

$$\begin{aligned} 0 &= F(-\infty), \\ 1 &= F(\infty), \\ P_{ba} &= F(b) - F(a). \end{aligned} \tag{20.3}$$

[3]This sentence and the rest of the section condense somewhat lengthy tracts of an introductory collegiate statistics text like [**?**][**?**][**?**][**?**], among others. If the sentence and section make little sense to you then so likely will the rest of the chapter, but any statistics text you might find conveniently at hand should fill the gap—which is less a mathematical gap than a conceptual one. Or, if defiant, you can stay here and work through the concepts on your own.

[4]We might as well have expressed the interval $a < x < b$ as $a \le x \le b$ or even as $a \le x < b$, except that such notational niceties would distract from the point the notation means to convey. The notation in this case is not really interested in the bounding points themselves. If *we* are interested the bounding points, as for example we would be if $f(x) = \delta(x)$ and $a = 0$, then we can always write in the style of $P_{(0^-)b}$, $P_{(0^+)b}$, $P_{(a-\epsilon)(b+\epsilon)}$, $P_{(a+\epsilon)(b-\epsilon)}$ or the like. We can even be most explicit and write in the style of $P\{a \le x \le b\}$, often met in the literature.

The *quantile* $F^{-1}(\cdot)$ inverts the CDF $F(x)$ such that

$$F^{-1}[F(x)] = x, \tag{20.4}$$

generally calculatable by a Newton-Raphson iteration (4.31) if by no other means.

It is easy enough to see that the product

$$P = P_1 P_2 \tag{20.5}$$

of two probabilities composes the single probability that not just one but both of two, independent events will occur. Harder to see, but just as important, is that the convolution

$$f(x) = f_1(x) * f_2(x) \tag{20.6}$$

of two probability density functions composes the single probability density function of the sum of two random variables $x = x_1 + x_2$, where, per Table 18.5,

$$f_2(x) * f_1(x) = f_1(x) * f_2(x) \equiv \int_{-\infty}^{\infty} f_1\left(\frac{x}{2} - \tau\right) f_2\left(\frac{x}{2} + \tau\right) d\tau.$$

That is, if you think about it in a certain way, the probability that $a < x_1 + x_2 < b$ cannot but be

$$\begin{aligned}
P_{ba} &= \lim_{\epsilon \to 0^+} \sum_{k=-\infty}^{\infty} \left\{ \left[ \int_{(k-1/2)\epsilon}^{(k+1/2)\epsilon} f_1(x)\,dx \right] \left[ \int_{a-k\epsilon}^{b-k\epsilon} f_2(x)\,dx \right] \right\} \\
&= \lim_{\epsilon \to 0^+} \sum_{k=-\infty}^{\infty} \left\{ \left[ \epsilon f_1(k\epsilon) \right] \left[ \int_{a}^{b} f_2(x - k\epsilon)\,dx \right] \right\} \\
&= \int_{-\infty}^{\infty} f_1(\tau) \left[ \int_{a}^{b} f_2(x - \tau)\,dx \right] d\tau \\
&= \int_{a}^{b} \left[ \int_{-\infty}^{\infty} f_1(\tau) f_2(x - \tau)\,d\tau \right] dx \\
&= \int_{a}^{b} \left[ \int_{-\infty}^{\infty} f_1\left(\frac{x}{2} + \tau\right) f_2\left(\frac{x}{2} - \tau\right) d\tau \right] dx \\
&= \int_{a}^{b} \left[ \int_{-\infty}^{\infty} f_1\left(\frac{x}{2} - \tau\right) f_2\left(\frac{x}{2} + \tau\right) d\tau \right] dx,
\end{aligned}$$

which in consideration of (20.1) implies (20.6).

## 20.2 The statistics of a distribution

A probability density function $f(x)$ describes a distribution whose *mean* $\mu$ and *standard deviation* $\sigma$ are defined such that

$$\begin{aligned} \mu &\equiv \langle x \rangle = \int_{-\infty}^{\infty} f(x) x \, dx, \\ \sigma^2 &\equiv \left\langle (x - \langle x \rangle)^2 \right\rangle = \int_{-\infty}^{\infty} f(x)(x - \mu)^2 \, dx; \end{aligned} \tag{20.7}$$

where $\langle \cdot \rangle$ indicates the *expected value* of the quantity enclosed, defined as the first line of (20.7) suggests. The mean $\mu$ is just the distribution's average, about which a random variable should center. The standard deviation $\sigma$ measures a random variable's typical excursion from the mean. The mean and standard deviation are *statistics* of the distribution.[5] When the chapter's introduction proposed that the average 25-year-old American male were $70 \pm 3$ inches tall, it was saying that his height could quantitatively be modeled as a random variable drawn from a distribution whose statistics are $\mu = 70$ inches and $\sigma = 3$ inches.

[5] Other statistics than the mean and standard deviation are possible, but these two are the most important ones and are the two this book treats.

## 20.3 The sum of random variables

The statistics of the sum of two random variables $x = x_1 + x_2$ are of interest. For the mean, substituting (20.6) into the first line of (20.7),

$$\begin{aligned}
\mu &= \int_{-\infty}^{\infty} [f_1(x) * f_2(x)] x \, dx \\
&= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f_1\left(\frac{x}{2} - \tau\right) f_2\left(\frac{x}{2} + \tau\right) d\tau \, x \, dx \\
&= \int_{-\infty}^{\infty} f_2(\tau) \int_{-\infty}^{\infty} f_1(x - \tau) x \, dx \, d\tau \\
&= \int_{-\infty}^{\infty} f_2(\tau) \int_{-\infty}^{\infty} f_1(x)(x + \tau) \, dx \, d\tau \\
&= \int_{-\infty}^{\infty} f_2(\tau) \left[ \int_{-\infty}^{\infty} f_1(x) x \, dx + \tau \int_{-\infty}^{\infty} f_1(x) \, dx \right] d\tau \\
&= \int_{-\infty}^{\infty} f_2(\tau) [\mu_1 + \tau] \, d\tau \\
&= \mu_1 \int_{-\infty}^{\infty} f_2(\tau) \, d\tau + \int_{-\infty}^{\infty} f_2(\tau) \tau \, d\tau.
\end{aligned}$$

That is,

$$\mu = \mu_1 + \mu_2, \tag{20.8}$$

which is no surprise, but at least it is nice to know that our mathematics is working as it should. The standard deviation of the sum of two random variables is such that, substituting (20.6) into the second line of (20.7),

$$\begin{aligned}
\sigma^2 &= \int_{-\infty}^{\infty} [f_1(x) * f_2(x)](x - \mu)^2 \, dx \\
&= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f_1\left(\frac{x}{2} - \tau\right) f_2\left(\frac{x}{2} + \tau\right) d\tau \, (x - \mu)^2 \, dx.
\end{aligned}$$

Applying (20.8),

$$\begin{aligned}
\sigma^2 &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_1\left(\frac{x}{2}-\tau\right) f_2\left(\frac{x}{2}+\tau\right)\, d\tau\,(x-\mu_1-\mu_2)^2\,dx \\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_1\left(\frac{x+\mu_1}{2}-\tau\right) f_2\left(\frac{x+\mu_1}{2}+\tau\right)\, d\tau\,(x-\mu_2)^2\,dx \\
&= \int_{-\infty}^{\infty} f_2(\tau)\int_{-\infty}^{\infty} f_1(x+\mu_1-\tau)(x-\mu_2)^2\,dx\,d\tau \\
&= \int_{-\infty}^{\infty} f_2(\tau)\int_{-\infty}^{\infty} f_1(x)[(x-\mu_1)+(\tau-\mu_2)]^2\,dx\,d\tau \\
&= \int_{-\infty}^{\infty} f_2(\tau)\Big\{\int_{-\infty}^{\infty} f_1(x)(x-\mu_1)^2\,dx \\
&\qquad + 2(\tau-\mu_2)\int_{-\infty}^{\infty} f_1(x)(x-\mu_1)\,dx \\
&\qquad + (\tau-\mu_2)^2\int_{-\infty}^{\infty} f_1(x)\,dx\Big\}\,d\tau \\
&= \int_{-\infty}^{\infty} f_2(\tau)\Big\{\int_{-\infty}^{\infty} f_1(x)(x-\mu_1)^2\,dx \\
&\qquad + 2(\tau-\mu_2)\int_{-\infty}^{\infty} f_1(x)x\,dx \\
&\qquad + (\tau-\mu_2)(\tau-\mu_2-2\mu_1)\int_{-\infty}^{\infty} f_1(x)\,dx\Big\}\,d\tau.
\end{aligned}$$

Applying (20.7) and (20.1),

$$\begin{aligned}
\sigma^2 &= \int_{-\infty}^{\infty} f_2(\tau)\left\{\sigma_1^2 + 2(\tau-\mu_2)\mu_1 + (\tau-\mu_2)(\tau-\mu_2-2\mu_1)\right\}\,d\tau \\
&= \int_{-\infty}^{\infty} f_2(\tau)\left\{\sigma_1^2 + (\tau-\mu_2)^2\right\}\,d\tau \\
&= \sigma_1^2\int_{-\infty}^{\infty} f_2(\tau)\,d\tau + \int_{-\infty}^{\infty} f_2(\tau)(\tau-\mu_2)^2\,d\tau.
\end{aligned}$$

Applying (20.7) and (20.1) again,

$$\sigma^2 = \sigma_1^2 + \sigma_2^2. \tag{20.9}$$

If this is right—as indeed it is—then the act of adding random variables together not only adds the means of the variables' respective distributions according to (20.8) but also, according to (20.9), adds the squares of the

standard deviations. It follows directly that, if $N$ independent instances $x_1, x_2, \ldots, x_N$ of a random variable are drawn from the same distribution $f_o(x_k)$, the distribution's statistics being $\mu_o$ and $\sigma_o$, then the statistics of their sum $x = \sum_{k=1}^{N} x_k = x_1 + x_2 + \cdots + x_N$ are

$$\begin{aligned} \mu &= N\mu_o, \\ \sigma &= \left(\sqrt{N}\right)\sigma_o. \end{aligned} \tag{20.10}$$

## 20.4 The transformation of a random variable

If $x_o$ is a random variable obeying the distribution $f_o(x_o)$ and $g(\cdot)$ is some invertible function whose inverse per (2.49) is styled $g^{-1}(\cdot)$, then

$$x \equiv g(x_o)$$

is itself a random variable obeying the distribution

$$f(x) = \left.\frac{f_o(x_o)}{|dg/dx_o|}\right|_{x_o = g^{-1}(x)}. \tag{20.11}$$

Another, suaver way to write the same thing is

$$f(x)\,|dx| = f_o(x_o)\,|dx_o|\,. \tag{20.12}$$

Either way, this is almost obvious if seen from just the right perspective, but can in any case be supported symbolically by

$$\int_a^b f_o(x_o)\,dx_o = \left|\int_{g(a)}^{g(b)} f_o(x_o)\frac{dx_o}{dx}\,dx\right| = \int_{g(a)}^{g(b)} f_o(x_o)\left|\frac{dx_o}{dg}\right|\,dx.$$

One of the most frequently useful transformations is the simple

$$x \equiv g(x_o) \equiv \alpha x_o, \quad \Im(\alpha) = 0,\ \Re(\alpha) > 0.$$

For this, evidently $dg/dx_o = \alpha$, so according to (20.11) or (20.12)

$$f(x) = \frac{1}{|\alpha|} f_o\left(\frac{x}{\alpha}\right). \tag{20.13}$$

If $\mu_o = 0$ and $\sigma_o = 1$, then $\mu = 0$ and, in train of (20.7),

$$\sigma^2 = \int_{-\infty}^{\infty} f(x)x^2\,dx = \int_{-\infty}^{\infty} f_o(x_o)(\alpha x_o)^2\,dx_o = \alpha^2;$$

whereby $\sigma = \alpha$ and we can rewrite the transformed PDF as

$$f(x) = \frac{1}{\sigma} f_o\left(\frac{x}{\sigma}\right) \text{ and } \mu = 0, \text{ if } \mu_o = 0 \text{ and } \sigma_o = 1. \tag{20.14}$$

Assuming null mean, (20.14) states that the act of scaling a random variable flattens out the variable's distribution and scales its standard deviation, all by the same factor—which, naturally, is what one would expect it to do.

## 20.5 The normal distribution

Combining the ideas of §§ 20.3 and 20.4 leads us now to ask whether a distribution does not exist for which, when independent random variables drawn from it are added together, *the sum obeys the same distribution,* only the standard deviations differing. More precisely, we should like to identify a distribution

$$f_o(x_o)\text{: } \mu_o = 0,\ \sigma_o = 1;$$

for which, if $x_1$ and $x_2$ are random variables drawn respectively from the distributions

$$f_1(x_1) = \frac{1}{\sigma_1} f_o\left(\frac{x_1}{\sigma_1}\right),$$
$$f_2(x_2) = \frac{1}{\sigma_2} f_o\left(\frac{x_2}{\sigma_2}\right),$$

as (20.14) suggests, then

$$x = x_1 + x_2$$

by construction is a random variable drawn from the distribution

$$f(x) = \frac{1}{\sigma} f_o\left(\frac{x}{\sigma}\right),$$

where per (20.9),

$$\sigma^2 = \sigma_1^2 + \sigma_2^2.$$

There are several distributions one might try, but eventually the Gaussian pulse $\Omega(x_o)$ of §§ 17.3 and 18.5,

$$\Omega(x) = \frac{\exp\left(-x^2/2\right)}{\sqrt{2\pi}}, \tag{20.15}$$

recommends itself. This works. The distribution $f_o(x_o) = \Omega(x_o)$ meets our criterion.

To prove that the distribution $f_o(x_o) = \Omega(x_o)$ meets our criterion we shall first have to show that it is indeed a distribution according to (20.1). Especially, we shall have to demonstrate that

$$\int_{-\infty}^{\infty} \Omega(x_o)\,dx_o = 1.$$

Fortunately, as it happens, we have already demonstrated this fact as (18.50); so, since $\Omega(x_o)$ evidently meets the other demands of (20.1), it apparently is a proper distribution. That $\mu_o = 0$ for $\Omega(x_o)$ is obvious by symmetry. That $\sigma_o = 1$ is shown by

$$\begin{aligned}
\int_{-\infty}^{\infty} \Omega(x_o)x_o^2\,dx_o &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \exp\left(-\frac{x_o^2}{2}\right) x_o^2\,dx_o \\
&= -\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} x_o\,d\left[\exp\left(-\frac{x_o^2}{2}\right)\right] \\
&= -\left.\frac{x_o \exp\left(-x_o^2/2\right)}{\sqrt{2\pi}}\right|_{-\infty}^{\infty} \\
&\qquad + \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \exp\left(-\frac{x_o^2}{2}\right)\,dx_o, \\
&= 0 + \int_{-\infty}^{\infty} \Omega(x_o)\,dx_o,
\end{aligned}$$

from which again according to (18.50)

$$\int_{-\infty}^{\infty} \Omega(x_o)x_o^2\,dx_o = 1 \tag{20.16}$$

as was to be shown. Now having justified the assertions that $\Omega(x_o)$ is a proper distribution and that its statistics are $\mu_o = 0$ and $\sigma_o = 1$, all that remains to be proved per (20.6) is that

$$\begin{gathered}
\left[\frac{1}{\sigma_1}\Omega\left(\frac{x_o}{\sigma_1}\right)\right] * \left[\frac{1}{\sigma_2}\Omega\left(\frac{x_o}{\sigma_2}\right)\right] = \frac{1}{\sigma}\Omega\left(\frac{x_o}{\sigma}\right), \\
\sigma_1^2 + \sigma_2^2 = \sigma^2,
\end{gathered} \tag{20.17}$$

which is to prove that the sum of Gaussian random variables is itself Gaussian. We will prove it in the Fourier domain of Ch. 18 as follows. According

to Tables 18.3 and 18.6, and to (20.15),

$$\begin{aligned}
&\left[\frac{1}{\sigma_1}\Omega\left(\frac{x_o}{\sigma_1}\right)\right] * \left[\frac{1}{\sigma_2}\Omega\left(\frac{x_o}{\sigma_2}\right)\right] \\
&= \mathcal{F}^{-1}\left\{\left(\sqrt{2\pi}\right)\mathcal{F}\left[\frac{1}{\sigma_1}\Omega\left(\frac{x_o}{\sigma_1}\right)\right]\mathcal{F}\left[\frac{1}{\sigma_2}\Omega\left(\frac{x_o}{\sigma_2}\right)\right]\right\} \\
&= \mathcal{F}^{-1}\left\{\left(\sqrt{2\pi}\right)\Omega(\sigma_1 x_o)\Omega(\sigma_2 x_o)\right\} \\
&= \mathcal{F}^{-1}\left\{\frac{\exp\left[-\sigma_1^2 x_o^2/2\right]\exp\left[-\sigma_2^2 x_o^2/2\right]}{\sqrt{2\pi}}\right\} \\
&= \mathcal{F}^{-1}\left\{\frac{\exp\left[-\left(\sigma_1^2+\sigma_2^2\right)x_o^2/2\right]}{\sqrt{2\pi}}\right\} \\
&= \mathcal{F}^{-1}\left\{\Omega\left[\left(\sqrt{\sigma_1^2+\sigma_2^2}\right)x_o\right]\right\} \\
&= \frac{1}{\sqrt{\sigma_1^2+\sigma_2^2}}\Omega\left(\frac{x_o}{\sqrt{\sigma_1^2+\sigma_2^2}}\right),
\end{aligned}$$

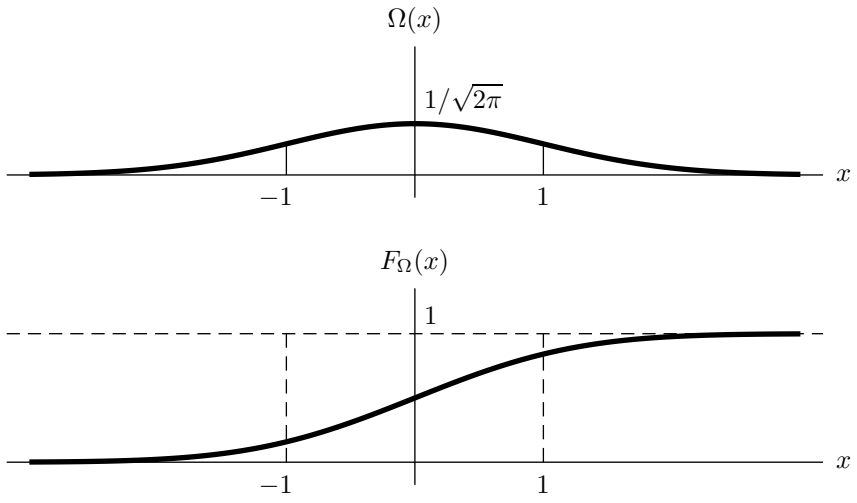
the last line of which is (20.17) in other notation, thus completing the proof.

In the Fourier context of Ch. 18 one usually names $\Omega(\cdot)$ the *Gaussian pulse,* as we have seen. The function $\Omega(\cdot)$ turns out to be even more prominent in probability theory than in Fourier theory, however, and in a probabilistic context it usually goes by the name of the *normal distribution.* This is what we will call $\Omega(\cdot)$ through the rest of the present chapter. Alternate conventional names include those of the *Gaussian distribution* and the *bell curve* (the Greek capital $\Omega$ vaguely, accidentally resembles a bell, as does the distribution's plot, and we will not be too proud to take advantage of the accident, so that is how you can remember it if you like). By whichever name, Fig. 20.1 plots the normal distribution $\Omega(\cdot)$ and its cumulative distribution function (20.2).

Regarding the cumulative normal distribution function, one way to calculate it numerically is to integrate the normal distribution's Taylor series term by term. As it happens, § 9.9 has worked a very similar integral as an example, so this section will not repeat the details, but the result is

$$\begin{aligned}
F_\Omega(x_o) = \int_{-\infty}^{x_o}\Omega(\tau)\,d\tau &= \frac{1}{2} + \frac{1}{\sqrt{2\pi}}\sum_{k=0}^{\infty}\frac{(-)^k x_o^{2k+1}}{(2k+1)2^k k!} \\
&= \frac{1}{2} + \frac{x_o}{\sqrt{2\pi}}\sum_{k=0}^{\infty}\frac{1}{2k+1}\prod_{j=1}^{k}\frac{-x_o^2}{2j}. \qquad (20.18)
\end{aligned}$$

Figure 20.1: The normal distribution $\Omega(x) \equiv (1/\sqrt{2\pi}) \exp(-x^2/2)$ and its cumulative distribution function $F_\Omega(x) = \int_{-\infty}^{x} \Omega(\tau)\, d\tau$.




Unfortunately, this Taylor series—though always theoretically correct—is practical only for small and moderate $|x_o| \lesssim 1$. For $|x_o| \gg 1$, see § 20.10.

The normal distribution tends to be the default distribution in applied mathematics. When one lacks a reason to do otherwise, one models a random quantity as a normally distributed random variable. See § 20.7 for the reason.

## 20.6 Inference of statistics

Suppose that several, concrete instances of a random variable—collectively called a *sample*—were drawn from a distribution $f(x)$ and presented to you, but that you were not told the shape of $f(x)$. Could you infer the shape?

The answer is that you could infer the shape with passable accuracy provided that the number $N$ of samples were large. Typically however one will be prepared to make some assumption about the shape such as that

$$f(x) = \mu + \frac{1}{\sigma}\Omega\left(\frac{x}{\sigma}\right), \tag{20.19}$$

which is to assume that $x$ were normally distributed with unknown statistics $\mu$ and $\sigma$. The problem then becomes to infer the statistics from the sample.

In the absence of additional information, one can hardly hardly suppose much about the mean other than that

$$\mu \approx \frac{1}{N} \sum_k x_k. \tag{20.20}$$

One infers the mean to be the average of the instances one has observed. One might think to infer the standard deviation in much the same way except that to calculate the standard deviation directly according to (20.7) would implicate our imperfect estimate (20.20) of the mean. If we wish to estimate the standard deviation accurately from the sample then we shall have to proceed more carefully than that.

It will simplify the standard-deviational analysis to consider the shifted random variable

$$u_k = x_k - \mu_{\text{true}}$$

instead of $x_k$ directly, where $\mu_{\text{true}}$ is not the estimated mean of (20.20) but the true, unknown mean of the hidden distribution $f(x)$. The distribution of $u$ then is $f(u + \mu_{\text{true}})$, a distribution which by construction has zero mean. (Naturally, we do not know—we shall never know—the actual value of $\mu_{\text{true}}$, but this does not prevent us from representing $\mu_{\text{true}}$ symbolically during analysis.) We shall presently find helpful the identities

$$\left\langle \sum_k u_k^2 \right\rangle = N\sigma^2,$$
$$\left\langle \sum_k {}^2 u_k \right\rangle = N\sigma^2,$$

the first of which is merely a statement of the leftward part of (20.7)'s second line with respect to the unknown distribution $f(u + \mu_{\text{true}})$ whose mean $\langle u \rangle$ is null by construction, the second of which considers the sum $\sum_k u_k$ as a random variable whose mean again is null but whose standard deviation $\sigma_\Sigma$ according to (20.9) is such that $\sigma_\Sigma^2 = N\sigma_{\text{true}}^2$.

With the foregoing definition and identities in hand, let us construct from the available sample the quantity

$$(\sigma')^2 \equiv \frac{1}{N} \sum_k \left( x_k - \frac{1}{N} \sum_k x_k \right)^2,$$

which would tend to approach $\sigma_{\text{true}}^2$ as $N$ grew arbitrarily large but which, unlike $\sigma_{\text{true}}$, is a quantity we can actually compute for any $N > 1$. By successive steps,

$$\begin{aligned}
\left(\sigma'\right)^2 &= \frac{1}{N}\sum_k \left( [u_k + \mu_{\text{true}}] - \frac{1}{N}\sum_k [u_k + \mu_{\text{true}}] \right)^2 \\
&= \frac{1}{N}\sum_k \left( u_k - \frac{1}{N}\sum_k u_k \right)^2 \\
&= \frac{1}{N}\sum_k \left( u_k^2 - \frac{2}{N} u_k \sum_k u_k + \frac{1}{N^2}\sum_k^2 u_k \right) \\
&= \frac{1}{N}\sum_k u_k^2 - \frac{2}{N^2}\sum_k^2 u_k + \frac{1}{N^2}\sum_k^2 u_k \\
&= \frac{1}{N}\sum_k u_k^2 - \frac{1}{N^2}\sum_k^2 u_k,
\end{aligned}$$

the expected value of which is

$$\left\langle \left(\sigma'\right)^2 \right\rangle = \frac{1}{N}\left\langle \sum_k u_k^2 \right\rangle - \frac{1}{N^2}\left\langle \sum_k^2 u_k \right\rangle,$$

Applying the identities of the last paragraph,

$$\left\langle \left(\sigma'\right)^2 \right\rangle = \sigma^2 - \frac{\sigma^2}{N} = \frac{N-1}{N}\sigma^2,$$

from which

$$\sigma^2 = \frac{N}{N-1}\left\langle \left(\sigma'\right)^2 \right\rangle.$$

Because the expected value $\langle(\sigma')\rangle$ is not a quantity whose value we know, we can only suppose that $\langle(\sigma')^2\rangle \approx \sigma^2$, whereby

$$\sigma^2 \approx \frac{N}{N-1}\left(\sigma'\right)^2,$$

and, substituting the definition of $(\sigma')^2$ into the last equation,

$$\sigma^2 \approx \frac{1}{N-1}\sum_k \left( x_k - \frac{1}{N}\sum_k x_k \right)^2. \tag{20.21}$$

The estimates (20.20) and (20.21) are known as *sample statistics.* They are the statistics one imputes to an unknown distribution based on the incomplete information of $N > 1$ samples.

This chapter generally assumes independent random variables when it speaks of probability. In statistical work however one must sometimes handle correlated quantities like the height and weight of a 25-year-old American male—for, obviously, if I point to some 25-year-old over there and say, "That's Pfufnik. The average is 160 pounds, but he weighs 250!" then your estimate of his probable height will change, because height and weight are not independent but correlated. The conventional statistical measure[6] of the correlation of a series $(x_k, y_k)$ of pairs of data, such as $([\text{height}]_k, [\text{weight}]_k)$ of the example, is the *correlation coefficient*

$$r \equiv \frac{\sum_k (x_k - \mu_x)(y_k - \mu_y)}{\sqrt{\sum_k (x_k - \mu_x)^2 \sum_k (x_y - \mu_y)^2}}, \tag{20.22}$$

a unitless quantity whose value is $\pm 1$, indicating perfect correlation, when $y_k = x_k$ or even when $y_k = a_1 x_k + a_0$; but whose value should be near zero when the paired data are unrelated. See Fig. 13.1 for another example of the kind of paired data in whose correlation one might be interested: in the figure, the correlation would be $+1$ if the points fell all right on the line. (Beware that the conventional correlation coefficient of eqn. 20.22 can overstate the relationship between paired data when $N$ is small. Consider for instance that $r = \pm 1$ always when $N = 2$. The coefficient as given nevertheless is conventional.)

If further elaborated, the mathematics of statistics rapidly grow much more complicated. The book will not pursue the matter further but will mention that the kinds of questions that arise tend to involve the statistics of the statistics themselves, treating the statistics as random variables. Such questions confound two, separate uncertainties: the uncertainty inherent by definition (20.1) in a random variable even were the variable's distribution precisely known; and the uncertain knowledge of the distribution.[7] Fortunately, if $N \gg 1$, one can usually tear the two uncertainties from one another without undue violence to accuracy, pretending that one knew the unknown $\mu$ and $\sigma$ to be exactly the values (20.20) and (20.21) respectively calculate, supposing that the distribution were the normal (20.19), and modeling on this basis.

[6] [**?**, § 9.9][**?**, eqns. 12-6 and 12-14]

[7] The subtle mathematical implications of this far exceed the scope of the present book but are developed to one degree or another in numerous collegiate statistics texts of which [**?**][**?**][**?**][**?**] are representative examples.

## 20.7 The random walk and its consequences

This section brings overview, insight and practical perspective. It also analyzes the simple but often encountered statistics of a series of all-or-nothing-type attempts.

### 20.7.1 The random walk

Matthew Sands gave a famous lecture [19, §§ I:6], on probability, on behalf of Richard P. Feynman at Caltech in the fall of 1961. The lecture is a classic and is recommended to every reader of this chapter who can conveniently lay hands on a printed copy—recommended among other reasons because it lends needed context to the rather abstruse mathematics this chapter has presented to the present point. One section of the lecture begins, "There is [an] interesting problem in which the idea of probability is required. It is the problem of the 'random walk.' In its simplest version, we imagine a 'game' in which a 'player' starts at the point $[D = 0]$ and at each 'move' is required to take a step *either* forward (toward $[+D]$) *or* backward (toward $[-D]$). The choice is to be made *randomly,* determined, for example, by the toss of a coin. How shall we describe the resulting motion?"

Sands goes on to observe that, though one cannot guess whether the 'player' will have gone forward or backward after $N$ steps—and, indeed, that in the absence of other information one must expect $\langle D_N \rangle = 0$, zero net progress—"[one has] the feeling that as $N$ increases, [the 'player'] is likely to have strayed farther from the starting point." Sands is right, but if $\langle D_N \rangle$ is not a suitable measure of this "likely straying," so to speak, then what would be?

The measure $\langle |D_N| \rangle$ might recommend itself, but this being nonanalytic (§§ 2.12.3 and 8.4) proves inconvenient in practice (you can try it if you like). The success of the least-squares technique of § 13.6 however encourages us to try the measure $\langle D_N^2 \rangle$. The squared distance $D_N^2$ is nonnegative in every instance and also is analytic, so its expected value $\langle D_N^2 \rangle$ proves a most convenient measure of "likely straying." It is moreover a measure universally accepted among scientists and engineers, and it is the measure this book will adopt.

Sands notes that, if the symbol $D_{N-1}$ represents the 'player's' position after $N - 1$ steps, his position after $N$ steps must be $D_N = D_{N-1} \pm 1$. The expected value $\langle D_N \rangle = 0$ is uninteresting as we said, but the expected value $\langle D_N^2 \rangle$ is interesting. And what is this expected value? Sands finds two

possibilities: either the 'player' steps forward on his $N$th step, in which case

$$\left\langle D_N^2 \right\rangle = \left\langle (D_{N-1} + 1)^2 \right\rangle = \left\langle D_{N-1}^2 \right\rangle + 2\left\langle D_{N-1} \right\rangle + 1;$$

or he steps backward on his $N$th step, in which case

$$\left\langle D_N^2 \right\rangle = \left\langle (D_{N-1} - 1)^2 \right\rangle = \left\langle D_{N-1}^2 \right\rangle - 2\left\langle D_{N-1} \right\rangle + 1.$$

Since forward and backward are equally likely, the actual expected value must be the average

$$\left\langle D_N^2 \right\rangle = \left\langle D_{N-1}^2 \right\rangle + 1$$

of the two possibilities. Evidently, the expected value increases by 1 with each step. Thus by induction, since $\langle D_0^2 \rangle = 0$,

$$\left\langle D_N^2 \right\rangle = N.$$

Observe that the PDF of a single step $x_k$ is $f_o(x_o) = [\delta(x_o + 1) + \delta(x_o - 1)]/2$, where $\delta(\cdot)$ is the Dirac delta of Fig. 7.10; and that the corresponding statistics are $\mu_o = 0$ and $\sigma_o = 1$. The PDF of $D_N$ is more complicated (though not especially hard to calculate in view of § 4.2), but its statistics are evidently $\mu_N = 0$ and $\sigma_N = \sqrt{N}$, agreeing with (20.10).

### 20.7.2 Consequences

An important variation of the random walk comes with the distribution

$$f_o(x_o) = (1 - p_o)\delta(x_o) + p_o\delta(x_o - 1), \tag{20.23}$$

which describes or governs an act whose probability of success is $p_o$. This distribution's statistics according to (20.7) are such that

$$\begin{aligned} \mu_o &= p_o, \\ \sigma_o^2 &= (1 - p_o)p_o. \end{aligned} \tag{20.24}$$

As an example of the use, consider a real-estate agent who expects to sell one house per 10 times he shows a prospective buyer a house: $p_o = 1/10 = 0.10$. The agent's expected result from a single showing, according to (20.24), is to sell $\mu_o \pm \sigma_o = 0.10 \pm 0.30$ of a house. The agent's expected result from $N = 400$ showings, according to (20.10), is to sell $\mu \pm \sigma = N\mu_o \pm \left(\sqrt{N}\right)\sigma_o = 40.0 \pm 6.0$ houses. Such a conclusion, of course, is valid only to the extent to which the model is valid—which in a real-estate agent's case may be *not very*—but that nevertheless is how the mathematics of it work.

As the number $N$ of attempts grows large one finds that the distribution $f(x)$ of the number of successes begins more and more to take on the bell-shape of Fig. 20.1's normal distribution. Indeed, this makes sense, for one would expect the aforementioned real-estate agent to have a relatively high probability of selling 39, 40 or 41 houses but a low probability to sell 10 or 70; thus one would expect $f(x)$ to take on something rather like the bell-shape. If for 400 showings the distribution is $f(x)$ then, according to (20.6), for $800 = 400 + 400$ showings the distribution must be $f(x) * f(x)$. Moreover, since the only known PDF which, when convolved with itself, does not change shape is the normal distribution of § 20.5, one infers that the normal distribution is the PDF toward which the real-estate agent's distribution—and indeed most other distributions of sums of random variables—must converge[8] as $N \to \infty$.

For such reasons, applications tend to approximate sums of several random variables as though the sums were normally distributed; and, moreover, tend to impute normal distributions to random variables whose true distributions are unnoticed, uninteresting or unknown. In the theory and application of probability, the normal distribution is the master distribution, the distribution of last resort, often the only distribution tried. The banal suggestion, "When unsure, go normal!" usually prospers in probabilistic work.

[8]Admittedly, the argument, which supposes that all (or at least most) aggregate PDFs must tend toward some common shape as $N$ grows large, is somewhat specious, or at least unrigorous—though on the other hand it is hard to imagine any plausible conclusion other than the correct one the argument reaches—but one can construct an alternate though tedious argument toward the normal distribution on the following basis. Counting permutations per § 4.2, derive an exact expression for the probability of $k$ successes in $N$ tries, which is $P_k = \binom{N}{k} p^k (1-p)^{N-k}$. Considering also the probabilities of $k-1$ and $k+1$ successes in $N$ tries, approximate the logarithmic derivative of $P_k$ per (4.22) as $(\partial P_k / \partial k)/P_k \approx (P_{k+1} - P_{k-1})/2P_k$ or, better—remembering that suitable arithmetical approximations are permissible in such work—as $(\partial P_k / \partial k)/P_k \approx (P_{k+1/2} - P_{k-1/2})/P_k$. Change $x \leftarrow (k - p_o N)/\sqrt{(1-p_o)p_o N}$. Discover a continuous, analytic function, which is $f(x) = C \exp(-ax^2)$, that for large $N$ has a similar logarithmic derivative in the distribution's peak region. To render the arithmetic tractable one might try first the specific case of $p = 1/2$ and make various arithmetical approximations as one goes, but to fill in the tedious details is left as an exercise to the interested (penitent?) reader. The author confesses that he prefers the specious argument of the narrative.

## 20.8 Other distributions

Many distributions other than the normal one of Fig. 20.1 are possible. This section will name a few of the most prominent.

### 20.8.1 The uniform distribution

The *uniform distribution* can be defined in any several forms, but the conventional form is

$$f(x) = \Pi\left(x - \frac{1}{2}\right) = \begin{cases} 1 & \text{if } 0 \le x < 1, \\ 0 & \text{otherwise.} \end{cases} \tag{20.25}$$

where $\Pi(\cdot)$ is the square pulse of Fig. 17.3. Besides sometimes being useful in its own right, this is also the distribution a computer's pseudorandom-number generator obeys. One can extract normally distributed (§ 20.5) or Rayleigh-distributed (§ 20.8.3) random variables from it by the Box-Muller transformation of § 20.9.

### 20.8.2 The exponential distribution

The *exponential distribution* is[9]

$$f(x) = \frac{1}{\mu} \exp\left(-\frac{x}{\mu}\right), \quad x \ge 0, \tag{20.26}$$

whose mean is

$$\frac{1}{\mu} \int_0^\infty \exp\left(-\frac{x}{\mu}\right) x \, dx = -\exp\left(-\frac{x}{\mu}\right)(x + \mu)\Big|_0^\infty = \mu$$

as advertised and whose standard deviation is such that

$$\begin{aligned} \sigma^2 &= \frac{1}{\mu} \int_0^\infty \exp\left(-\frac{x}{\mu}\right)(x - \mu)^2 \, dx \\ &= -\exp\left(-\frac{x}{\mu}\right)(x^2 + \mu^2)\Big|_0^\infty, \end{aligned}$$

[9]Unlike the section's other subsections, this one explicitly includes the mean $\mu$ in the expression (20.26) of its distribution. The inclusion of $\mu$ here is admittedly inconsistent. The reader who prefers to do so can mentally set $\mu = 1$ and read the section in that light. However, in typical applications the entire point of choosing the exponential distribution may be to specify $\mu$, or to infer it. The exponential distribution is inherently "$\mu$-focused," so to speak. The author prefers to leave the $\mu$ in the expression for this reason.

(the integration by the method of unknown coefficients of § 9.4), which implies that

$$\sigma = \mu. \tag{20.27}$$

The exponential's CDF (20.2) and quantile (20.4) are evidently

$$\begin{aligned} F(x) &= 1 - \exp\left(-\frac{x}{\mu}\right), \\ F^{-1}(u) &= -\mu \ln(1-u). \end{aligned} \tag{20.28}$$

Among other effects, the exponential distribution models the delay until some imminent event like a mechanical bearing's failure or the arrival of a retail establishment's next customer.

### 20.8.3 The Rayleigh distribution

The *Rayleigh distribution* is a generalization of the normal distribution for position in a plane. Let each of the $x$ and $y$ coordinates be drawn independently from a normal distribution of zero mean and unit standard deviation, such that

$$\begin{aligned} dP &\equiv [\Omega(x)\,dx]\,[\Omega(y)\,dy] \\ &= \frac{1}{2\pi}\exp\left(-\frac{x^2+y^2}{2}\right)dx\,dy \\ &= \frac{1}{2\pi}\exp\left(-\frac{\rho^2}{2}\right)\rho\,d\rho\,d\phi, \end{aligned}$$

whence

$$\begin{aligned} P_{ba} &\equiv \int_{\phi=-\pi}^{\pi}\int_{\rho=a}^{b} dP \\ &= \frac{1}{2\pi}\int_{-\pi}^{\pi}\int_{a}^{b}\exp\left(-\frac{\rho^2}{2}\right)\rho\,d\rho\,d\phi \\ &= \int_{a}^{b}\exp\left(-\frac{\rho^2}{2}\right)\rho\,d\rho, \end{aligned}$$

which implies the distribution

$$f(\rho) = \rho\exp\left(-\frac{\rho^2}{2}\right), \quad \rho \geq 0. \tag{20.29}$$

This is the Rayleigh distribution. That it is a proper distribution according to (20.1) is proved by evaluating the integral

$$\int_0^\infty f(\rho)\,d\rho = 1 \tag{20.30}$$

using the method of § 18.5. Rayleigh's CDF (20.2) and quantile (20.4) are evidently

$$\begin{aligned} F(\rho) &= 1 - \exp\left(-\frac{\rho^2}{2}\right), \\ F^{-1}(u) &= \sqrt{-2\ln(1-u)}. \end{aligned} \tag{20.31}$$

The Rayleigh distribution models among others the distance $\rho$ by which a missile might miss its target.

Incidentally, there is nothing in the mathematics to favor any particular value of $\phi$ over another, $\phi$ being the azimuth toward which the missile misses, for the integrand $\exp(-\rho^2/2)\rho\,d\rho\,d\phi$ above includes no $\phi$; so, unlike $\rho$, $\phi$ by symmetry will be uniformly distributed.

### 20.8.4 The Maxwell distribution

The *Maxwell distribution* extends the Rayleigh from two to three dimensions. Maxwell's derivation closely resembles Rayleigh's, with the difference that Maxwell uses all three of $x$, $y$ and $z$ and then transforms to spherical rather than cylindrical coordinates. The distribution which results, the Maxwell distribution, is

$$f(r) = \frac{2r^2}{\sqrt{2\pi}} \exp\left(-\frac{r^2}{2}\right), \quad r \geq 0, \tag{20.32}$$

which models, among others, the speed at which an air molecule might travel.[10]

### 20.8.5 The log-normal distribution

In the *log-normal distribution,* it is not $x$ but

$$x_o \equiv \frac{\ln x}{\alpha} \tag{20.33}$$

[10] [19, eqn. I:40.7]

that is normally distributed, a fairly common case. Setting $x = g(x_o) = \exp \alpha x_o$ and $f_o(x_o) = \Omega(x_o)$ in (20.11), one can express the log-normal distribution in the form[11]

$$f(x) = \frac{1}{\alpha x} \Omega \left( \frac{\ln x}{\alpha} \right). \tag{20.34}$$

## 20.9 The Box-Muller transformation

The quantiles (20.28) and (20.31) imply easy conversions from the uniform distribution to the exponential and Rayleigh. Unfortunately, we lack a quantile formula for the normal distribution. However, we can still convert uniform to normal by way of Rayleigh as follows.

Section 20.8.3 has shown how Rayleigh gives the distance $\rho$ by which a missile misses a target when each of $x$ and $y$ are normally distributed and, interestingly, how the azimuth $\phi$ is uniformly distributed under these conditions. Because we know the quantiles, to convert a pair of instances $u$ and $v$ of a uniformly distributed random variable to Rayleigh's distance and azimuth is thus straightforward:[12]

$$\begin{aligned} \rho &= \sqrt{-2 \ln(1 - u)}, \\ \phi &= (2\pi) \left( v - \frac{1}{2} \right). \end{aligned} \tag{20.35}$$

But for the reason just given,

$$\begin{aligned} x &= \rho \cos \phi, \\ y &= \rho \sin \phi, \end{aligned} \tag{20.36}$$

must then constitute two independent instances of a normally distributed random variable with $\mu = 0$ and $\sigma = 1$. Evidently, though we lack an easy way to convert a single uniform instance to a single normal instance, we can convert a *pair* of uniform instances to a pair of normal instances. Equations (20.35) and (20.36) are the *Box-Muller transformation.*[13]

[11] [**?**, Ch. 5]

[12] One can eliminate a little trivial arithmetic by appropriate changes of variable in (20.35) like $u' \leftarrow 1 - u$, but to do so saves little computational time and makes the derivation harder to understand. Still, the interested reader might complete the improvement as an exercise.

[13] [65]

## 20.10 The normal cumulative distribution function at large arguments

The Taylor series (20.18) in theory correctly calculates the normal CDF $F_\Omega(x)$, an entire function, for any argument $x$. In practice however—consider the Taylor series

$$1 - F_\Omega(6) \approx -\text{0x0.8000} + \text{0x2.64C6} - \text{0xE.5CA7} + \text{0x4D.8DEC} - \cdots$$

Not promising, is it? Using a computer's standard, `double`-type floating-point arithmetic, this calculation fails, swamped by rounding error.

One can always calculate in greater precision,[14] of course, asking the computer to carry extra bits; and, actually, this is not necessarily a bad approach. There remain however several reasons one might prefer a more efficient formula.

- One might wish to evaluate the CDF at thousands or millions of points, not just one. At some scale, even with a computer, the calculation grows expensive.
- One might wish to evaluate the CDF on a low-power "embedded device."
- One might need to evaluate the CDF under a severe time constraint measured in microseconds, as in aircraft control.
- Hard though it might be for some to imagine, one might actually wish to evaluate the CDF with a pencil! Or with a slide rule. (Besides that one might not have a suitable electronic computer conveniently at hand, that electronic computers will never again be scarce is a proposition whose probability the author is not prepared to evaluate.)
- The mathematical method by which a more efficient formula is derived is most instructive.[15]
- One might regard a prudent measure of elegance, even in applications, to be its own reward.

[14] [**?**]

[15] Such methods prompt one to wonder how much useful mathematics our civilization should have forgone had Leonhard Euler (1707–1783), Carl Friedrich Gauss (1777–1855) and other hardy mathematical minds of the past computers to lean upon.

Here is the method.[16] Beginning from

$$\begin{aligned} 1 - F_\Omega(x) &= \frac{1}{\sqrt{2\pi}} \int_x^\infty \exp\left(-\frac{\tau^2}{2}\right) d\tau \\ &= \frac{1}{\sqrt{2\pi}} \left\{ -\int_{\tau=x}^\infty \frac{d\left[e^{-\tau^2/2}\right]}{\tau} \right\} \end{aligned}$$

and integrating by parts,

$$\begin{aligned} 1 - F_\Omega(x) &= \frac{1}{\sqrt{2\pi}} \left\{ \frac{e^{-x^2/2}}{x} - \int_x^\infty \frac{e^{-\tau^2/2}\, d\tau}{\tau^2} \right\} \\ &= \frac{1}{\sqrt{2\pi}} \left\{ \frac{e^{-x^2/2}}{x} + \int_{\tau=x}^\infty \frac{d\left[e^{-\tau^2/2}\right]}{\tau^3} \right\}. \end{aligned}$$

Integrating by parts again,

$$\begin{aligned} 1 - F_\Omega(x) &= \frac{1}{\sqrt{2\pi}} \left\{ \frac{e^{-x^2/2}}{x} - \frac{e^{-x^2/2}}{x^3} + 3\int_x^\infty \frac{e^{-\tau^2/2}\, d\tau}{\tau^4} \right\} \\ &= \frac{1}{\sqrt{2\pi}} \left\{ \frac{e^{-x^2/2}}{x} - \frac{e^{-x^2/2}}{x^3} - 3\int_{\tau=x}^\infty \frac{d\left[e^{-\tau^2/2}\right]}{\tau^5} \right\}. \end{aligned}$$

Integrating by parts repeatedly,

$$\begin{aligned} 1 - F_\Omega(x) = \frac{1}{\sqrt{2\pi}} \Bigg\{ & \frac{e^{-x^2/2}}{x} - \frac{e^{-x^2/2}}{x^3} + \frac{3e^{-x^2/2}}{x^5} - \cdots \\ & + \frac{(-)^{n-1}(2n-3)!!e^{-x^2/2}}{x^{2n-1}} \\ & + (-)^n (2n-1)!! \int_x^\infty \frac{e^{-\tau^2/2}\, d\tau}{\tau^{2n}} \Bigg\}, \end{aligned}$$

in which the convenient notation

$$m!! \equiv \begin{cases} \prod_{j=1}^{(m+1)/2} (2j-1) = (m)(m-2)\cdots(5)(3)(1) & \text{for odd } m, \\ \prod_{j=1}^{m/2} (2j) = (m)(m-2)\cdots(6)(4)(2) & \text{for even } m, \end{cases} \tag{20.37}$$

[16] [43, § 2.2]

is introduced.[2, Exercise 2.2.15] The last expression for $1 - F_\Omega(x)$ is better written

$$\begin{aligned} 1 - F_\Omega(x) &= \frac{\Omega(x)}{x}[S_n(x) + R_n(x)], \\ S_n(x) &\equiv \sum_{k=0}^{n-1}\left[\prod_{j=1}^{k}\frac{2j-1}{-x^2}\right] = \sum_{k=0}^{n-1}\frac{(-)^k(2k-1)!!}{x^{2k}}, \\ R_n(x) &\equiv (-)^n(2n-1)!!x\int_x^\infty \frac{e^{(x^2-\tau^2)/2}\,d\tau}{\tau^{2n}}. \end{aligned} \tag{20.38}$$

The series $S_n(x)$ is an *asymptotic series,* also called an *semiconvergent series.*[17] So long as $x \gg 1$, the first several terms of the series will successively shrink in magnitude but, no matter how great the argument $x$ might be, eventually the terms will insist on growing again, growing without limit. Unlike a Taylor series, $S_\infty(x)$ diverges for all $x$.

Fortunately, nothing requires us to let $n \to \infty$, and we remain free to choose $n$ strategically as we like—for instance to exclude from $S_n$ the series' least term in magnitude and all the terms following. So excluding leaves us with the problem of evaluating the integral $S_n$, but see:

$$\begin{aligned} |R_n(x)| &\le (2n-1)!!\,|x|\int_x^\infty \left|\frac{e^{(x^2-\tau^2)/2}\,d\tau}{\tau^{2n}}\right| \\ &\le \frac{(2n-1)!!}{|x|^{2n}}\int_x^\infty \left|e^{(x^2-\tau^2)/2}\tau\,d\tau\right|, \end{aligned}$$

because $|x| \le |\tau|$, so $|x|^{2n+1} \le |\tau|^{2n+1}$. Changing $\xi^2 \leftarrow \tau^2 - x^2$, whereby $\xi\,d\xi = \tau\,d\tau$,

$$|R_n(x)| \le \frac{(2n-1)!!}{|x|^{2n}}\int_0^\infty \left|e^{-\xi^2/2}\xi\,d\xi\right|.$$

Using (20.29) and (20.30),

$$|R_n(x)| \le \frac{(2n-1)!!}{|x|^{2n}}, \quad \Im(x) = 0, \tag{20.39}$$

which in view of (20.38) has that the magnitude $|R_n|$ of the error due to truncating the series after $n$ terms does not exceed the magnitude of the first omitted term. Equation (20.38) thus provides the efficient means we have sought to estimate the CDF accurately for large arguments.

[17] As professional use them, the adjectives *asymptotic* and *semiconvergent* apparently can differ slightly in meaning [2]. We'll not worry about that here.

## 20.11 The normal quantile

Though no straightforward quantile (20.4) formula for the normal distribution seems to be known, nothing prevents one from calculating the quantile via the Newton-Raphson iteration (4.31)[18]

$$\begin{aligned} x_{k+1} &= x_k - \frac{F_\Omega(x_k) - u}{\Omega(x_k)}, \\ F_\Omega^{-1}(u) &= \lim_{k \to \infty} x_k, \\ x_0 &= 0, \end{aligned} \tag{20.40}$$

where $F_\Omega(x)$ is as given by (20.18) and/or (20.38) and $\Omega(x)$, naturally, is as given by (20.15). The shape of the normal CDF as seen in Fig. 20.1—curving downward traveling right from $x = 0$, upward when traveling left, evidently guarantees convergence per Fig. 4.5.

In the large-argument limit,

$$\begin{aligned} 1 - u &\ll 1, \\ x &\gg 1; \end{aligned}$$

so, according to (20.38),

$$F_\Omega(x) \approx 1 - \frac{\Omega(x)}{x}\left(1 - \frac{1}{x^2} + \cdots\right).$$

Substituting this into (20.40) yields, by successive steps,

$$\begin{aligned} x_{k+1} &\approx x_k - \frac{1}{\Omega(x_k)}\left[1 - u - \frac{\Omega(x_k)}{x_k}\left(1 - \frac{1}{x_k^2} + \cdots\right)\right] \\ &\approx x_k - \frac{1-u}{\Omega(x_k)} + \frac{1}{x_k} - \frac{1}{x_k^3} + \cdots \\ &\approx x_k - \frac{\left(\sqrt{2\pi}\right)(1-u)}{1 - x_k^2/2 + \cdots} + \frac{1}{x_k} - \frac{1}{x_k^3} + \cdots \\ &\approx x_k - \left(\sqrt{2\pi}\right)\left(1-u\right)\left(1 + \frac{x_k^2}{2} + \cdots\right) + \frac{1}{x_k} - \frac{1}{x_k^3} + \cdots \\ &\approx x_k - \left(\sqrt{2\pi}\right)\left(1-u\right) + \frac{1}{x_k} + \cdots, \end{aligned}$$

[18] When implementing numerical algorithms like these on the computer one should do it intelligently. For example, if $F_\Omega(x_k)$ and $u$ are both likely to be close to 1, do not ask the computer to calculate and/or store these quantities. Rather, ask it to calculate and/or store $1 - F_\Omega(x_k)$ and $1 - u$. Then, when (20.40) instructs you to calculate a quantity like $F_\Omega(x_k) - u$, let the computer instead calculate $[1-u]-[1-F_\Omega(x_k)]$, which is arithmetically no different but numerically, on the computer, much more precise.

suggesting somewhat lazy, but usually acceptable convergence in domains of typical interest (the convergence might be unacceptable if, for example, $x > 0\text{x}40$, but the writer has never encountered an application of the normal distribution $\Omega(x)$ or its incidents at such large values of $x$). If unacceptable, various stratagems might be tried to accelerate the Newton-Raphson, or—if you have no need to impress anyone with the pure elegance of your technique but only want the right answer reasonably fast—you might just search for the root in the naïve way, trying $F_\Omega(2^0)$, $F_\Omega(2^1)$, $F_\Omega(2^2)$ and so on until identifying a bracket $F_\Omega(2^{k-1}) < u \leq F_\Omega(2^k)$; then dividing the bracket in half, then in half again, then again and again until satisfied with the accuracy thus achieved, or until the bracket were strait enough for you to set $x_0$ to the bracket's lower (not upper) limit and to switch over to (20.40) which performs well when it starts close enough to the root. In truth, though not always stylish, the normal quantile of a real argument is relatively quick, easy and accurate to calculate once you have (20.15), (20.18) and (20.38) in hand, even when the performance of (20.40) might not quite suit. You only must remain a little flexible as to the choice of technique.[19]

[19] See also § 8.10.4.

# Plan

The following chapters are tentatively planned to complete the book.

21. The wave equation[1]
22. Cylinder functions
23. Orthogonal polynomials[2,3]
24. Transformations to speed series convergence[4]
25. The conjugate-gradient algorithm
26. Remarks

Chapters are likely yet to be inserted, removed, divided, combined and shuffled, but that's the planned outline at the moment.

The book means to stop short of hypergeometric functions, parabolic cylinder functions, selective-dimensional (Weyl and Sommerfeld) Fourier transforms, wavelets, and iterative techniques more advanced than the conjugate-gradient (the advanced iterative techniques being too active an area of research for such a book as this yet to treat). However, acknowledging the uniquely seminal historical importance Kepler's laws, the book would like to add an appendix on the topic, to precede the existing Appendix D.

[1] Chapter 21 might begin with Poisson's equation and the corresponding static case. After treating the wave equation proper, it might end with the parabolic wave equation.

[2] Chapter 23 would be pretty useless if it did not treat Legendre polynomials, so presumably it will do at least this.

[3] The author has not yet decided how to apportion the treatment of the wave equation in spherical geometries between Chs. 21, 22 and 23.

[4] Chapter 24 is tentatively to treat at least the Poisson sum formula, Mosig's summation-by-parts technique and, the author believes, the Watson transformation; plus maybe some others as seems appropriate. This might also be a good chapter in which to develop the infinite-product forms of the sine and the cosine and thence Euler's and Andrews' clever closed-form series summations from [2, § 1.7 and exercises] and maybe from other, similar sources.

# Appendices

# Appendix A

# Hexadecimal and other notational matters

The importance of conventional mathematical notation is hard to overstate. Such notation serves two distinct purposes: it conveys mathematical ideas from writer to reader; and it concisely summarizes complex ideas on paper to the writer himself. Without the notation, one would find it difficult even to think clearly about the math; to discuss it with others, nearly impossible.

The right notation is not always found at hand, of course. New mathematical ideas occasionally find no adequate preëstablished notation, when it falls to the discoverer and his colleagues to establish new notation to meet the need. A more difficult problem arises when old notation exists but is inelegant in modern use.

Convention is a hard hill to climb, and rightly so. Nevertheless, slavish devotion to convention does not serve the literature well; for how else can notation improve over time, if writers will not incrementally improve it? Consider the notation of the algebraist Girolamo Cardano in his 1539 letter to Tartaglia:

> [T]he cube of one-third of the coefficient of the unknown is greater in value than the square of one-half of the number. [46]

If Cardano lived today, surely he would express the same thought in the form

$$\left(\frac{a}{3}\right)^3 > \left(\frac{x}{2}\right)^2.$$

Good notation matters.

Although this book has no brief to overhaul applied mathematical notation generally, it does seek to aid the honorable cause of notational evolution

in a few specifics. For example, the book sometimes treats $2\pi$ implicitly as a single symbol, so that (for instance) the quarter revolution or right angle is expressed as $2\pi/4$ rather than as the less evocative $\pi/2$.

As a single symbol, of course, $2\pi$ remains a bit awkward. One wants to introduce some new symbol $\xi = 2\pi$ thereto. However, it is neither necessary nor practical nor desirable to leap straight to notational Utopia in one great bound. It suffices in print to improve the notation incrementally. If this book treats $2\pi$ sometimes as a single symbol—if such treatment meets the approval of slowly evolving convention—then further steps, the introduction of new symbols $\xi$ and such, can safely be left incrementally to future writers.

## A.1 Hexadecimal numerals

Treating $2\pi$ as a single symbol is a small step, unlikely to trouble readers much. A bolder step is to adopt from the computer science literature the important notational improvement of the hexadecimal numeral. No incremental step is possible here; either we leap the ditch or we remain on the wrong side. In this book, we choose to leap.

Traditional decimal notation is unobjectionable for measured quantities like 63.7 miles, \$1.32 million or $9.81\,\mathrm{m/s^2}$, but its iterative tenfold structure meets little or no aesthetic support in mathematical theory. Consider for instance the decimal numeral 127, whose number suggests a significant idea to the computer scientist, but whose decimal notation does nothing to convey the notion of the largest signed integer storable in a byte. Much better is the base-sixteen hexadecimal notation 0x7F, which clearly expresses the idea of $2^7 - 1$. To the reader who is not a computer scientist, the aesthetic advantage may not seem immediately clear from the one example, but consider the decimal number 2,147,483,647, which is the largest signed integer storable in a standard thirty-two bit word. In hexadecimal notation, this is 0x7FFF FFFF, or in other words $2^{0\mathrm{x}1\mathrm{F}} - 1$. The question is: which notation more clearly captures the idea?

To readers unfamiliar with the hexadecimal notation, to explain very briefly: hexadecimal represents numbers not in tens but rather in sixteens. The rightmost place in a hexadecimal numeral represents ones; the next place leftward, sixteens; the next place leftward, sixteens squared; the next, sixteens cubed, and so on. For instance, the hexadecimal numeral 0x1357 means "seven, plus five times sixteen, plus thrice sixteen times sixteen, plus once sixteen times sixteen times sixteen." In hexadecimal, the sixteen symbols 0123456789ABCDEF respectively represent the numbers zero through

fifteen, with sixteen being written 0x10.

All this raises the sensible question: why sixteen?[1] The answer is that sixteen is $2^4$, so hexadecimal (base sixteen) is found to offer a convenient shorthand for binary (base two, the fundamental, smallest possible base). Each of the sixteen hexadecimal digits represents a unique sequence of exactly four bits (binary digits). Binary is inherently theoretically interesting, but direct binary notation is unwieldy (the hexadecimal number 0x1357 is binary 0001 0011 0101 0111), so hexadecimal is written in proxy.

The conventional hexadecimal notation is admittedly a bit bulky and unfortunately overloads the letters A through F, letters which when set in italics usually represent coefficients not digits. However, the real problem with the hexadecimal notation is not in the notation itself but rather in the unfamiliarity with it. The reason it is unfamiliar is that it is not often encountered outside the computer science literature, but it is not encountered because it is not used, and it is not used because it is not familiar, and so on in a cycle. It seems to this writer, on aesthetic grounds, that this particular cycle is worth breaking, so this book uses the hexadecimal for integers larger than 9. If you have never yet used the hexadecimal system, it is worth your while to learn it. For the sake of elegance, at the risk of challenging entrenched convention, this book employs hexadecimal throughout.

Observe that in some cases, such as where hexadecimal numbers are arrayed in matrices, this book may omit the cumbersome hexadecimal prefix "0x." Specific numbers with physical units attached appear seldom in this book, but where they do naturally decimal not hexadecimal is used: $v_{\text{sound}} = 331$ m/s rather than the silly-looking $v_{\text{sound}} = \text{0x14B}$ m/s.

Combining the hexadecimal and $2\pi$ ideas, we note here for interest's sake that

$$2\pi \approx \text{0x6.487F}.$$

## A.2 Avoiding notational clutter

Good applied mathematical notation is not cluttered. Good notation does not necessarily include every possible limit, qualification, superscript and

[1] An alternative advocated by some eighteenth-century writers was twelve. In base twelve, one quarter, one third and one half are respectively written 0.3, 0.4 and 0.6. Also, the hour angles (§ 3.6) come in neat increments of $(0.06)(2\pi)$ in base twelve, so there are some real advantages to that base. Hexadecimal, however, besides having momentum from the computer science literature, is preferred for its straightforward proxy of binary.

subscript. For example, the sum

$$S = \sum_{i=1}^{M} \sum_{j=1}^{N} a_{ij}^2$$

might be written less thoroughly but more readably as

$$S = \sum_{i,j} a_{ij}^2$$

if the meaning of the latter were clear from the context.

When to omit subscripts and such is naturally a matter of style and subjective judgment, but in practice such judgment is often not hard to render. The balance is between showing few enough symbols that the interesting parts of an equation are not obscured visually in a tangle and a haze of redundant little letters, strokes and squiggles, on the one hand; and on the other hand showing enough detail that the reader who opens the book directly to the page has a fair chance to understand what is written there without studying the whole book carefully up to that point. Where appropriate, this book often condenses notation and omits redundant symbols.

# Appendix B

# The Greek alphabet

Mathematical experience finds the Roman alphabet to lack sufficient symbols to write higher mathematics clearly. Although not completely solving the problem, the addition of the Greek alphabet helps. See Table B.1.

When first seen in mathematical writing, the Greek letters take on a wise, mysterious aura. Well, the aura is fine—the Greek letters are pretty—but don't let the Greek letters throw you. They're just letters. We use them not because we want to be wise and mysterious[1] but rather because we simply do not have enough Roman letters. An equation like

$$\alpha^2 + \beta^2 = \gamma^2$$

says no more than does an equation like

$$a^2 + b^2 = c^2,$$

after all. The letters are just different (though naturally one prefers to use the letters one's audience expects when one can).

Applied as well as professional mathematicians tend to use Roman and Greek letters in certain long-established conventional sets: $abcd$; $fgh$; $ijk\ell$; $mn$; $pqr$; $st$; $uvw$; $xyz$. For the Greek: $\alpha\beta\gamma$; $\delta\epsilon$; $\kappa\lambda\mu\nu\xi$; $\rho\sigma\tau$; $\phi\chi\psi\omega$. Greek

[1] Well, you can use them to be wise and mysterious if you want to. It's kind of fun, actually, when you're dealing with someone who doesn't understand math—if what you want is for him to go away and leave you alone. Otherwise, we tend to use Roman and Greek letters in various conventional ways: Greek minuscules (lower-case letters) for angles; Roman capitals for matrices; $e$ for the natural logarithmic base; $f$ and $g$ for unspecified functions; $i$, $j$, $k$, $m$, $n$, $M$ and $N$ for integers; $P$ and $Q$ for metasyntactic elements; $t$, $T$ and $\tau$ for time; $d$, $\delta$ and $\Delta$ for change; $A$, $B$ and $C$ for unknown coefficients; etc.

Table B.1: The Roman and Greek alphabets.

ROMAN

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| $Aa$ | Aa | $Gg$ | Gg | $Mm$ | Mm | $Tt$ | Tt |
| $Bb$ | Bb | $Hh$ | Hh | $Nn$ | Nn | $Uu$ | Uu |
| $Cc$ | Cc | $Ii$ | Ii | $Oo$ | Oo | $Vv$ | Vv |
| $Dd$ | Dd | $Jj$ | Jj | $Pp$ | Pp | $Ww$ | Ww |
| $Ee$ | Ee | $Kk$ | Kk | $Qq$ | Qq | $Xx$ | Xx |
| $Ff$ | Ff | $L\ell$ | Ll | $Rr$ | Rr | $Yy$ | Yy |
| | | | | $Ss$ | Ss | $Zz$ | Zz |

GREEK

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| $A\alpha$ | alpha | $H\eta$ | eta | $N\nu$ | nu | $T\tau$ | tau |
| $B\beta$ | beta | $\Theta\theta$ | theta | $\Xi\xi$ | xi | $\Upsilon\upsilon$ | upsilon |
| $\Gamma\gamma$ | gamma | $I\iota$ | iota | $O o$ | omicron | $\Phi\phi$ | phi |
| $\Delta\delta$ | delta | $K\kappa$ | kappa | $\Pi\pi$ | pi | $X\chi$ | chi |
| $E\epsilon$ | epsilon | $\Lambda\lambda$ | lambda | $P\rho$ | rho | $\Psi\psi$ | psi |
| $Z\zeta$ | zeta | $M\mu$ | mu | $\Sigma\sigma$ | sigma | $\Omega\omega$ | omega |

letters are frequently paired with their Roman congeners as appropriate: $a\alpha$; $b\beta$; $cg\gamma$; $d\delta$; $e\epsilon$; $f\phi$; $k\kappa$; $\ell\lambda$; $m\mu$; $n\nu$; $p\pi$; $r\rho$; $s\sigma$; $t\tau$; $x\chi$; $z\zeta$.[2]

Mathematicians usually avoid letters like the Greek capital H (eta), which looks just like the Roman capital H, even though H (eta) is an entirely

[2]The capital pair $Y\Upsilon$ is occasionally seen but is awkward both because the Greek minuscule $\upsilon$ is visually almost indistinguishable from the unrelated (or distantly related) Roman minuscule $v$; and because the ancient Romans regarded the letter $Y$ not as a congener but as the Greek letter itself, seldom used but to spell Greek words in the Roman alphabet. To use $Y$ and $\Upsilon$ as separate symbols is to display an indifference to, easily misinterpreted as an ignorance of, the Graeco-Roman sense of the thing—which is silly, arguably, if you think about it, since no one objects when you differentiate $j$ from $i$, or $u$ and $w$ from $v$—but, anyway, one is probably the wiser to tend to limit the mathematical use of the symbol $\Upsilon$ to the very few instances in which established convention decrees it. (In English particularly, there is also an old typographical ambiguity between $Y$ and a Germanic, non-Roman letter named "thorn" which has practically vanished from English today, to the point that the typeface in which you are reading these words lacks a glyph for it—but which sufficiently literate writers are still expected to recognize on sight. This is one more reason to tend to avoid $\Upsilon$ when you can, a Greek letter that makes you look ignorant when you use it wrong and pretentious when you use it right. You can't win.)

The history of the alphabets is extremely interesting. Unfortunately, a footnote in an appendix to a book on derivations of applied mathematics is probably not the right place for an essay on the topic, so we'll let the matter rest there.

proper member of the Greek alphabet. The Greek minuscule $\upsilon$ (upsilon) is avoided for like reason, for mathematical symbols are useful only insofar as we can visually tell them apart.

# Appendix C

# A bare sketch of the pure theory of the complex variable

At least three of the various disciplines of pure mathematics stand out for their pedagogical intricacy and the theoretical depth of their core results. The first of the three is number theory which, except for the simple results of § 6.1, scientists and engineers tend to get by largely without. The second is matrix theory (Chs. 11 through 14), a bruiser of a discipline the applied mathematician of the computer age—try though he might—can hardly escape. The third is the pure theory of the complex variable.

The introduction's § 1.3 admires the beauty of the pure theory of the complex variable even while admitting that "its arc takes off too late and flies too far from applications for such a book as this." To develop the pure theory properly is a worthy book-length endeavor of its own requiring moderately advanced preparation on its reader's part which, however, the reader who has reached the end of the present book's Ch. 9 possesses. If the writer doubts the strictly applied *necessity* of the pure theory, still, he does not doubt its health to one's overall mathematical formation. It provides another way to think about complex numbers. Scientists and engineers with advanced credentials occasionally expect one to be acquainted with it for technical-social reasons, regardless of its practical use. Besides, the pure theory is interesting. This alone recommends some attention to it.

The pivotal result of pure complex-variable theory is the Taylor series by Cauchy's impressed residue theorem. If we will let these few pages of appendix replace an entire book on the pure theory, then Cauchy's and

Taylor's are the results we will sketch. The bibliography lists presentations far more complete.

*Cauchy's impressed residue theorem*[1] is that

$$f(z) = \frac{1}{i2\pi} \oint \frac{f(w)}{w - z}\, dw \tag{C.1}$$

if $z$ lies within the closed complex contour about which the integral is taken and if $f(z)$ is everywhere analytic (§ 8.4) within and along the contour. More than one proof of the theorem is known, depending on the assumptions from which the mathematician prefers to start, but this writer is partial to an instructively clever proof he has learned from D.N. Arnold[2] which goes as follows. Consider the function

$$g(z,t) \equiv \frac{1}{i2\pi} \oint \frac{f[z + (t)(w - z)]}{w - z}\, dw,$$

whose derivative with respect to the parameter $t$ is[3]

$$\frac{\partial g}{\partial t} = \frac{1}{i2\pi} \oint f'[z + (t)(w - z)]\, dw.$$

We notice that this is

$$\begin{aligned} \frac{\partial g}{\partial t} &= \frac{1}{i2\pi} \oint \frac{\partial}{\partial w} \left\{ \frac{f[z + (t)(w - z)]}{t} \right\} dw \\ &= \frac{1}{i2\pi} \left\{ \frac{f[z + (t)(w - z)]}{t} \right\}_a^b, \end{aligned}$$

where $a$ and $b$ respectively represent the contour integration's beginning and ending points. But this integration ends where it begins, so $a = b$ and the factor $\{\cdot\}_a^b$ in braces vanishes, whereupon

$$\frac{\partial g}{\partial t} = 0,$$

[1] This is not a standard name. Though they name various associated results after Cauchy in one way or another, neither [31] nor [4] seems to name this particular result, though both do feature it. Since (C.1) impresses a pole and thus also a residue on a function $f(z)$ which in the domain of interest lacks them, the name *Cauchy's impressed residue theorem* ought to serve this appendix's purpose ably enough.

[2] [4, § III]

[3] The book does not often employ Newton's notation $f'(\cdot) \equiv [(d/d\zeta)f(\zeta)]_{\zeta=(\cdot)}$ of § 4.4 but the notation is handy here because it evades the awkward circumlocution of changing $\zeta \leftarrow z$ in (C.1) and then writing

$$\frac{\partial g}{\partial t} = \frac{1}{i2\pi} \oint \frac{[(d/d\zeta)f(\zeta)]_{\zeta=z+(t)(w-z)}}{w - z}\, dw.$$

meaning that $g(z,t)$ does not vary with $t$. Observing per (8.26) that

$$\frac{1}{i2\pi}\oint\frac{dw}{w-z}=1,$$

we have that

$$f(z)=\frac{f(z)}{i2\pi}\oint\frac{dw}{w-z}=g(z,0)=g(z,1)=\frac{1}{i2\pi}\oint\frac{f(w)}{w-z}\,dw$$

as was to be proved. (There remains a basic question as to whether the paragraph's integration is even valid. Logically, it ought to be valid, since $f[z]$ being analytic is infinitely differentiable,[4] but when the integration is used as the sole theoretical support for the entire calculus of the complex variable, well, it seems an awfully slender reed to carry so heavy a load. Admittedly, maybe this is only a psychological problem, but a professional mathematician will devote many pages to preparatory theoretical constructs before even attempting the integral, the result of which lofty effort is not in the earthier spirit of applied mathematics. On the other hand, now that the reader has followed the book along its low road and the high integration is given only in reserve, now that the integration reaches a conclusion already believed and, once there, is asked to carry the far lighter load of this appendix only, the applied reader may feel easier about trusting it.)

One could follow Arnold hence toward the proof of the theorem of one Goursat and further toward various other interesting results, a path of study the writer recommends to sufficiently interested readers: see [4]. Being in a tremendous hurry ourselves, however, we will leave Arnold and follow F.B. Hildebrand[5] directly toward the Taylor series. Positing some expansion point $z_o$ and then expanding (C.1) geometrically per (2.34) about it, we have

[4]The professionals minimalistically actually require only that the function be once differentiable under certain conditions, from which they prove infinite differentiability, but this is a fine point which will not concern us here.

[5][31, § 10.7]

that

$$\begin{aligned} f(z) &= \frac{1}{i2\pi}\oint \frac{f(w)}{(w-z_o)-(z-z_o)}\,dw \\ &= \frac{1}{i2\pi}\oint \frac{f(w)}{(w-z_o)[1-(z-z_o)/(w-z_o)]}\,dw \\ &= \frac{1}{i2\pi}\oint \frac{f(w)}{w-z_o}\sum_{k=0}^{\infty}\left[\frac{z-z_o}{w-z_o}\right]^k dw \\ &= \sum_{k=0}^{\infty}\left\{\left[\frac{1}{i2\pi}\oint \frac{f(w)}{(w-z_o)^{k+1}}\,dw\right](z-z_o)^k\right\}, \end{aligned}$$

which, being the power series

$$\begin{aligned} f(z) &= \sum_{k=0}^{\infty}(a_k)(z-z_o)^k, \\ a_k &\equiv \frac{1}{i2\pi}\oint \frac{f(w)}{(w-z_o)^{k+1}}\,dw, \end{aligned} \tag{C.2}$$

by definition constitutes the Taylor series (8.19) for $f(z)$ about $z = z_o$, assuming naturally that $|z - z_o| < |w - z_o|$ for all $w$ along the contour so that the geometric expansion above will converge.

The important theoretical implication of (C.2) is that *every function has a Taylor series about any point across whose immediate neighborhood the function is analytic.* There evidently is no such thing as an analytic function without a Taylor series—a fact we already knew if we have read and believed Ch. 8, but some readers may find it more convincing this way.

Comparing (C.2) against (8.19), incidentally, we have also that

$$\left.\frac{d^k f}{dz^k}\right|_{z=z_o} = \frac{k!}{i2\pi}\oint \frac{f(w)}{(w-z_o)^{k+1}}\,dw, \tag{C.3}$$

which is an alternate way to write (8.31).

Close inspection of the reasoning by which we have reached (C.2) reveals, quite by the way, at least one additional result which in itself tends to vindicate the pure theory's technique. It is this: that *a Taylor series remains everywhere valid out to the distance of the nearest nonanalytic point.* The proposition is explained and proved as follows. For the aforementioned contour of integration nothing prevents one from choosing a circle, centered in the Argand plane on the expansion point $z = z_o$, the circle's radius just as

large as it can be while still excluding all nonanalytic points. The requirement that $|z - z_o| < |w - z_o|$ for all $w$ along the contour evidently is met for all $z$ inside such a circle, which means that the Taylor series (C.2) converges for all $z$ inside the circle, which—precisely because we have stipulated that the circle be the largest possible centered on the expansion point—implies and thus proves the proposition in question. As an example of the proposition's use, consider the Taylor series Table 8.1 gives for $-\ln(1 - z)$, whose nearest nonanalytic point at $z = 1$ lies at unit distance from the series' expansion point $z = 0$: according to the result of this paragraph, the series in question remains valid over the Argand circle out to unit distance, $|1 - z| < 1$.

# Appendix D

# Manuscript history

The book in its present form is based on various unpublished drafts and notes of mine, plus some of my wife Kristie's (née Hancock), going back to 1983 when I was fifteen years of age. What prompted the contest I can no longer remember, but the notes began one day when I challenged a high-school classmate to prove the quadratic formula. The classmate responded that he didn't need to prove the quadratic formula because the proof was in the class math textbook, then counterchallenged me to prove the Pythagorean theorem. Admittedly obnoxious (I was fifteen, after all) but not to be outdone, I whipped out a pencil and paper on the spot and started working. But I found that I could not prove the theorem that day.

The next day I did find a proof in the school library,[1] writing it down, adding to it the proof of the quadratic formula plus a rather inefficient proof of my own invention to the law of cosines. Soon thereafter the school's chemistry instructor happened to mention that the angle between the tetrahedrally arranged four carbon-hydrogen bonds in a methane molecule was $109^\circ$, so from a symmetry argument I proved that result to myself, too, adding it to my little collection of proofs. That is how it started.[2]

The book actually has earlier roots than these. In 1979, when I was twelve years old, my father bought our family's first eight-bit computer. The computer's built-in *BASIC* programming-language interpreter exposed

[1] A better proof is found in § 2.10.

[2] Fellow gear-heads who lived through that era at about the same age might want to date me against the disappearance of the slide rule. Answer: in my country, or at least at my high school, I was three years too young to use a slide rule. The kids born in 1964 learned the slide rule; those born in 1965 did not. I wasn't born till 1967, so for better or for worse I always had a pocket calculator in high school. My family had an eight-bit computer at home, too, as we shall see.

functions for calculating sines and cosines of angles. The interpreter's manual included a diagram much like Fig. 3.1 showing what sines and cosines were, but it never explained how the computer went about calculating such quantities. This bothered me at the time. Many hours with a pencil I spent trying to figure it out, yet the computer's trigonometric functions remained mysterious to me. When later in high school I learned of the use of the Taylor series to calculate trigonometrics, into my growing collection of proofs the series went.

Five years after the Pythagorean incident I was serving the U.S. Army as an enlisted troop in the former West Germany. Although those were the last days of the Cold War, there was no shooting war at the time, so the duty was peacetime duty. My duty was in military signal intelligence, frequently in the middle of the German night when there often wasn't much to do. The platoon sergeant wisely condoned neither novels nor cards on duty, but he did let the troops read the newspaper after midnight when things were quiet enough. Sometimes I used the time to study my German—the platoon sergeant allowed this, too—but I owned a copy of Richard P. Feynman's *Lectures on Physics* [19] which I would sometimes read instead.

Late one night the battalion commander, a lieutenant colonel and West Point graduate, inspected my platoon's duty post by surprise. A lieutenant colonel was a highly uncommon apparition at that hour of a quiet night, so when that old man appeared suddenly with the sergeant major, the company commander and the first sergeant in tow—the last two just routed from their sleep, perhaps—surprise indeed it was. The colonel may possibly have caught some of my unlucky fellows playing cards that night—I am not sure—but me, he caught with my boots unpolished, reading the *Lectures.*

I snapped to attention. The colonel took a long look at my boots without saying anything, as stormclouds gathered on the first sergeant's brow at his left shoulder, then asked me what I had been reading.

"Feynman's *Lectures on Physics,* sir."

"Why?"

"I am going to attend the university when my three-year enlistment is up, sir."

"I see." Maybe the old man was thinking that I would do better as a scientist than as a soldier? Maybe he was remembering when he had had to read some of the *Lectures* himself at West Point. Or maybe it was just the singularity of the sight in the man's eyes, as though he were a medieval knight at bivouac who had caught one of the peasant levies, thought to be illiterate, reading Cicero in the original Latin. The truth of this, we shall never know. What the old man actually said was, "Good work, son. Keep

it up."

The stormclouds dissipated from the first sergeant's face. No one ever said anything to me about my boots (in fact as far as I remember, the first sergeant—who saw me seldom in any case—never spoke to me again). The platoon sergeant thereafter explicitly permitted me to read the *Lectures* on duty after midnight on nights when there was nothing else to do, so in the last several months of my military service I did read a number of them. It is fair to say that I also kept my boots better polished.

In Volume I, Chapter 6, of the *Lectures* there is a lovely introduction to probability theory. It discusses the classic problem of the "random walk" in some detail, then states without proof that the generalization of the random walk leads to the Gaussian distribution

$$p(x) = \frac{\exp(-x^2/2\sigma^2)}{\sigma\sqrt{2\pi}}.$$

For the derivation of this remarkable theorem, I scanned the book in vain. One had no Internet access in those days, but besides a well-equipped gym the Army post also had a tiny library, and in one yellowed volume in the library—who knows how such a book got there?—I did find a derivation of the $1/\sigma\sqrt{2\pi}$ factor.[3] The exponential factor, the volume did not derive. Several days later, I chanced to find myself in Munich with an hour or two to spare, which I spent in the university library seeking the missing part of the proof, but lack of time and unfamiliarity with such a German site defeated me. Back at the Army post, I had to sweat the proof out on my own over the ensuing weeks. Nevertheless, eventually I did obtain a proof which made sense to me. Writing the proof down carefully, I pulled the old high-school math notes out of my military footlocker (for some reason I had kept the notes and even brought them to Germany), dusted them off, and added to them the new Gaussian proof.

That is how it has gone. To the old notes, I have added new proofs from time to time, and although somehow I have misplaced the original high-school leaves I took to Germany with me the notes have nevertheless grown with the passing years. These years have brought me the good things years can bring: marriage, family and career; a good life gratefully lived, details of which interest me and mine but are mostly unremarkable as seen from the outside. A life however can take strange turns, reprising earlier themes. I had become an industrial building construction engineer for a living (and, appropriately enough, had most lately added to the notes a

[3]The citation is now unfortunately long lost.

mathematical justification of the standard industrial building construction technique to measure the resistance-to-ground of a new building's electrical grounding system), when at a juncture between construction projects an unexpected opportunity arose to pursue a Ph.D. in engineering at Virginia Tech, courtesy (indirectly, as it developed) of a research program not of the United States Army as last time but this time of the United States Navy. The Navy's research problem turned out to be in the highly mathematical fields of theoretical and computational electromagnetics. Such work naturally brought a blizzard of new formulas, whose proofs I sought or worked out and, either way, added to the notes—whence the manuscript and, in due time, this book.

The book follows in the honorable tradition of Courant's and Hilbert's 1924 classic *Methods of Mathematical Physics* [14]—a tradition subsequently developed by, among others, Jeffreys and Jeffreys [34], Arfken and Weber [3], and Weisstein[4] [65]. The present book's chief intended contribution to the tradition lies in its applied-level derivations of the many results it presents. Its author always wanted to know why the Pythagorean theorem was so. The book is presented in this spirit.

A book can follow convention or depart from it; yet, though occasional departure might render a book original, frequent departure seldom renders a book good. Whether this particular book is original or good, neither or both, is for the reader to tell, but in any case the book does both follow and depart. Convention is a peculiar thing: at its best, it evolves or accumulates only gradually, patiently storing up the long, hidden wisdom of generations past; yet herein arises the ancient dilemma. Convention, in all its richness, in all its profundity, can, sometimes, stagnate at a local maximum, a hillock whence higher ground is achievable not by gradual ascent but only by descent first—or by a leap. Descent risks a bog. A leap risks a fall. One ought not

[4]Weisstein lists results encyclopedically, alphabetically by name. I organize results more traditionally by topic, leaving alphabetization to the book's index, that readers who wish to do so can coherently read the book from front to back.

There is an ironic personal story in this. As children in the 1970s, my brother and I had a 1959 World Book encyclopedia in our bedroom, about twenty volumes. The encyclopedia was then a bit outdated (in fact the world had changed tremendously in the fifteen or twenty years following 1959, so it was more than a bit outdated) but the two of us still used it sometimes. Only years later did I learn that my father, who in 1959 was fourteen years old, had bought the encyclopedia with money he had earned delivering newspapers daily before dawn, *and then had read the entire encyclopedia, front to back.* My father played linebacker on the football team and worked a job after school, too, so where he found the time or the inclination to read an entire encyclopedia, I'll never know. Nonetheless, it does prove that even an encyclopedia can be read from front to back.

run such risks without cause, even in such an inherently unconservative discipline as mathematics.

Well, the book does risk. It risks one leap at least: it employs hexadecimal numerals.

This book is bound to lose at least a few readers for its unorthodox use of hexadecimal notation ("The first primes are $2, 3, 5, 7, 0\text{xB}, \ldots$"). Perhaps it will gain a few readers for the same reason; time will tell. I started keeping my own theoretical math notes in hex a long time ago; at first to prove to myself that I could do hexadecimal arithmetic routinely and accurately with a pencil, later from aesthetic conviction that it was the right thing to do. Like other applied mathematicians, I've several own private notations, and in general these are not permitted to burden the published text. The hex notation is not my own, though. It existed before I arrived on the scene and, since I know of no math book better positioned to risk its use, I have with hesitation and no little trepidation resolved to let this book use it. Some readers will approve; some will tolerate; undoubtedly some will do neither. The views of the last group must be respected, but in the meantime the book has a mission; and crass popularity can be only one consideration, to be balanced against other factors. The book might gain even more readers, after all, had it no formulas, and painted landscapes in place of geometric diagrams! I like landscapes, too, but anyway you can see where that line of logic leads.

More substantively: despite the book's title, adverse criticism from some quarters for lack of rigor is probably inevitable; nor is such criticism necessarily improper from my point of view. Still, serious books by professional mathematicians tend to be *for* professional mathematicians, which is understandable but does not always help the scientist or engineer who wants to use the math to model something. The ideal author of such a book as this would probably hold two doctorates: one in mathematics and the other in engineering or the like. The ideal author lacking, I have written the book.

So here you have my old high-school notes, extended over twenty-five years and through the course of two-and-a-half university degrees, now partly typed and revised for the first time as a LaTeX manuscript. Where this manuscript will go in the future is hard to guess. Perhaps the revision you are reading is the last. Who can say? The manuscript met an uncommonly enthusiastic reception at Debconf 6 [15] May 2006 at Oaxtepec, Mexico; and in August of the same year it warmly welcomed Karl Sarnow and Xplora Knoppix [69] aboard as the second official distributor of the book. Such developments augur well for the book's future at least. But in the meantime, if anyone should challenge you to prove the Pythagorean theorem on the

spot, why, whip this book out and turn to § 2.10. That should confound 'em.

THB

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